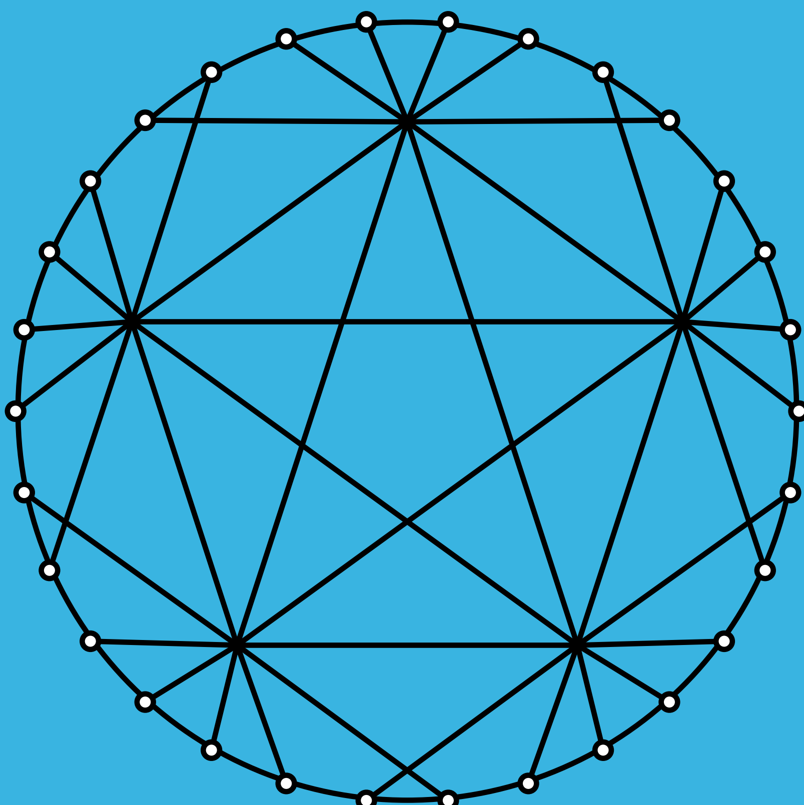
Volume 100
February 2024

BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS



Duluth, Minnesota, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **100** (2024), 138–150

Neighborhood face-magic labelings of ladders, fans, and wheels

Bryan Freyberg and Alexa Hedtke

Abstract. We introduce the following variation of type (a, b, c) magic labelings. Let $G = (V, E, F)$ be a planar graph and $f : V \cup E \cup F \to \{1, 2, \ldots, |V \cup E \cup F|\}$ be a bijection. Define the weight of a face as the sum of the labels of the vertices, edges, and faces bordering that face. If the weight of every s-sided face is equal to some fixed constant μ_s for all s, we call f a neighborhood face-magic labeling. We show that ladders, fans, wheels, and subdivided versions of all these families admit neighborhood face-magic labelings.

1 Introduction

Assume $a, b, c \in \{0, 1\}$ and let G be a simple planar graph with vertex, edge, and face sets denoted by V, E, and F, respectively. A *labeling of type* (a, b, c) is an injective assignment of labels from $\{1, 2, \ldots, a|V| + b|E| + c|F|\}$ to $V \cup E \cup F$ such that every vertex receives a labels, every edge receives b labels, and every face receives c labels. The *weight* of a face $f \in F$ is the sum of the label of f with the labels of the vertices and edges adjacent to f. The labeling is considered *face-magic* if the weight of every s-sided face is equal to some fixed constant for every s (different constants are allowed for different values of s).

The notion of face-magic labeling of type $(1, 1, 0)$ was first introduced by Lih in 1983 and later generalized to the definition just given [5]. For a survey of results in this area (and many other areas in graph labeling), we refer the reader to [2]. Figure 1 shows a face-magic labeling of type $(1, 1, 0)$. The weight of every triangular face is 36 and the weight of every 5-sided face (there is only one) is 63.

Received: 22 May 2023
Accepted: 19 July 2023

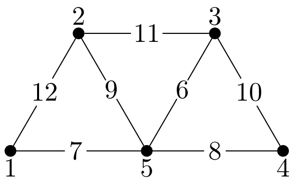


Figure 1: A Face-magic labeling of type $(1, 1, 0)$

One may wonder why the weight of a face f should be determined by the neighborhood of f with respect to vertices and edges but not with respect to faces. We introduce the following alternative way to define the weight of a face so that it is a function of all neighboring elements of the face.

Given a type $(1, 1, 1)$ labeling of a graph G and its planar embedding, define the *neighborhood weight* (from here on we omit the word "neighborhood") of a face f as the sum of the labels of the vertices, edges, and faces that border f. If the weights of every s-sided face is equal to some fixed constant for every s (different constants are allowed for different values of s), we will call the labeling *neighborhood face-magic* and the graph a *neighborhood face-magic graph*.

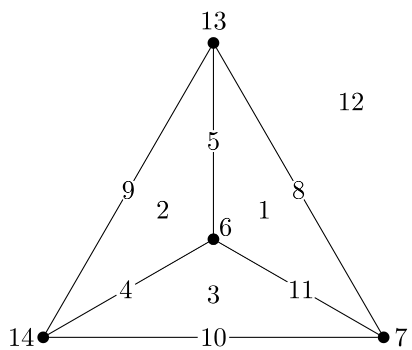


Figure 2: A neighborhood face-magic labeling of W_3

In general, it should be at least as difficult to find a neighborhood face-magic labeling as it is to find a type $(1, 1, 1)$ face-magic labeling of a graph. This is partly due to the fact that the weight of a face under the former labeling is a function of more labels than the weight of the same face under the latter labeling. Figure 2 shows a neighborhood face-magic labeling of the wheel graph W_3. The weight of all four triangular faces is 67.

2 Tools

An $a \times b$ *magic rectangle* $MR(a,b)$ is an $a \times b$ array containing each of the first ab positive integers exactly once such that the sum of elements in each row and the sum of elements in each column is equal to fixed constants ρ and σ, respectively. An $MR(3,5)$ is shown below.

$$\begin{array}{ccccc|c} 7 & 5 & 4 & 10 & 14 & 40 \\ 15 & 13 & 8 & 3 & 1 & 40 \\ 2 & 6 & 12 & 11 & 9 & 40 \\ \hline 24 & 24 & 24 & 24 & 24 & \end{array}$$

Harmuth proved the following in 1881.

Theorem 2.1 (Harmuth [3], [4]). *An $a \times b$ magic rectangle exists if and only if $a \equiv b \pmod 2$, except if $a = 1$, $b = 1$, or $a = b = 2$.*

To address the case when $a \not\equiv b \pmod 2$, Froncek, Paramasivam, and Prajeesh introduced quasi-magic rectangles [1] in 2022. Let a be an odd integer and b an even integer. An $a \times b$ *quasi-magic rectangle* $QMR(a, b : \lambda)$ is an a by b array such that each of the integers $1, 2, \ldots, \lambda - 1, \lambda + 1, \ldots, ab + 1$ appears exactly once, the sum of the elements in each row is equal to some constant ρ, and the sum of the elements in each column is equal to some constant σ.

Theorem 2.2 (Froncek et al. [1]). *A quasi-magic rectangle $QMR(a, 2t : at + 1)$ exists for all odd $a \geq 1$ and $t \geq 1$ except when $t = 1$ and $a \equiv 1 \pmod 4$.*

Another well studied magic-type labeling that will be of use is the following. Let $G = (V, E)$ be a simple graph and $g : V \to \{1, 2, \ldots, |V|\}$ a bijection. If there exists a constant k such that $w(v) = \sum_{uv \in E} g(u) = k$ for all $v \in V$, then we say g is a *distance magic* labeling and the graph G is a *distance magic graph.* For example, Figure 3 shows a distance magic labeling of the wheel graph W_4 with $k = 10$.

Given a planar embedding of G with face set F, the dual of G, denoted G^*, is the graph with vertex set F and two vertices are adjacent in G^* if and only if the corresponding two faces share an edge in G. The graph shown in Figure 3 is the dual of the graph in Figure 4. The neighborhood face-magic labeling of the graph in the latter figure was obtained from the following observation.

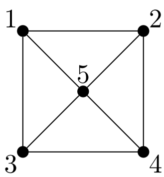


Figure 3: A distance magic labeling of W_4

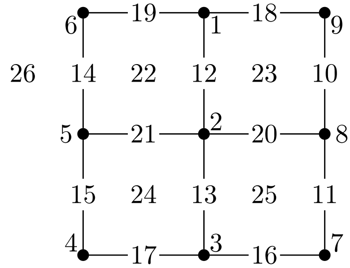


Figure 4: Neighborhood face-magic labeling of a grid graph

Observation 2.3. *If a graph G admits a face-magic labeling of type $(1, 1, 0)$ and its dual G^* admits a distance magic labeling, then G is a neighborhood face-magic graph.*

Proof. Let $G = (V, E, F)$ have a face-magic labeling g of type $(1, 1, 0)$ where the weight of every s-sided face is k_s, and G^* have distance magic labeling h with magic constant k_h. The labeling $\ell : V \cup E \cup F \to \{1, 2, \ldots, |V \cup E \cup F|\}$ defined as

$$\begin{array}{rcl} \ell(v) & = & g(v) \\ \ell(e) & = & g(e) \\ \ell(f) & = & h(f') + |V| + |E| \end{array}$$

where $v \in V$, $e \in E$, and $f' \in V(G^*)$ corresponds to $f \in F$, is a neighborhood face-magic labeling of G since the weight of every s-sided face is $k_s + k_h + s(|V| + |E|)$. $\square$

Unfortunately, distance magic graphs are very rare, so this observation leads to limited results. For example, the wheel W_n is a self-dual graph (see Section 5 for the definition of W_n). It is known that all wheels are face-magic of type $(1, 1, 0)$ and W_n is distance magic if and only if $n = 4$ [6, 8, 7]. So we obtain the following corollary.

Corollary 2.4. *The wheel W_4 is a neighborhood face-magic graph.*

A more fruitful observation is the following.

Observation 2.5. *If G is a neighborhood face-magic graph, then subdividing every edge of G r times forms a graph H that is also neighborhood face-magic.*

Proof. Leave the labels of G unchanged. Subdividing each edge of G by r creates $r|E|$ new edges and $r|E|$ new vertices. Each new vertex-edge pair (v, e) corresponding to the same edge in G can be labeled in a complementary way (symmetric about the middle) from the set $[|V| + |E| + |F| + 1, |V| + (2r + 1)|E| + |F|]$ so that the labels of v and e sum to $2|V| + (2r + 2)|E| + 2|F| + 1$. Therefore, each s-sided face of G corresponds to an $(r + 1)s$-sided face of H whose weight has been increased by $r(2|V| + (2r + 2)|E| + 2|F| + 1)$. Since this number is constant, we have described a neighborhood face-magic labeling of H. $\square$

Figure 5 shows a neighborhood face-magic labeling of the wheel graph W_6 and Figure 6 shows the labeling for the corresponding subdivided graph with $r = 1$.

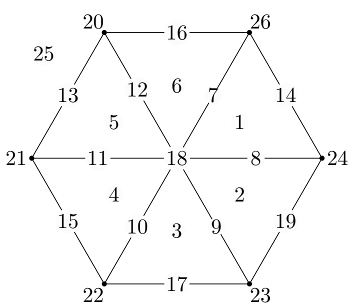


Figure 5: Neighborhood face-magic labeling of W_6

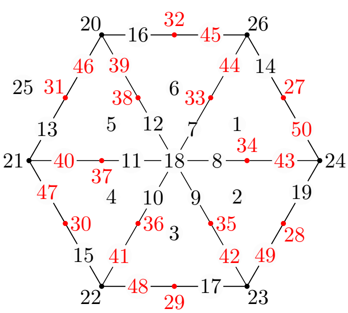


Figure 6: Construction from Observation 2.5 with $r = 1$

3 Ladders

For $n \geq 2$, the *ladder graph* $L_n \cong P_n \square P_2$ is the graph with vertex set $V = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. Let F be the set of faces where $f_i \in F$ is the face bounded by the four edges $u_i v_i, u_i u_{i+1}, u_{i+1} v_{i+1}$, and $v_i v_{i+1}$ for $1 \leq i \leq n - 1$. We denote the $2n$-sided exterior face as f_∞. We will refer to the set $R_i = \{u_i, v_i, u_iv_i\}$ as the i^{th} *rung* of the ladder and the sets $\{u_1, u_1u_2, u_2, u_2u_3, \ldots, u_n\}$ and $\{v_1, v_1v_2, v_2, v_2v_3, \ldots, v_n\}$ as the two *rails* of the ladder. In total, we have $|V| = 2n, |E| = 3n-2$, and $|F| = n$.

Theorem 3.1. *The ladder graph $L_n \cong P_n \square P_2$ is a neighborhood face-magic graph if and only if $n \geq 3$.*

Proof. Let $G \cong L_n$ be embedded in the plane in the natural way. There does not exist a neighborhood face-magic labeling of L_2, for if there did, the label of the two faces would be the same. So we may assume $n \geq 3$ from now on. Let $S = [1, 6n-2]$ and form the partition $S = A \cup B$ where $A = [1, 3n-3]$ and $B = [3n-2, 6n-2]$. Define $\ell : V \cup E \cup F \to S$ as follows. Let

$$\begin{array}{lcl} \ell(f_i) & = & i, \text{ for } 1 \leq i \leq n-1, \\ \ell(u_iu_{i+1}) & = & 2n-2-i, \text{ for } 1 \leq i \leq n-2, \\ \ell(u_{n-1}u_n) & = & 2n-2, \text{ and} \\ \ell(v_iv_{i+1}) & = & 3n-2-i, \text{ for } 1 \leq i \leq n-1. \end{array}$$

Thus far, we have assigned all the labels from A and only the rungs and f_∞ remain to be labeled.

If n is odd, let M be a magic rectangle $MR(3, n)$ and assume WLOG the largest element $3n$ lies in the last column of M. Otherwise, let M be a quasi-magic rectangle $QMR(3, n : \lambda)$ such that the element $\lambda - 1$ lies in the last column. Such an M exists by Theorem 2.1 or Theorem 2.2. Let σ be the column sum of M. Increase every element in M by $3n-3$. If n is odd, replace the entry $6n-3$ with $6n-2$. If n is even, replace the entry $\lambda + 3n - 4$ with $\lambda + 3n - 3$. Let M' denote this new $3 \times n$ array and let ℓ map column i of M' to the i^{th} rung of L_n in the natural way. Finally, let

$$\ell(f_\infty) = \begin{cases} 6n-3 & n \text{ is odd} \\ \lambda + 3n - 4 & n \text{ is even} \end{cases}.$$

We claim that ℓ is a neighborhood face-magic labeling of L_n. It is clear that ℓ is a bijection, so it remains to check the weights of the 4-sided faces. Let $i \in [1, n-1]$ be given. If $i = 1$, we have

$$\begin{array}{rcl} w(f_1) & = & \ell(f_2) + \ell(u_1u_2) + \ell(v_1v_2) + \ell(R_1) + \ell(R_2) + \ell(f_\infty) \\ & = & 2 + (2n-3) + (3n-3) + 2(\sigma + 3(3n-3)) + \ell(f_\infty) \\ & = & 23n - 22 + 2\sigma + \ell(f_\infty). \end{array}$$

If $2 \leq i \leq n-2$, we have

$$\begin{aligned} w(f_i) &= \ell(f_{i\pm 1}) + \ell(u_i u_{i+1}) + \ell(v_i v_{i+1}) + \ell(R_i) + \ell(R_{i+1}) + \ell(f_\infty) \\ &= 2i + (2n-2-i) + (3n-2-i) + 2(\sigma + 3(3n-3)) + \ell(f_\infty) \\ &= 23n - 22 + 2\sigma + \ell(f_\infty). \end{aligned}$$

Finally, if $i = n-1$, we obtain

$$\begin{aligned} w(f_{n-1}) &= \ell(f_{n-2}) + \ell(u_{n-1}u_n) + \ell(v_{n-1}v_n) + \\ &\quad\ \ell(R_{n-1}) + \ell(R_n) + \ell(f_\infty) \\ &= (3n-4) + (3n-2-(n-1)) + \\ &\quad\ 2\sigma + 1 + 6(3n-3)) + \ell(f_\infty) \\ &= 23n - 22 + 2\sigma + \ell(f_\infty). \end{aligned}$$

Since the weight of every 4-sided face is the same, we have proved the claim. □

Figure 7 shows the neighborhood face-magic labeling of L_5 produced by the construction of Theorem 3.1 in conjunction with the 3×5 magic rectangle from Section 2. The weight of each 4-sided face is 168.

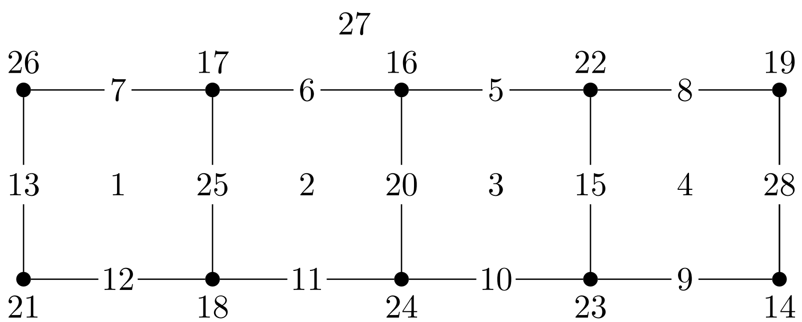


Figure 7: A neighborhood face-magic labeling of L_5

4 Fans

The *fan graph* $F_n \cong P_n + K_1$ is the join of a path on $n \geq 2$ vertices and a single vertex called the *hub*. Let

$$\begin{aligned} V(F_n) &= \{v_i : 1 \leq i \leq n\} \cup \{h\}, \\ E(F_n) &= \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{hv_i : 1 \leq i \leq n\}, \text{ and} \\ F(F_n) &= \{f_i : 1 \leq i \leq n-1\} \cup f_\infty, \end{aligned}$$

where f_i is the triangular face bound by the edges hv_i, v_iv_{i+1}, and hv_{i+1}, and f_∞ is the exterior $n+1$-sided face.

Theorem 4.1. *The fan F_n is a neighborhood face-magic graph if and only if $n \geq 3$.*

Proof. Assume $G \cong F_n$ is embedded in the plane as its namesake suggests. If $n = 2$, the graph consists of a single triangle f_1 with exterior triangle f_∞. Suppose the graph has a neighborhood face-magic labeling. Since both triangles share all 6 vertices and edges, the label of each face must be the same. But this is a contradiction, since the labeling is an injection. So, from now on we may assume $n \geq 3$.

We describe a labeling $\ell : V \cup E \cup F \to [1, 4n]$ as follows. Let $\ell(f_i) = i$ for $i \in [1, n-1]$, $\ell(v_iv_{i+1}) = \lceil \frac{7n-2}{2} \rceil - 2i$ for $i \in [1, n-2]$, and $\ell(h) = n$. If n is odd, let

$$\begin{array}{lcl} \ell(f_\infty) & = & \frac{5n-1}{2}, \\ \ell(v_{n-1}v_n) & = & \frac{5n+3}{2}, \text{ and} \\ \ell(\{v_i, hv_i\}) & = & \{s_i, t_i : s_i \in S_j, t_i \in T_j, s_i + t_i = 5n+1\} \text{ for } i \in [1, n], \end{array}$$

where $S_1 = [n+1, \frac{3n+5}{2}]$, $S_2 = \{\frac{3n+9}{2}, \frac{3n+13}{2}, \ldots, \frac{5n-5}{2}\}$, $T_1 = [\frac{7n-3}{2}, 4n]$, and $T_2 = \{\frac{5n+7}{2}, \frac{5n+11}{2}, \ldots, \frac{7n-7}{2}\}$. Notice, $|S_1| = |T_1| = \frac{n+5}{2}$ and $|S_2| = |T_2| = \frac{n-5}{2}$, so the n pairs (s_i, t_i) may be chosen so they form a (disjoint) partition of $S_1 \cup S_2 \cup T_1 \cup T_2$.

Similarly, if n is even, let

$$\begin{array}{lcl} \ell(f_\infty) & = & 4n-1, \\ \ell(v_{n-1}v_n) & = & \frac{5n}{2}, \\ \ell(\{v_i, hv_i\}) & = & \{s'_i, t'_i : s'_i \in S'_j, t'_i \in T'_j, s'_i + t'_i = 5n\} \text{ for } i \in [1, n-1], \\ \ell(v_n) & = & 4n, \text{ and} \\ \ell(hv_n) & = & n+1, \end{array}$$

where $S'_1 = [n+2, \frac{3n}{2}+2]$, $S'_2 = \{\frac{3n}{2}+4, \frac{3n}{2}+6, \ldots, \frac{5n}{2}-2\}$, $T'_1 = [\frac{7n}{2}-2, 4n-2]$, and $T'_2 = \{\frac{5n}{2}+2, \frac{5n}{2}+4, \ldots, \frac{7n}{2}-4\}$. Notice, $|S'_1| = |T'_1| = \frac{n}{2}+1$ and $|S'_2| = |T'_2| = \frac{n}{2}-2$, so the $n-1$ pairs (s'_i, t'_i) may be chosen so they form a partition of $S'_1 \cup S'_2 \cup T'_1 \cup T'_2$.

With the labeling completed, we may now compute the weights. Let $f_i \in F$ be given. If $i \in [2, n-2]$, then the weight of f_i is

$$\begin{array}{rcl} w(f_i) & = & \ell(f_{i-1}) + \ell(f_{i+1}) + \ell(h) + \ell(v_iv_{i+1}) + \\ & & \ell(v_i) + \ell(hv_i) + \ell(v_{i+1}) + \ell(hv_{i+1}) + \ell(f_\infty). \end{array}$$

If n is odd, then this is

$$\begin{aligned} w(f_i) &= 2i + n + \tfrac{7n-1}{2} - 2i + 2(5n+1) + \tfrac{5n-1}{2} \\ &= 17n + 1. \end{aligned}$$

On the other hand, if n is even it is

$$\begin{aligned} w(f_i) &= 2i + n + \tfrac{7n}{2} - 1 - 2i + 2(5n) + 4n - 1 \\ &= \tfrac{37n}{2} - 2. \end{aligned}$$

Finally, for $i \in \{1, n-1\}$, we have

$$\begin{aligned} w(f_1) &= \ell(f_2) + \ell(h) + \ell(v_1v_2) + \ell(f_\infty) + \\ &\quad \ell(v_1) + \ell(hv_1) + \ell(v_2) + \ell(hv_2), \end{aligned}$$

and

$$\begin{aligned} w(f_{n-1}) &= \ell(f_{n-2}) + \ell(h) + \ell(v_{n-1}v_n) + \ell(f_\infty) + \\ &\quad \ell(v_{n-1}) + \ell(hv_{n-1}) + \ell(v_n) + \ell(hv_n). \end{aligned}$$

It is straightforward to check that

$$w(f_1) = w(f_{n-1}) = \begin{cases} 17n+1 & n \text{ is odd} \\ \frac{37n}{2} - 2 & n \text{ is even} \end{cases}.$$

Since the weight of every triangular face of G is the same, we have described a neighborhood face-magic labeling of G. $\square$

Figures 8 and 9 show the construction from Theorem 4.1 applied to the fans F_5 and F_6, respectively.

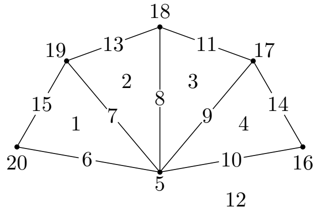


Figure 8: Neighborhood face-magic labeling of F_5

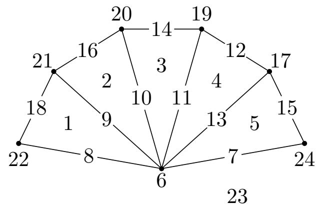


Figure 9: Neighborhood face-magic labeling of F_6

5 Wheels

The *wheel graph* $W_n \cong C_n + K_1$ is the join of a cycle of length $n \geq 3$ and a single vertex called the *hub*. The edges incident with the hub are called *spokes* and the edges around the perimeter are called *rim* edges. Figures 2, 3, 10, 5, and 11 show W_3, W_4, W_5, W_6, and W_8, respectively.

Theorem 5.1. *The wheel W_n is a neighborhood face-magic graph if and only if $n \geq 3$.*

Proof. Let $G = (V, E, F) \cong W_n$ be the natural embedding of the wheel graph with vertex set $V = \{v_i : i \in [1, n]\} \cup \{h\}$ and edge set $E = E_s \cup E_r$ where $E_s = \{hv_i : i \in [1, n]\}$ are the spoke edges and $E_r = \{v_i v_{i+1} : i \in [1, n]\}$ (with arithmetic performed modulo n) are the rim edges. Denote the triangular face enclosed by the set of edges $\{hv_i, hv_{i+1}, v_i v_{i+1}\}$ as f_i for $i \in [1, n]$, and the n-sided exterior face by f_∞. By the labeling shown in Figure 2, we may assume $n \geq 4$. We describe a type $(1, 1, 1)$ labeling $\ell : V \cup E \cup F \to [1, 4n + 2]$ as follows.

Case 1. n is odd and $n \geq 5$.
For $i \in [1, n]$ let $\ell(f_i) = i$. Define $m = \frac{5n+3}{2}$ and label the rim edges

$$\begin{aligned} \ell(v_1 v_2) &= m - 1, \\ \ell(v_1 v_n) &= m + 1, \\ \ell(v_i v_{i+1}) &= m + n + 1 - 2i, \end{aligned}$$

for $i \in [2, n - 1]$.

The labels that remain form the set $A \cup B \cup C$ where

$$\begin{aligned} A &= [n + 1, m - n + 2], \\ B &= \{m - n + 4, m - n + 6, \ldots, m - 5, m - 3\} \cup \\ &\quad\ \{m + 3, m + 5, \ldots, m + n - 6, m + n - 4\}, \text{ and} \\ C &= [m + n - 2, 4n + 2]. \end{aligned}$$

It is easy to see that $|A| = |C| = \frac{n+7}{2}$ and $|B| = n - 5$. Form the set of pairs S such that

$$S = \{(a, c) : a + c = 5n + 3, a \in A, c \in C\} \cup \{(b, b') : b + b' = 5n + 3, b, b' \in B\}.$$

Notice that S partitions the remaining labels into $\frac{n+7}{2} + \frac{n-5}{2} = n + 1$ pairs with common sum $5n + 3$. We complete the assignment ℓ by forming an arbitrary bijection between $\{(h, f_\infty), (v_i, hv_i) : i \in [1, n]\}$ and S.

Let $i \in [1, n]$. We have

$$\begin{aligned} w(f_i) &= \ell(v_i) + \ell(hv_i) + \ell(v_{i+1}) + \ell(hv_{i+1}) + \ell(h) + \ell(f_\infty) + \\ &\quad\ \ell(v_i v_{i+1}) + \ell(f_{i-1}) + \ell(f_{i+1}) \\ &= 3(5n + 3) + (m + n + 1) \\ &= \tfrac{37n+23}{2}. \end{aligned}$$

The next case follows in much the same fashion.

Case 2. n is even and $n \geq 4$.
For $i \in [1, n]$ let $\ell(f_i) = i$. Define $m = \frac{5n+2}{2}$ and let

$$\begin{array}{lcl} \ell(f_\infty) & = & 4n+1, \\ \ell(h) & = & m+2, \\ \ell(v_1) & = & 4n+2, \\ \ell(hv_1) & = & n+1, \\ \ell(v_1v_2) & = & m-2, \text{ and} \\ \ell(v_1v_n) & = & m. \end{array}$$

For $i \in [2, n-1]$, let

$$\ell(v_iv_{i+1}) = m + n - 2i + 1.$$

The labels that remain form the set $A \cup B \cup C$ where

$$\begin{array}{lcl} A & = & [n+2, m-n+2], \\ B & = & \{m-n+4, m-n+6, \ldots, m-4\} \\ & \cup & \{m+4, m+6, \ldots, m+n-6, m+n-4\}, \text{ and} \\ C & = & [m+n-2, 4n]. \end{array}$$

It is easy to see that $|A| = |C| = \frac{n+4}{2}$ and $|B| = n-6$. Form the set of pairs S such that

$$S = \{(a,c) : a+c = 5n+2, a \in A, c \in C\} \cup \{(b,b') : b+b' = 5n+2, b, b' \in B\}.$$

Notice that S partitions the remaining labels into $\frac{n+4}{2} + \frac{n-6}{2} = n-1$ pairs with common sum $5n+2$. We complete the assignment ℓ by forming an arbitrary bijection between $\{(v_i, hv_i) : i \in [2, n]\}$ and S.

Let $i \in [1, n]$. We have

$$\begin{array}{lcl} w(f_i) & = & \ell(v_i) + \ell(hv_i) + \ell(v_{i+1}) + \ell(hv_{i+1}) + \ell(v_iv_{i+1}) + \ell(f_{i\pm1}) + \\ & & \ell(h) + \ell(f_\infty) \\ & = & (11n + m + 5) + (4n + m + 3) \\ & = & 20n + 10. \end{array}$$

In both cases we have shown that the weight of every triangular face is equal to the same constant. The weight of the n-sided exterior face is irrelevant since $n > 3$. Hence, we have proved the claim. $\square$

Figures 10 and 11 demonstrate the labeling just described for $n = 5$ and $n = 8$, respectively.

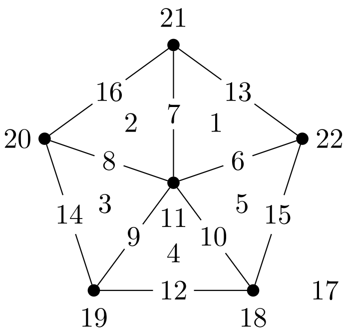


Figure 10: Neighborhood face-magic labeling of W_5

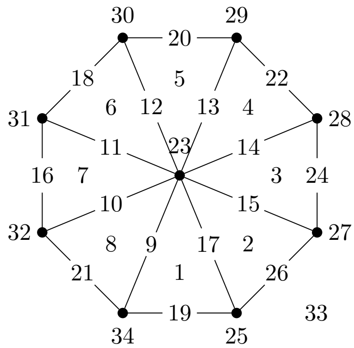


Figure 11: Neighborhood face-magic labeling of W_8

6 Concluding remarks

We have shown that all ladders, fans, and wheels are neighborhood face-magic graphs. By Observation 2.5, any subdivided version of these graphs is also neighborhood face-magic. One obvious path forward is to consider other infinite families of graphs. Another problem to consider is the type $(a, b, 1)$ analog of neighborhood face-magic labeling where $a = 0$ or $b = 0$.

7 Acknowledgements

This work was partially supported by the University of Minnesota Office of Undergraduate Research.

References

[1] D. Froncek, K. Paramasivam, and A.V. Prajeesh, Quasi-magic rectangles, *J. Combin. Des.*, **30** (2022), 193–202.

[2] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, **5** (1998), Dynamic Survey 6, 43 pp.

[3] T. Harmuth, Ueber magische Quadrate und ähnliche Zahlenfiguren, *Arch. Math. Phys.*, **66** (1881), 286–313.

[4] T. Harmuth, Ueber magische Rechtecke mit ungeraden Seitenzahlen, *Arch. Math. Phys.*, **66** (1881), 413–447.

[5] K. Lih, On magic and consecutive labelings of plane graphs, *Util. Math.*, **24** (1983), 165–197.

[6] A. Lladó, J. Moragas. Cycle-magic graphs, *Discrete Math.*, **307** (2007), 2925–2933.

[7] M. Miller, C. Rodger, R. Simanjuntak. Distance magic labelings of graphs, *Austral. J. Combin.*, **28** (2003), 305–315.

[8] M. Roswitha, E.T. Baskoro, T.K. Maryati, N.A. Kurdhi, I. Susanti. Further results on cycle-supermagic labeling, *AKCE Int. J. Graphs Comb.*, **10** (2013), 211–220.

BRYAN FREYBERG AND ALEXA HEDTKE
UNIVERSITY OF MINNESOTA DULUTH, DULUTH MN, USA
frey0031@d.umn.edu, hedtk041@d.umn.edu