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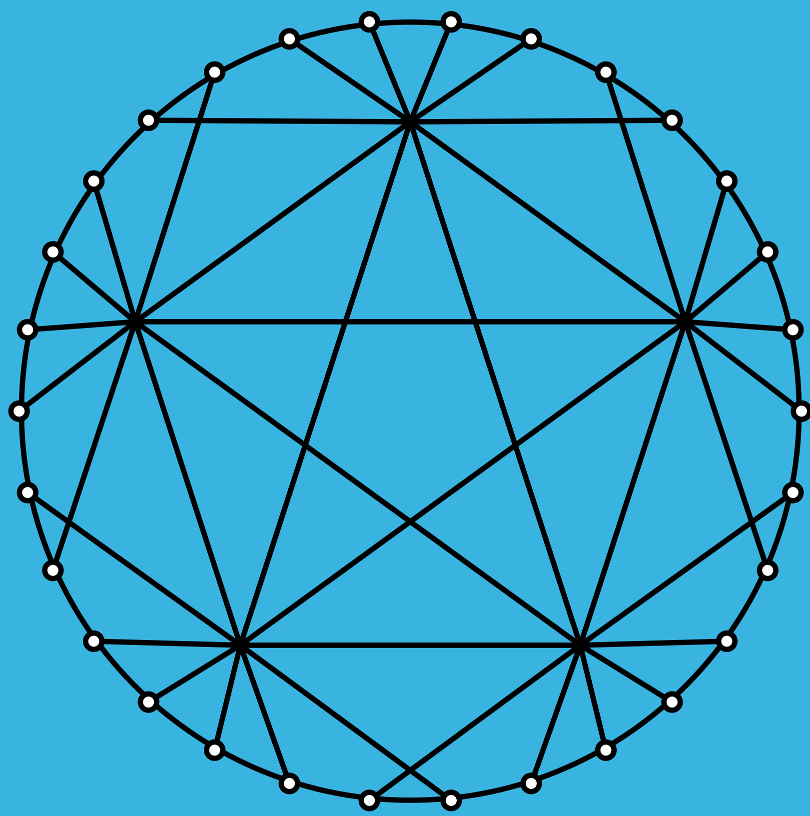

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Group-supermagic labeling of Cartesian products of two odd cycles

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Abstract. A Γ-supermagic labeling of a graph $G = (V, E)$ with $|E| = k$ is a bijection from E to an Abelian group Γ of order k such that the sum of labels of all incident edges of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. An existence of a Γ-supermagic labeling of Cartesian product of two cycles, $C_n \square C_m$ for every $n, m \geq 3$ by Z_{2mn} was proved recently. In this paper we present a labeling method for all Abelian groups of order $2mn$ where m, n are odd and greater than one.

1 Motivation

The Cartesian product of cycles $C_{n_1} \square C_{n_2} \square \cdots \square C_{n_s}$ can be viewed as the Cayley graph of group $Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_s}$ generated by group elements $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$.

It is then natural to ask the following question. Given the Cartesian product $G = C_{n_1} \square C_{n_2} \square \cdots \square C_{n_s}$ with vertex set V and edge set E and $\Gamma = Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_t}$ a finite Abelian group or order $|V|$, $|E|$, or $|V| + |E|$, can the vertices, edges, or both, respectively, be labeled bijectively with elements of Γ so that the weight of every vertex is the same element of Γ? The weight of a vertex is the sum of the labels of the elements incident and/or adjacent to every vertex. Exact definitions of the above notions are given in Section 2.

The question was originally studied for labelings of products of two cycles by positive integers. The vertex version was fully settled by Rao, Singh,

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and Parameswaran in [11], and the edge version was partially solved by Ivančo [9]. We give an overview of the results for integers and Abelian groups for both vertex and edge labelings in Sections 3 and 4.

In Section 5 we present a recursive construction for Γ-supermagic labeling of the Cartesian product $C_m \Box C_n$ of any two odd cycles for all Abelian groups Γ of order $2mn$.

2 Definitions and background results

Although the Cartesian product of graphs is a well known notion, we define it here for completeness.

Definition 2.1. The *Cartesian product* $G = G_1 \Box G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2, respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

We start with definitions of distance magic and Γ-distance magic labelings. The notion of distance magic labeling was introduced independently by several authors under different names. We use the term distance magic as it became the most common form.

Definition 2.2. A *distance magic labeling* of a graph $G(V, E)$ with $|V| = p$ is a bijection g from V to the set $\{1, 2, \ldots, p\}$ such that the sum of labels of all adjacent vertices of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same positive constant c, called the *magic constant*. That is,

$$w(x) = \sum_{y:xy \in E} g(y) = c$$

for every vertex $x \in V$. A graph that admits a distance magic labeling is called a *distance magic graph*.

The group version of the labeling, called a Γ-*distance magic labeling*, was introduced in [3].

Definition 2.3. A Γ-*distance magic labeling* of a graph $G(V, E)$ with $|V| = p$ is a bijection g from V to an Abelian group Γ of order p such that the sum of labels of all adjacent vertices of every vertex $x \in V$, called the *weight* of

x and denoted $w(x)$, is equal to the same element $\mu \in \Gamma$, called the *magic constant.* That is,

$$w(x) = \sum_{y:xy \in E} g(y) = \mu$$

for every vertex $x \in V$. A graph that admits a Γ-distance magic labeling is called a Γ-*distance magic graph.*

The notion of supermagic labeling was also studied under the name of *vertex-magic edge labeling.*

Definition 2.4. A *supermagic labeling* of a graph $G(V, E)$ with $|E| = q$ is a bijection f from E to the set $\{1, 2, \ldots, q\}$ such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same positive constant c, called the *magic constant.* That is,

$$w(x) = \sum_{xy \in E} f(xy) = c$$

for every vertex $x \in V$. A graph that admits a supermagic labeling is called a *supermagic graph.*

A more general form of edge labeling, called *magic labeling*, dates back to Sedláček [12] who allowed the labels to be real numbers. Stanley [14, 15] used the term for labeling with non-negative integers. Stewart [16] introduced the notion of supermagic labeling, where the set of labels consisted of $|E|$ consecutive integers. It is easy to observe that when a supermagic graph is regular, then the edge labels can start with any positive integer, and therefore are always considered to be $1, 2, \ldots, |E|$.

Definition 2.5. A Γ-*supermagic labeling* of a graph $G(V, E)$ with $|E| = q$ is a bijection f from E to an Abelian group Γ of order q such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same element $\mu \in \Gamma$, called the *magic constant.* That is,

$$w(x) = \sum_{xy \in E} f(xy) = \mu$$

for every vertex $x \in V$. A graph that admits a Γ-supermagic labeling is called a Γ-*supermagic graph.*

We also state here some well known group theory results, which we will use later, starting with *The Fundamental Theorem of Finite Abelian Groups.*

Theorem 2.6 (The Fundamental Theorem of Finite Abelian Groups)**.** *Let Γ be an Abelian group of order $n = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$, where $k \geq 1$, $p_1, p_2, \dots, p_k$ are primes, not necessarily distinct, and $s_1, s_2, \dots, s_k$ positive integers.*

Then Γ is isomorphic to $Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \dots \oplus Z_{p_k^{s_k}}$ and if we moreover require that for every $i = 1, 2, \dots, k-1$ we have $p_i \leq p_{i+1}$ and if $p_i = p_{i+1}$, then $s_i \leq s_{i+1}$, then the expression determined by the k-tuple $(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k})$ is unique and we call it the canonical form *of the group Γ and denote by $\Gamma(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k})$.*

Another well known theorem is the following.

Theorem 2.7. *An Abelian group*

$$\Gamma(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k}) = Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \dots \oplus Z_{p_k^{s_k}}$$

is cyclic if and only if all primes p_i, $i = 1, 2, \dots, k$ are distinct.

3 Related results

We first list results on distance magic and Γ-distance magic labelings of Cartesian cycle products, since there has been more research in the direction of non-cyclic Abelian groups.

Rao, Singh, and Parameswaran proved the following [11].

Theorem 3.1 (Rao et al., [11])**.** *The graph $C_n \square C_m$ is distance magic if and only if $n, m \equiv 2 \pmod 4$ and $n = m \geq 6$.*

Froncek introduced the concept of Γ-distance magic labeling [3] and proved a complete result on Γ-distance magic labeling of Cartesian product of two cycles with cyclic groups.

Theorem 3.2 (Froncek, [3])**.** *The Cartesian product $C_m \square C_n$ has a Z_{mn}-distance magic labeling if and only if $m, n \geq 3$ and mn is even.*

Cichacz [1] proved a more general result for other Abelian groups.

Theorem 3.3 (Cichacz, [1])**.** *Let m, n, s, t be positive integers, $m, n \geq 3$ and $l = \operatorname{lcm}(m, n)$. Let $\Gamma = Z_{lt} \times A$, where A is an Abelian group of order s and $mn = lst$. Then the Cartesian product $C_m \square C_n$ has a Γ-distance magic labeling.*

This was recently slightly improved by Cichacz, Dyrlaga, and Froncek.

Theorem 3.4 (Cichacz et al., [2]). *Let mn be even and $l = \mathrm{lcm}(m, n)$. Then $C_m \square C_n$ has a $Z_\alpha \times A$-magic labeling for any $\alpha \equiv 0 \pmod{l/2}$ and any Abelian group A of order mn/α.*

A stronger result holds for $C_n \square C_n$.

Theorem 3.5 (Cichacz et al., [2]). *Let n be even. The graph $G = C_n \square C_n$ has a Γ-distance magic labeling for any Abelian group Γ of order n^2.*

As opposed to Γ-supermagic labeling of $C_m \square C_n$, where we show in Section 5 that such a labeling of odd cycles exists for each Abelian group Γ of an appropriate order, this is not the case for Γ-distance magic labeling.

Recall that for a given Abelian group Γ, the *exponent* of Γ, $\exp(\Gamma)$, is the smallest positive integer r such that for every element $g \in \Gamma$, $g^r = e$, the identity element. There are two bounds for the existence of Γ-distance magic labeling of $C_m \square C_n$ related to $\exp(\Gamma)$.

For a positive integer m define a function

$$f(m) = \begin{cases} m/4 & \text{if} \quad m \equiv 0 \pmod 4, \\ m/2 & \text{if} \quad m \equiv 2 \pmod 4, \\ m & \text{if} \quad m \equiv 1 \pmod 2. \end{cases}$$

Theorem 3.6 (Cichacz et al., [2]). *Let Γ be an Abelian group of an even order mn with exponent r. If $2r \min\{f(m), f(n)\} < \mathrm{lcm}(m, n)$, then there does not exist a Γ-distance magic labeling of the Cartesian product $C_m \square C_n$.*

Theorem 3.7 (Cichacz et al., [2]). *Let Γ be an Abelian group of an even order mn with exponent r. If $2r \gcd(m, n) < \mathrm{lcm}(m, n)$, then there does not exist a Γ-distance magic labeling of the Cartesian product $C_m \square C_n$.*

Each of the two theorems is giving slightly different results. When $m \equiv 0 \pmod 4$ and $n = mq$, then Theorem 3.6 gives a stronger result; when m is odd and $n \neq mq$, then Theorem 3.7 is stronger.

4 Known results

Similar results to Theorem 3.1 for supermagic labeling with positive integers were proved by Ivančo [9].

Theorem 4.1 (Ivančo, [9]). *Let $n \geq 3$. Then the Cartesian product $C_n \square C_n$ has a supermagic labeling.*

Theorem 4.2 (Ivančo, [9]). *Let $m, n \geq 4$ be even integers. Then $C_m \square C_n$ has a supermagic labeling.*

Ivančo also conjectured that there exists a supermagic labeling for all Cartesian products $C_m \square C_n$.

Conjecture 4.3 (Ivančo, [9]). *The Cartesian product $C_m \square C_n$ allows a supermagic labeling for any $m, n \geq 3$.*

In an unpublished manuscript, Froncek [4] verified that the conjecture is true when m, n are both odd and not relatively prime.

Froncek, McKeown, McKeown, and McKeown [6] proved that a Γ-supermagic labeling equivalent of the conjecture is true for the cyclic group Z_{2mn} when at least one of m, n is odd.

Theorem 4.4 (Froncek et al., [6]). *The Cartesian product $C_m \square C_n$ admits a Z_{2mn}-supermagic labeling for all odd $m \geq 3$ and any $n \geq 3$.*

This along with Ivančo's theorem provided a complete result for cyclic groups.

Theorem 4.5 (Froncek et al., [6]). *The Cartesian product $C_m \square C_n$ admits a Z_{2mn}-supermagic labeling for all $m, n \geq 3$.*

Froncek [5] recently extended the result to direct and strong products of cycles.

Theorem 4.6 (Froncek, [5]). *The direct product $C_m \times C_n$ admits a Z_{2mn}-supermagic labeling for all $m, n \geq 3$.*

Theorem 4.7 (Froncek, [5]). *The strong product $C_m \boxtimes C_n$ admits a Z_{4mn}-supermagic labeling for all $m, n \geq 3$.*

The results obtained in Theorem 4.4 were based on two similar constructions for m, n both odd, and m odd and n even. The main building blocks in these constructions were labelings of the horizontal cycles consecutively with the even elements of Z_{2mn} and the vertical cycles with the odd elements. The last case of m, n both even relied on Ivančo's existential result in Theorem 4.2.

Because Ivančo did not provide a construction of the labeling whose existence he proved, Froncek and McKeown [7] later supplied a new unified construction for all parity combinations of m and n. In this construction, the main building block is the *diagonal* of the product $C_m \Box C_n$, which is a cycle alternating vertical and horizontal edges. Then the vertical edges are labeled with one coset of Z_{2mn} in increasing order, while the horizontal edges are labeled by another coset in decreasing order.

The diagonal construction was then used by Sorensen [13] and Paananen [10] to obtain the following.[1]

Theorem 4.8 (Paananen and Sorensen, [10, 13])**.** *For integers $m \geq 3$ and $n \geq 3$, $C_m \Box C_n$ can be labeled with group elements from the group $Z_{mn} \oplus Z_2$ to form a Γ-supermagic labeling.*

When m, n are both odd, the above is equivalent to Theorem 4.5.

Theorem 4.9 (Paananen and Sorensen, [10, 13])**.** *Let $m, n \geq 3$ and A_k be an Abelian group of order k. Then $C_m \Box C_n$ can be labeled with group elements from the group $\Gamma = Z_{\mathrm{lcm}(m,n)} \oplus A_{2\gcd(m,n)}$ to form a Γ-supermagic labeling.*

Theorem 4.10 (Paananen and Sorensen, [10, 13])**.** *Let $n \equiv 0 \pmod 4$, $n = 4k$, and A_{2n} be any Abelian group of order $2n$. Then the Cartesian product $C_n \Box C_n$ can be labeled with group elements from the group $Z_{2k} \oplus Z_2 \oplus A_{2n}$ to form a Γ-supermagic labeling.*

We remark here that in [13] there was a typo in Theorem 4.10 (which was Theorem 4.3 in [13]); the group was given as $Z_k \oplus Z_2 \oplus A_{2n}$ rather than $Z_{2k} \oplus Z_2 \oplus A_{2n}$.

The following two results by Sorensen [13] and Paananen [10] used a recursive construction, building labelings of larger Cartesian products combining several copies of smaller ones.

Theorem 4.11 (Paananen and Sorensen, [10, 13])**.** *For $m, n \geq 2$ the graph $C_{2^m} \Box C_{2^n}$ has a Γ-supermagic labeling for $\Gamma = Z_2^{m+n+1}$.*

[1] Paananen [10] (2021) and Sorensen [13] (2020) worked on a joint project for their MS theses. While all results cited here are their joint work, their theses were written and defended independently. Both theses contain Theorems 4.8–4.12.

Theorem 4.12 (Paananen and Sorensen, [10, 13])**.** *For $m,n \geq 3$ and $j,k \geq 1$, the graph $C_{m^j}\square C_{n^k}$ has Γ-supermagic labeling for $\Gamma = Z_{mn} \oplus Z_m^{j-1} \oplus Z_n^{k-1} \oplus Z_2$.*

In the following section we prove that a Γ-supermagic labeling equivalent to Conjecture 4.3 for odd cycles is true for any Abelian group Γ of the proper order.

5 Construction

When the primes $p_1, p_2, \dots, p_k$ in $\Gamma(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k})$ are all distinct, then Γ is cyclic, and the labeling exists by Theorem 4.5.

Our construction for the remaining case when $p_i = p_{i+1}$ for some i is recursive and the proofs are by induction on the order of the group

$$\Gamma(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k})$$

(or the number of edges in $C_m\square C_n$, which is equivalent). We first show that for some i, $p_i^{s_i}$ divides one of m,n and then construct a labeling of the graph $G = C_m\square C_n$ using a labeling of a graph $G' = C_m\square C_t$, where t divides n and $n = tp_i^{s_i}$ for some $p_i^{s_i}$ in the canonical expression of the group $\Gamma(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k})$.

We take $p_i^{s_i} = q$ copies $G'_1, G'_2, \dots, G'_q$ of $G' = C_m\square C_t$ labeled by the group

$$\begin{aligned}\Gamma'(p_1^{s_1}, p_2^{s_2}, \dots, p_{i-1}^{s_{i-1}}, p_{i+1}^{s_{i+1}}, \dots, p_k^{s_k})&\\ = Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \dots Z_{p_{i-1}^{s_{i-1}}} \oplus Z_{p_{i+1}^{s_{i+1}}} \oplus \dots \oplus Z_{p_k^{s_k}},&\end{aligned}$$

cut and unfold each of them and glue them together to obtain the graph $G = C_m\square C_n$ labeled non-bijectively with the group Γ'. Then we add a different element of the group $Z_{p_i^{s_i}}$ (thus achieving bijectivity) to the label of each edge $e_1, e_2, \dots, e_q$ belonging to $G'_1, G'_2, \dots, G'_q$, respectively, all corresponding to the same edge e in G' so that the resulting labeling is Γ-supermagic.

We start with a useful lemma. We prove it in full generality, although we only need it for m,n odd and thus the case of $p_1 = p_2 = 2$ would not be necessary.

Lemma 5.1. *Let $G = C_m \square C_n$ and $\Gamma(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k})$ be of order $2mn$. If $p_{i-1} < p_i = p_{i+1}$ for some $i \in \{2, 3, \dots, k-1\}$ or $i = 1$ and $p_i = p_1 = p_2 = 2$, then $p_i^{s_i}$ divides one of m, n.*

Proof. Let $m = p_i^u m'$ and $n = p_i^v n'$. Assume without loss of generality that $0 \le u \le v$. But then when $p_1 = p_2 = 2$, we have $s_1 \le s_2$ and

$$2s_1 \le s_1 + s_2 \le u + v + 1 < u + v + 2 = (u+1) + (v+1) \le 2(v+1),$$

yielding $s_1 < v + 1$ and hence $s_1 \le v$. When $p_i \ge 3$, we have $s_i \le s_{i+1}$ and

$$2s_i \le s_i + s_{i+1} \le u + v \le 2v,$$

yielding $s_i \le v$. Hence, $p_i^{s_i}$ divides n, as desired. □

The following is a contrapositive of the Lemma above.

Corollary 5.2. *Let $G = C_m \square C_n$ and $\Gamma(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k})$ be of order $2mn$. If no $p_i^{s_i}$ for any $i \in \{1, 2, \dots, k\}$ divides either m or n, then all primes p_i are distinct and $\Gamma(p_1^{s_1}, p_2^{s_2}, \dots, p_k^{s_k})$ is cyclic.*

This can happen when for all $i = 1, 2, \dots, k$ we have $m = p_i^{u_i} m_i$ and $n = p_i^{v_i} n_i$ where $0 \le u_i, v_i < s_i$.

Throughout our construction we use the following notation. The vertices of $C_m \square C_n$ will be denoted by $x_{i,j}$ with $0 \le i \le m-1, 0 \le j \le n-1$. Every vertex $x_{i,j}$ has neighbors $x_{i-1,j}, x_{i,j+1}, x_{i+1,j}, x_{i,j-1}$ joined by *vertical edges*

$$v_{i-1,j} = x_{i-1,j}x_{i,j} \text{ and } v_{i,j} = x_{i,j}x_{i+1,j}$$

and *horizontal edges*

$$h_{i,j-1} = x_{i,j-1}x_{i,j} \text{ and } h_{i,j} = x_{i,j}x_{i,j+1}.$$

When we use several copies of the same graph G' glued together into the resulting graph G, we denote the image of $x_{i,j}, v_{i,j}$ or $h_{i,j}$ in the r-th copy G'_r simply by $x_{i,j}^r, v_{i,j}^r$ and $h_{i,j}^r$. The subscripts are taken modulo m and n, respectively.

To perform our recursive construction, we need two base cases, namely $C_p \square C_q$, where p and q are distinct odd primes, and $C_p \square C_p$, where p is an arbitrary odd prime. Notice that when $p \ne q$, then the group $Z_2 \oplus Z_p \oplus Z_q$ is cyclic and there exists a Z_{2pq}-supermagic labeling by Theorem 4.5 proved in [6]. By the same result, $C_p \square C_p$ can be labeled by the cyclic group Z_{2p^2}.

Although the results are quoted in Section 4, we re-state them here as our starting cases in a simple Lemma.

Lemma 5.3 (Froncek et al., [6]). *Let p, q be odd primes, not necessarily distinct. Then the graph $C_p \Box C_q$ admits a Z_{2pq}-supermagic labeling.*

The remaining base case of $C_n \Box C_n$ labeled by the group $Z_2 \oplus Z_n \oplus Z_n$ is treated in the following.

Lemma 5.4. *Let n be an odd integer and $\Gamma = Z_2 \oplus Z_n \oplus Z_n$. Then the graph $C_n \Box C_n$ admits a Γ-supermagic labeling.*

Proof. We label the vertical edges as

$$f(v_{i,j}) = (0, 2i, 2j)$$

and the horizontal edges as

$$f(h_{i,j}) = (1, 1-2i, n-2j) = (1, 1-2i, -2j).$$

The labeling function is clearly bijective. Then

$$\begin{aligned} w(x_{i,j}) &= f(v_{i-1,j}) + f(h_{i,j}) + f(v_{i,j}) + f(h_{i,j-1}) \\ &= (0, 2i-2, 2j) + (1, 1-2i, -2j) + (0, 2i, 2j) + (1, 1-2i, -2j+2) \\ &= (0, 0, 2) \end{aligned}$$

for every i, j in the range which proves that f is a Γ-supermagic labeling. $\square$

Now we are ready to prove our main result.

Theorem 5.5. *Let m, n be odd and greater than one. Then there exists a Γ-supermagic labeling of the graph $C_m \Box C_n$ by any Abelian group Γ of order $2mn$.*

Proof. Recall that $\Gamma(p_1^{s_1}, p_2^{s_2}, \ldots, p_k^{s_k}) = Z_{p_1^{s_1}} \oplus Z_{p_2^{s_2}} \oplus \cdots \oplus Z_{p_k^{s_k}}$ and because both C_m and C_n are odd, we have $p_1^{s_1} = 2$ and $p_2 \geq 3$. When $k = 2$ or $k = 3$ and $p_2 < p_3$, $\Gamma = Z_{2mn}$ is cyclic and the result follows from Theorem 4.5.

For $k = 3$ and $p_2 = p_3$, when $s_2 < s_3$ or $m < n$, we have $n = p_2^{s_2} t$ for some $t > 1$ and we proceed by induction as described below. When $k = 3, p_2 = p_3$ and $s_2 = s_3$ we have $p_2^{s_2} = p_3^{s_3} = m = n$ and the labeling exists by Lemma 5.4. If $k > 3$ and all primes are distinct, Γ is again cyclic and the labeling exists by Theorem 4.5.

Therefore, we now assume that Γ is not cyclic and proceed by strong induction on the number of vertices. Base cases are treated by Lemmas 5.3 and 5.4.

When $\Gamma(p_1^{s_1}, p_2^{s_2}, \ldots, p_k^{s_k})$ is not cyclic, then we must have a subscript $1 < z < k$ such that $p_{z-1} < p_z = p_{z+1}$ and by Lemma 5.1, $p_z^{s_z}$ divides one of m, n.

We denote $p_z^{s_z} = q$ and assume without loss of generality that $n = qt$ for some odd t, $1 < t < n$. By inductive hypothesis, $G' = C_m \Box C_t$ can be labeled by any Abelian group of order $2mt$. In particular, it can be labeled by the group $\Gamma''(p_1^{s_1}, p_2^{s_2}, \ldots, p_{z-1}^{s_{z-1}}, p_{z+1}^{s_{z+1}}, \ldots, p_k^{s_k})$. This group is isomorphic to

$$\Gamma'(p_1^{s_1}, p_2^{s_2}, \ldots, p_{z-1}^{s_{z-1}}, 1, p_{z+1}^{s_{z+1}}, \ldots, p_k^{s_k}),$$

which we will use instead for convenience. We will call that labeling f' and the magic constant μ'. Notice that all elements of Γ' have zero in the z-th position, as the entry 1 indicates the trivial group Z_1 in the external direct product.

Now we take q copies $G^0, G^1, \ldots, G^{q-1}$ of G' and label each of them using labeling f'. For each $r = 0, 1, \ldots, q-1$, the vertical edge of G^r corresponding to the edge $v_{i,j}$ of G' will be denoted by $v_{i,j}^r$, and similarly the horizontal edge corresponding to $h_{i,j}$ will be denoted by $h_{i,j}^r$.

We cut in each copy the edges $h_{i,t-1}^r$ for $i = 0, 1, \ldots, m-1$ and glue the copies together in a natural way, joining vertices $x_{i,t-1}^r$ and $x_{i,0}^{r+1}$ by edge $h_{i,t-1}^r$. We do not change notation as it should be clear from the context when such edge belongs to G'_r or to G. The labels of these edges do not change.

So far, all vertices $x_{i,j}^r$, $r = 1, 2, \ldots, t$ for fixed i and j have the same weight. However, while the labeling is not bijective, the weight of each vertex is still μ', since all vertices with "new" neighbors have the new incident edge $h_{i,t-1}^r$ in G labeled with the same element as the original edge $h_{i,t-1}^r$ in G'.

To obtain a bijective labeling f, we now replace the entry zero in the z-th position by an element of Z_q in each part (subgraph) G'_r of G so that the resulting sum of labels at each vertex is still the same.

For all vertical edges except $v_{i,0}^r$ we replace the zero in the z-th position by $2r-1$ and for all horizontal edges $h_{i,j}^r$ we replace it by the inverse of $2r-1$, which is $q-2r+1$. The sum of these entries for each $x_{i,j}^r$ where $j \neq 0$ is zero.

For a vertex $x_{i,0}^r$ the already labeled horizontal edges incident with it are $h_{i,t-1}^{r-1}$ and $h_{i,0}^r$ labeled in the z-th position by $q-2(r-1)+1$ and $q-2r+1$, respectively, contributing $2q-4r+4 = -4r+4$ to the weight of $x_{i,0}^r$. Now for every $r = 0, 1, \ldots, q-1$ and $i = 0, 1, \ldots, m-1$ we label all vertical edges $v^r_{i,0}$ in the z-th position by $2(r-1)$ and the sum of the partial labels is zero, as desired.

It should be obvious that the new labeling is a bijection, and the proof is complete. $\square$

We illustrate the inductive step in the following.

Example 5.6. We present a labeling of $C_3 \square C_9$ by $\Gamma(2,3,3^2)$ using the inductive construction from Theorem 5.5.

We observe that $\Gamma(2,3,3^2) \cong Z_{18} \oplus Z_3$ and first label $C_3 \square C_3$ with the cyclic group Z_{18}. Such a labeling exists by Theorem 4.4. The red numbers are the weights of vertices in Z_{18}.

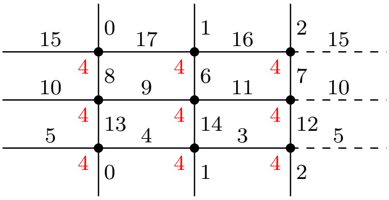

Figure 1: $C_3 \square C_3$ labeled by $Z_{18} \oplus Z_1$

Then we change the labeling using the group $\Gamma(2,1,3^2) \cong Z_{18} \oplus Z_1$ instead of just Z_{18} and glue together three copies of G', denoted by G^0, G^1, G^2. We print the labels of the horizontal "cut and glue" edges in red, and the vertical edges $v^r_{i,j}$ whose label is changed from the original one in G' are printed in blue.

6 Conclusion

Between the submission and publication of this paper, the authors continued their effort to completely classify the spectrum of all Cartesian products of two cycles with Γ-supermagic labeling. They completely solved the case of two even cycles [8]. The case of one odd and one even cycle remains wide open except for labeling with cyclic groups (see Theorem 4.4).

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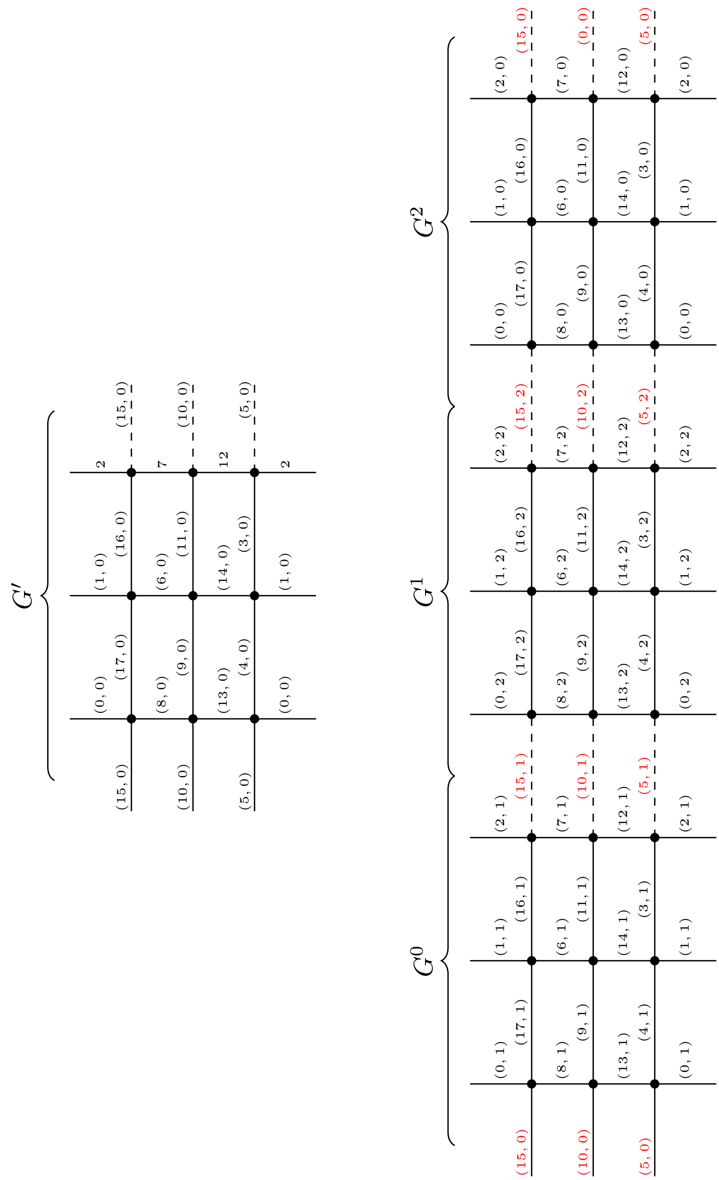

Figure 2: $C_3 \square C_9$ labeled by $Z_{18} \oplus Z_3$

particular, the third author is grateful to the referee for catching a typo (mentioned in Section 4) even in his Master's thesis [13]!

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