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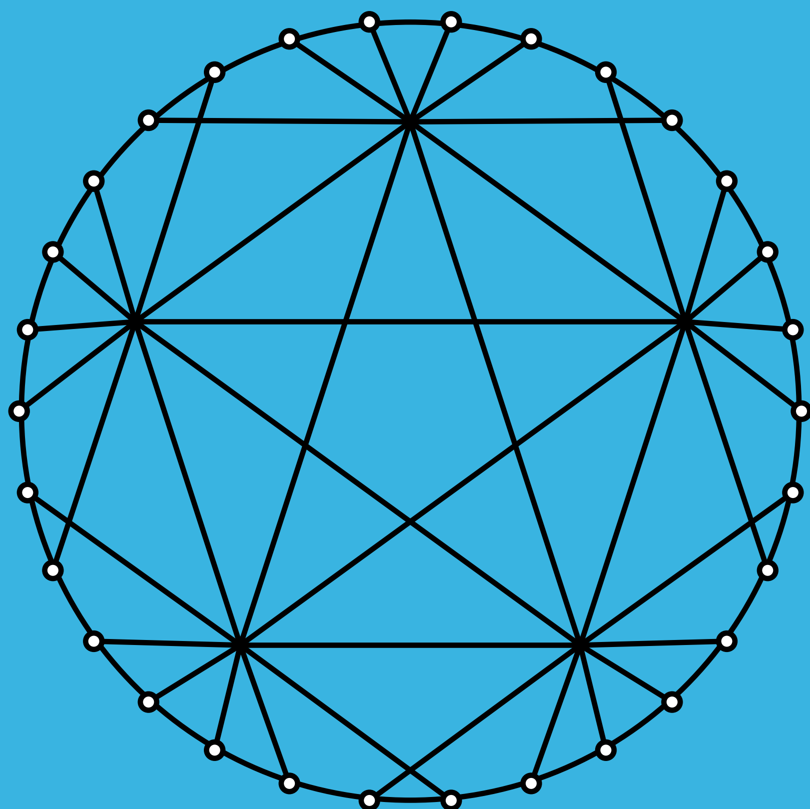

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On tight 6-cycle and tight 9-cycle decompositions of complete 3-uniform hypergraphs minus a 1-factor

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Abstract. The complete 3-uniform hypergraph of order v, denoted by $K_v^{(3)}$, has a set V of size v as its vertex set and the set of all 3-element subsets of V as its edge set. If $v \equiv 0 \pmod 3$, then the edge set of $K_v^{(3)}$ contains a collection I of $v/3$ vertex-disjoint edges, called a 1-factor. Let $K_v^{(3)} - I$ denote any hypergraph isomorphic to the one obtained by removing the edge set of a 1-factor from that of $K_v^{(3)}$. For $m > 3$, a 3-uniform tight m-cycle, denoted TC_m, is any hypergraph isomorphic to the one with vertex set $\mathbb{Z}_m$ and edge set $\big\{\{i, i+1, i+2\} : i \in \mathbb{Z}_m\big\}$. Necessary and sufficient conditions for the existence of TC_6- and TC_9-decompositions of $K_v^{(3)}$ have previously been found. We show that there exists a TC_6-decomposition of $K_v^{(3)} - I$ if and only if $v \equiv 0, 3$, or $6 \pmod{12}$ and that there exists a TC_9-decomposition of $K_v^{(3)} - I$ if and only if $v \equiv 0 \pmod 3$ and $v \neq 6$. Results similar to ours were obtained independently and simultaneously by Keszler and Tuza (*Spectrum of* 3*-uniform* 6*- and* 9*-cycle systems over* $K_v^{(3)} - I$, arXiv:2212.11058.)

1 Introduction

In this work, we find necessary and sufficient conditions for the existence of decompositions of $K_v^{(3)} - I$, the complete 3-uniform hypergraph of order v minus a 1-factor, into the 3-uniform tight cycle on 6 vertices and also

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into the tight cycle on 9 vertices. Our results are a subset of similar ones obtained simultaneously and independently by Keszler and Tuza [17].

Before proceeding with the motivation for this problem, we first note that this paper is part of a sequence of results obtained by participants in a Research Experiences for Undergraduates (REU) and Teachers program at the first author's home institution. Because there have been multiple papers on similar problems by different groups of REU participants, definitions, motivation and summaries of known results are sometimes stated in these papers in similar ways to earlier ones in the sequence. As we are limited in the number of ways we can vary definitions and statements of known results without appearing to copy them, we will state upfront that we will use the style of presentation of two of the earlier papers from the sequence (see [1] and [8]). However, in all cases, the standard notations and definitions should be credited to a 2014 groundbreaking paper by Bryant, Herke, Maenhaut, and Wannasit [6], which also provides a template for settling decomposition problems of complete 3-uniform hypergraphs into subgraphs of small order.

1.1 Motivating results

A popular area of study in combinatorics pertains to partitions or decompositions of certain classes of incidence structures into isomorphic copies of smaller regular structures. For example, an affine plane of order v corresponds to a decomposition of K_{v^2}, the complete graph on v^2 vertices, into copies of K_v. Another example is the Oberwolfach problem, which seeks to find a decomposition of K_v or of $K_v - I$, where I is a 1-factor, into isomorphic 2-factors on v vertices. Interest in m-cycle decompositions of K_v can be traced back to the 1890's and Walecki's solution for decomposing K_v into Hamiltonian cycles (see Lucas [19] and Alspach [2]). In 2001, Alspach and Gavlas [3] fully settled the problem of decomposing K_v and $K_v - I$ into cycles of odd length. In 2002, Šajna [23] settled the corresponding problem for even cycle lengths. In 2003, Buratti [9] gave an alternative solution for the odd case. Relaxing the condition that requires edge sets in graphs to be 2-element subsets of the vertices leads to the concept of a hypergraph and many of the same classical problems for graphs yield corresponding decomposition problems.

A *hypergraph* H consists of a finite nonempty set V of *vertices* and a set E of nonempty subsets of V called *hyperedges*, or simply *edges*. As with graphs, the number of edges that contain a vertex u is the *degree* of u

and a hypergraph H is *k-regular* if every vertex in H has degree k. A k-regular spanning subhypergraph of H is called a *k-factor* in H. If H' is a subhypergraph of H, then $H - H'$ denotes the hypergraph obtained from H by deleting the edges of H'. If for each $e \in E$, we have $|e| = t$, then H is said to be *t-uniform.* Thus graphs are 2-uniform hypergraphs. Let V be a nonempty set and let $t \geq 2$ be an integer. Let $K_V^{(t)}$ denote the hypergraph with vertex set V and edge set the set of all t-element subsets of V. If $|V| = v$, let $K_v^{(t)}$ denote any hypergraph isomorphic to $K_V^{(t)}$. We refer to $K_v^{(t)}$ as the *complete t-uniform hypergraph of order v.*

A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of K such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s) = E(K)$ and $E(H_i) \cap E(H_j) = \varnothing$ for all $1 \leq i < j \leq s$. If each element H_i of Δ is isomorphic to a fixed hypergraph H, then H_i is called an *H-block*, and Δ is called an *H-decomposition* of K or a *(K, H)-design.* We may in this case say that H *decomposes* K or that K is *decomposable* into copies of H. A $\big(K_v^{(t)}, H\big)$-design is also known as an *H-design of order v.* The problem of determining all values of v for which there exists an H-design of order v is known as the *spectrum problem* for H.

A $(K_v^{(t)}, K_k^{(t)})$-design is a generalization of a Steiner system and is equivalent to what is known as an $S(t, k, v)$-design. A summary of results on $S(t, k, v)$-designs appears in [10]. Keevash [16] has recently shown that for all t and k the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of v. Similar results were obtained by Glock, Kühn, Lo, and Osthus [12, 13] and extended to include the corresponding asymptotic results for H-designs of order v for all uniform hypergraphs H. These results for t-uniform hypergraphs mirror Wilson's asymptotic results for graphs [24]. Although Glock et al.'s [12, 13] assure the existence of H-designs of order v for sufficiently large values of v for any uniform hypergraph H, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2. As stated earlier, a 2014 groundbreaking paper by Bryant et al. [6] provides a template for settling the spectrum problem for 3-uniform H-designs, especially in the case when H is 3-colorable (i.e., H has chromatic number 3).

There are multiple ways defining an m-cycle in a t-uniform hypergraph. In this work, we focus on the concept of a tight m-cycle, which generalizes the Katona-Kierstead [15] definition of a Hamilton cycle (also called a Hamiltonian chain in [15]). For $m > t \geq 2$, let $\mathbb{Z}_m$ denote the group of integers modulo m and let $TC_m^{(t)}$ denote any hypergraph isomorphic to the t-uniform hypergraph with vertex set $\mathbb{Z}_m$ and edge set $\big\{ \{i, i+1, \ldots, i+t-1\} : i \in \mathbb{Z}_m \big\}$.

We call $TC_m^{(t)}$ a *t-uniform tight m-cycle*. We note that $TC_{t+1}^{(t)}$ is also the complete hypergraph $K_{t+1}^{(t)}$. As we deal exclusively with 3-uniform tight cycles in this manuscript, we will henceforth use TC_m to denote a $TC_m^{(3)}$.

The 3-uniform tight m-cycle decomposition problem was first introduced by Bailey and Stevens [4] in the context of investigations of Hamilton cycle decompositions of $K_m^{(3)}$. Also in 2009, Meszka and Rosa [22] added to the results from [4] and introduced the idea of TC_m-decompositions of $K_v^{(3)}$ with particular focus on the case $m = 5$. They also noted that, as a consequence of Hanani's classic result on the existence of Steiner quadruple systems (i.e., $S(3,4,v)$-designs) [14], there exists a $(K_v^{(3)}, TC_4)$-design if and only if $v \equiv 2$ or $4 \pmod 6$. More recently, several additional partial results on $(K_v^{(3)}, TC_5)$-designs and $(K_v^{(3)}, TC_7)$-designs were given by multiple various authors (see [18], [11], and [20]).

Using an approach similar to the one by Bryant et al. [6], Akin et al. [1] recently settled the spectrum problem for 3-uniform tight 6-cycles by showing that there exists a TC_6-decomposition of $K_v^{(3)}$ if and only if $v \equiv 1$, 2, 10, 20, 28, or 29 (mod 36) and $v \geq 10$. Similarly, Bunge et al. [8] settled the corresponding problem for 3-uniform tight 9-cycles and showed that there exists a TC_9-decomposition of $K_v^{(3)}$ if and only if $v \equiv 1$ or 2 (mod 27).

Because TC_m is 3-regular and $K_v^{(3)}$ is $\binom{v-1}{2}$-regular, a necessary degree condition for the existence of a TC_m-decomposition of $K_v^{(3)}$ is $v \equiv 1$ or 2 (mod 3). In the case where $v \equiv 0 \pmod 3$, we have $\binom{v-1}{2} \equiv 1 \pmod 3$, and hence the removal of a 1-factor, say I, from $K_v^{(3)}$ results in a hypergraph $K_v^{(3)} - I$ that satisfies the degree condition for a TC_m-decomposition. For convenience, we may denote $K_v^{(3)} - I$ by $K_v^{(3)*}$. Meszka and Rosa [22] were the first to propose investigating tight cycle decompositions of $K_v^{(3)*}$ and gave a TC_6-decomposition of $K_6^{(3)*}$, a TC_9-decomposition of $K_9^{(3)*}$, and a TC_{12}-decomposition of $K_{12}^{(3)*}$.

In this manuscript, we find necessary and sufficient conditions for the existence of TC_6- and TC_9-decompositions of $K_v^{(3)*}$. As noted earlier, our results are a subset of similar ones obtained simultaneously and independently by Keszler and Tuza [17].

1.2 Additional notation

If a and b are integers, we define $[a, b]$ to be $\{r \in \mathbb{Z} : a \leq r \leq b\}$. For any edge-disjoint hypergraphs G and H, we use $G \cup H$ to indicate the hypergraph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Similarly, if H is a hypergraph and r is a nonnegative integer, we let rH denote the edge-disjoint union of r copies of H.

We next define additional notation for certain types of multipartite-like 3-uniform hypergraphs. Our notation is a variation of the one used by Bryant et al. in [6] and has been incorporated into the majority of the prior papers from the beforementioned undergraduate research program (e.g., [1] and [8]). Let A, B, C be pairwise-disjoint sets. The hypergraph with vertex set $A \cup B \cup C$ and edge set consisting of all 3-element sets having exactly one vertex in each of A, B, C is denoted by $K^{(3)}_{A,B,C}$. The hypergraph with vertex set $A \cup B$ and edge set consisting of all 3-element sets having at most 2 vertices in each of A and B is denoted by $L^{(3)}_{A,B}$. If $|A| = a$, $|B| = b$, and $|C| = c$, we may use $K^{(3)}_{a,b,c}$ to denote any hypergraph that is isomorphic to $K^{(3)}_{A,B,C}$ and $L^{(3)}_{a,b}$ to denote any hypergraph that is isomorphic to $L^{(3)}_{A,B}$. We further let $L^{(2,1)}_{A,B}$ denote the subgraph of $L^{(3)}_{A,B}$ where each edge contains two vertices from A and one vertex from B. We define $L^{(2,1)}_{a,b}$, $L^{(1,2)}_{A,B}$, and $L^{(1,2)}_{a,b}$, correspondingly.

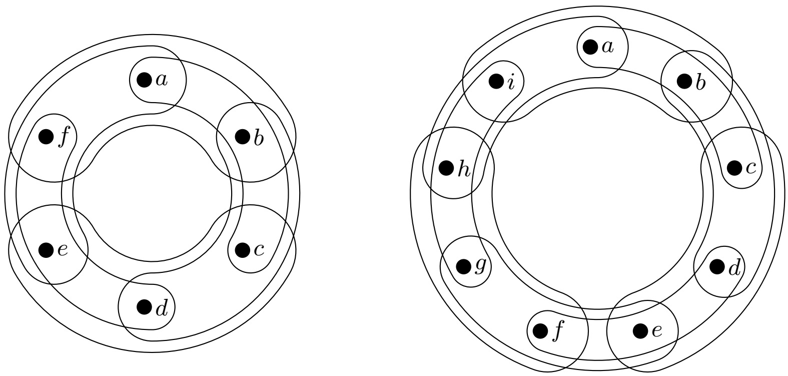
It is simple to see and has been observed previously (see [1] and [8]) that if A, A', B, and C are pairwise-disjoint, then $K^{(3)}_{A\cup A',B,C} = K^{(3)}_{A,B,C} \cup K^{(3)}_{A',B,C}$ and $L^{(3)}_{A\cup A',B} = L^{(3)}_{A,B} \cup L^{(3)}_{A',B} \cup K^{(3)}_{A,A',B}$. Thus we have the following basic lemmas.

Lemma 1.1. *If a, b, c, x, y, z are positive integers, then $K^{(3)}_{xa,yb,zc}$ is decomposable into xyz copies of $K^{(3)}_{a,b,c}$.*

Lemma 1.2. *If a, b, x, y are positive integers, then $L^{(3)}_{xa,yb}$ is decomposable into xy copies of $L^{(3)}_{a,b}$, and $\binom{x}{2}y$ copies of $K^{(3)}_{a,a,b}$, and $x\binom{y}{2}$ copies of $K^{(3)}_{a,b,b}$.*

2 Foundational decompositions

In this section we show several examples of TC_6- and TC_9-decompositions that are used in proving our main result. As noted earlier, Meszka and



(a) the TC_6 denoted by $[a, b, c, d, e, f]$

(b) the TC_9 denoted by $[a, b, c, d, e, f, g, h, i]$

Figure 1: The 3-uniform tight 6- and 9-cycles shown with the notation used in the examples throughout Section 2.

Rosa [22] gave a TC_6-decomposition of $K_6^{(3)*}$ and a TC_9-decomposition of $K_9^{(3)*}$. We give our own decompositions here in Examples 2.1 and 2.5, respectively. Similarly, the decompositions in Examples 2.3 and 2.7 can be found in [1] and [8], respectively. We include these example decompositions here for completeness of the results. Also, the approach in Example 2.9 was suggested by Meszka [21].

We use $[a, b, c, d, e, f]$ to denote the hypergraph (isomorphic to TC_6) with vertex set $\{a, b, c, d, e, f\}$ and edge set $\big\{\{a, b, c\}, \{b, c, d\}, \{c, d, e\}, \{d, e, f\}, \{e, f, a\}, \{f, a, b\}\big\}$. Similarly, we use $[a, b, c, d, e, f, g, h, i]$ to denote the hypergraph (isomorphic to TC_9) with vertex set $\{a, b, c, d, e, f, g, h, i\}$ and edge set $\big\{\{a, b, c\}, \{b, c, d\}, \{c, d, e\}, \{d, e, f\}, \{e, f, g\}, \{f, g, h\}, \{g, h, i\}, \{h, i, a\}, \{i, a, b\}\big\}$. This notation is used for brevity in the many example decompositions of this section. It is also demonstrated in Figure 1.

2.1 TC_6-decompositions

Example 2.1. Let $V\left(K_6^{(3)*}\right) = [0, 5]$ with $\big\{\{0, 2, 4\}, \{1, 3, 5\}\big\}$ as the edge set of the removed 1-factor. Then

$$B = \big\{[0, 1, 2, 3, 4, 5],\ [0, 1, 4, 5, 2, 3],\ [0, 3, 4, 1, 2, 5]\big\}$$

is a TC_6-decomposition of $K_6^{(3)*}$.

Example 2.2. Let $V\left(K_{15}^{(3)*}\right) = \mathbb{Z}_{15}$ with $\big\{\{0,5,10\}, \{1,6,11\}, \{2,7,12\}, \{3,8,13\}, \{4,9,14\}\big\}$ as the edge set of the removed 1-factor and let

$$B_1 = \big\{[0,1,4,3,2,5],\ [0,2,4,12,9,6],\ [0,7,1,12,2,9],\ [0,8,7,1,2,11],\ [0,8,10,5,1,12]\big\}.$$

Then a TC_6-decomposition of $K_{15}^{(3)*}$ consists of the orbits of the TC_6-blocks in B_1 under the action of the map $j \mapsto j+i \pmod{15}$ for $i \in \mathbb{Z}_{15}$.

Example 2.3. Let $V\left(K_{2,3,3}^{(3)}\right) = \{0,1,\ldots,7\}$ with vertex partition $\big\{\{0,1\}, \{2,3,4\}, \{5,6,7\}\big\}$. Then

$$B = \big\{[0,2,5,1,3,6],\ [0,3,7,1,4,5],\ [0,4,6,1,2,7]\big\}$$

is a TC_6-decomposition of $K_{2,3,3}^{(3)}$.

Example 2.4. Let $V\left(L_{6,6}^{(3)}\right) = \mathbb{Z}_{12}$ with vertex partition $\big\{\{0,2,4,6,8,10\}, \{1,3,5,7,9,11\}\big\}$ and let

$$B_1 = \big\{[0,1,9,4,3,7],\ [0,2,11,8,10,5]\big\},$$
$$B_2 = \big\{[0,1,2,6,7,8]\big\}.$$

Then a TC_6-decomposition of $L_{6,6}^{(3)}$ consists of the orbits of the TC_6-blocks in B_1 under the action of the map $j \mapsto j+i \pmod{12}$ for $i \in \mathbb{Z}_{12}$ and the orbit of the TC_6-block in B_2 under the action of the map $j \mapsto j+i \pmod{12}$ for $i \in \{0,1,2,3,4,5\}$.

2.2 TC_9-decompositions

Example 2.5. Let $V\left(K_{9}^{(3)*}\right) = \mathbb{Z}_{9}$ with $\big\{\{0,3,6\}, \{1,4,7\}, \{2,5,8\}\big\}$ as the edge set of the removed 1-factor and let

$$B_1 = \big\{[0,2,5,1,4,3,8,6,7]\big\}.$$

Then a TC_9-decomposition of $K_{9}^{(3)*}$ consists of the orbit of the TC_9-block in B_1 under the action of the map $j \mapsto j+i \pmod{9}$ for $i \in \mathbb{Z}_{9}$.

Example 2.6. Let $V\left(K_{12}^{(3)*}\right) = \mathbb{Z}_{12}$ with $\big\{\{0,4,8\}, \{1,5,9\}, \{2,6,10\}, \{3,7,11\}\big\}$ as the edge set of the removed 1-factor and let

$$B_1 = \big\{[0,2,7,4,6,1,5,8,11],\ [0,4,11,10,5,1,3,2,6]\big\}.$$

Then a TC_9-decomposition of $K_{12}^{(3)*}$ consists of the orbits of the TC_9-blocks in B_1 under the action of the map $j \mapsto j + i \pmod{12}$ for $i \in \mathbb{Z}_{12}$.

Example 2.7. Let $V\left(K_{3,3,3}^{(3)}\right) = \{0, 1, \ldots, 8\}$ with the vertex partition $\{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}\}$. Then

$$B = \big\{[0, 1, 2, 3, 4, 5, 6, 7, 8], [0, 2, 4, 6, 8, 1, 3, 5, 7], [0, 4, 8, 3, 7, 2, 6, 1, 5]\big\}$$

is a TC_9-decomposition of $K_{3,3,3}^{(3)}$.

Example 2.8. Let $V\left(L_{6,6}^{(3)}\right) = \mathbb{Z}_{12}$ with vertex partition $\{\{0, 2, 4, 6, 8, 10\}, \{1, 3, 5, 7, 9, 11\}\}$ and let

$$\begin{aligned} B_1 &= \big\{[0, 1, 4, 6, 9, 2, 10, 7, 11]\big\}, \\ B_2 &= \big\{[0, 3, 6, 4, 7, 10, 8, 11, 2], [0, 7, 2, 8, 3, 10, 4, 11, 6]\big\}. \end{aligned}$$

Then a TC_9-decomposition of $L_{6,6}^{(3)}$ consists of the orbit of the TC_9-block in B_1 under the action of the map $j \mapsto j + i \pmod{12}$ for $i \in \mathbb{Z}_{12}$ and the orbits of the TC_9-blocks in B_2 under the action of the map $j \mapsto j + i \pmod{12}$ for $i \in \{0, 1, 2, 3\}$.

Example 2.9. Let $V\left(L_{9,3}^{(2,1)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3\}$ with the obvious vertex partition and let

$$\begin{aligned} B_1 = \big\{&[0, 3, \infty_1, 1, 4, \infty_2, 2, 5, \infty_3], [0, \infty_2, 3, 1, \infty_3, 4, 2, \infty_1, 5], \\ &[0, 1, \infty_1, 5, 3, \infty_2, 7, 8, \infty_3], [0, \infty_2, 1, 5, \infty_3, 3, 7, \infty_1, 8]\big\}. \end{aligned}$$

Then a TC_9-decomposition of $L_{9,3}^{(2,1)}$ consists of the orbits of the TC_9-blocks in B_1 under the action of the map $\infty_k \mapsto \infty_k$, for $k \in \{1, 2, 3\}$, and $j \mapsto j + 3i \pmod 9$ for $i \in \{0, 1, 2\}$.

3 Main results

In order for $K_v^{(3)}$ to contain a 1-factor, we must have $3 \mid v$. That is, $K_v^{(3)*}$, or $K_v^{(3)} - I$, is only defined for $v \equiv 0 \pmod 3$. Also, as mentioned in the introduction, the degree condition for a decomposition of $K_v^{(3)*}$ into a 3-regular graph is necessarily satisfied when $v \equiv 0 \pmod 3$. We set out to prove that this one necessary condition is almost sufficient for the existence of TC_6- and TC_9-decompositions of $K_v^{(3)*}$. First, we establish all necessary conditions for each decomposition.

Lemma 3.1. *Let v be a positive integer such that $v \equiv 0 \pmod 3$. If there exists a TC_6-decomposition of $K_v^{(3)*}$, then $v \not\equiv 9 \pmod{12}$.*

Proof. Let $v = 3m$ for some positive integer m. Note that TC_6 has 6 edges while $K_{3m}^{(3)*}$ has $\binom{3m}{3} - m$ edges. Hence, for a TC_6-decomposition of $K_{3m}^{(3)*}$ to exist, we must have 6 divides $3m(3m-1)(3m-2)/6 - m = \frac{9}{2}m^2(m-1)$. Therefore, we must have $4 \mid m^2(m-1)$. Thus, $m \equiv 0$, 1, or $2 \pmod 4$. □

Lemma 3.2. *Let v be a positive integer such that $v \equiv 0 \pmod 3$. If there exists a TC_9-decomposition of $K_v^{(3)*}$, then $v \neq 6$.*

Proof. Note that TC_9 has 9 vertices, and $K_6^{(3)*}$ has a nonempty edge set. However, $K_6^{(3)*}$ has too few vertices to admit a subhypergraph isomorphic to TC_9. □

Next, we prove a lemma that is fundamental to our constructions that establish sufficient conditions on TC_6- and TC_9-decompositions of $K_v^{(3)*}$. This lemma is an extension of variations which have used in prior REU work (e.g., [1] and [8]), but this approach first appears in the paper by Bryant et al. in [6].

Lemma 3.3. *Let n, x be positive integers and let r be a nonnegative integer such that $n \equiv r \equiv 0 \pmod 3$. There exists a decomposition of $K_{nx+r}^{(3)*}$ that is comprised of the following hypergraphs under the given conditions:*

- *1 copy of $K_{n+r}^{(3)*}$,*
- *$x-1$ copies of $L_{r,n}^{(3)} \cup K_n^{(3)*}$ if $x \geq 2$,*
- *$\binom{x}{2}$ copies of $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$ if $x \geq 2$,*
- *$\binom{x}{3}$ copies of $K_{n,n,n}^{(3)}$ if $x \geq 3$.*

Proof. Let $R, V_1, V_2, \ldots, V_x$ be pairwise-disjoint sets of vertices with $|R| = r$ and $|V_1| = |V_2| = \cdots = |V_x| = n$ and let $V = R \cup V_1 \cup V_2 \cup \cdots \cup V_x$. Since $n \equiv r \equiv 0 \pmod 3$, each of $K_{V_1 \cup R}^{(3)}, K_{V_2}^{(3)}, K_{V_3}^{(3)}, \ldots, K_{V_x}^{(3)}$ admits a 1-factor, say $I_1, I_2, I_3, \ldots, I_x$, respectively. Let $I = I_1 \cup I_2 \cup \cdots \cup I_x$. Now, $K_V^{(3)} - I$ can be viewed as the edge-disjoint union

$$\left(K_{V_1 \cup R}^{(3)} - I_1\right) \cup \bigcup_{2 \leq i \leq x} \left(L_{V_i,R}^{(3)} \cup \left(K_{V_i}^{(3)} - I_i\right)\right)$$
$$\cup \bigcup_{1 \leq i < j \leq x} \left(K_{R,V_i,V_j}^{(3)} \cup L_{V_i,V_j}^{(3)}\right) \cup \bigcup_{1 \leq i < j < k \leq x} \left(K_{V_i,V_j,V_k}^{(3)}\right),$$

and the result thus follows. □

Regarding the hypergraphs mentioned in the above lemma and its proof, we note that, if $r = 0$, then $L^{(3)}_{r,n}$ and $K^{(3)}_{r,n,n}$ are empty. Moreover, if $r = 3$, then the 1-factor removed from $K^{(3)}_{V_1 \cup R}$ could (without loss of generality) include the edge on the 3 vertices in R, and thus $K^{(3)*}_{n+r}$ is isomorphic to $L^{(3)}_{V_i,R} \cup K^{(3)*}_{V_i}$ for any value of i as in the above proof. This leads us to the following corollary to Lemma 3.3.

Corollary 3.4. *Let n, x be positive integers such that $n \equiv 0 \pmod 3$ and let $r \in \{0, 3\}$. There exists a decomposition of $K^{(3)*}_{nx+r}$ consisting of isomorphic copies of $K^{(3)*}_{n+r}$, $L^{(3)}_{n,n}$ (if $x \geq 2$), $K^{(3)}_{n,n,n}$ (if $x \geq 3$), and $K^{(3)}_{3,n,n}$ (if $r = 3$ and $x \geq 2$).*

Besides being necessary for our proofs of the main results, the following lemma and its proof employ the above corollary in a similar fashion to how we show sufficiency in the more general case.

Lemma 3.5. *There exists a TC_9-decomposition of $K^{(3)*}_{18}$.*

Proof. We note that $18 = 9 \cdot 2 + 0$, so by Corollary 3.4 the result follows from TC_9-decompositions of $K^{(3)*}_9$ and $L^{(3)}_{9,9}$. First, a TC_9-decomposition of $K^{(3)*}_9$ is given in Example 2.5. Second, consider pairwise-disjoint vertex sets $A_1, A_2, A_3, B_1, B_2, B_3$ with three vertices each and let $A = A_1 \cup A_2 \cup A_3$ and $B = B_1 \cup B_2 \cup B_3$. Then $|A| = |B| = 9$ and $L^{(3)}_{A,B} = \bigcup_{i=1}^{3} L^{(2,1)}_{A,B_i} \cup \bigcup_{i=1}^{3} L^{(1,2)}_{A_i,B}$. Therefore, a TC_9-decomposition of $L^{(3)}_{9,9}$ follows from a TC_9-decomposition of $L^{(2,1)}_{9,3}$, which is given in Example 2.9. □

The next two lemmas are beneficial in applying the fundamental construction.

Lemma 3.6. *Let x, y, z be positive integers. There exist TC_6- and TC_9-decompositions of $K^{(3)}_{6x,3y,3z}$.*

Proof. Note that TC_6 decomposes $K^{(3)}_{2,3,3}$ (see Example 2.3). Also, TC_9 decomposes $K^{(3)}_{3,3,3}$ (see Example 2.7). Hence, both results follow from Lemma 1.1. □

Lemma 3.7. *Let x, y be positive integers. There exist TC_6- and TC_9-decompositions of $L^{(3)}_{6x,6y}$.*

Proof. Note that both TC_6 and TC_9 decompose $L^{(3)}_{6,6}$ (see Examples 2.4 and 2.8). Also, both TC_6 and TC_9 decompose $K^{(3)}_{6,6,6}$ by Lemma 3.6. Hence, both results follow from Lemma 1.2. □

Finally, we have all that we need to prove our main results.

Theorem 3.8. *There exists a TC_6-decomposition of $K_v^{(3)*}$ if and only if $v \equiv 0$, 3, or 6 (mod 12).*

Proof. We note that $K_v^{(3)*}$ is only defined for positive integers $v \equiv 0 \pmod 3$. Furthermore, the result is vacuously true when $v = 3$. Otherwise, the necessary conditions for the decomposition are established in Lemma 3.1. Thus, we need only establish their sufficiency for $v \geq 6$.

First, if $v \equiv 0, 6 \pmod{12} \equiv 0 \pmod 6$, then we let $v = 6x$ for some positive integer x. By Corollary 3.4, it suffices to show that TC_6 decomposes $K_6^{(3)*}$ (see Example 2.1), $L_{6,6}^{(3)}$ (see Lemma 3.7), and $K_{6,6,6}^{(3)}$ (see Lemma 3.6).

Second, if $v \equiv 3 \pmod{12}$, then we let $v = 12x + 3$ for some positive integer x. By Corollary 3.4, it suffices to show that TC_6 decomposes $K_{15}^{(3)*}$ (see Example 2.2), $L_{12,12}^{(3)}$ (see Lemma 3.7), $K_{12,12,12}^{(3)}$ (see Lemma 3.6), and $K_{3,12,12}^{(3)}$ (see Lemma 3.6). □

Theorem 3.9. *There exists a TC_9-decomposition of $K_v^{(3)*}$ if and only if $v \equiv 0 \pmod 3$ and $v \neq 6$.*

Proof. As before, we note that $K_v^{(3)*}$ is only defined for positive integers $v \equiv 0 \pmod 3$, and the result is vacuously true when $v = 3$. Otherwise, the necessary conditions for the decomposition are established in Lemma 3.2. Thus, we need only establish their sufficiency for $v \geq 9$.

First, if $v \equiv 3 \pmod 6$, then we let $v = 6x + 3$ for some positive integer x. By Corollary 3.4, it suffices to show that TC_9 decomposes $K_9^{(3)*}$ (see Example 2.5), $L_{6,6}^{(3)}$ (see Lemma 3.7), $K_{6,6,6}^{(3)}$ (see Lemma 3.6), and $K_{3,6,6}^{(3)}$ (see Lemma 3.6).

Second, if $v \equiv 0 \pmod 6$, then we let $v = 12x + r$ for some positive integer x and with $r \in \{0, 6\}$. By Lemma 3.3, it suffices to show that TC_9 decomposes $K_{12}^{(3)*}$ (see Example 2.6), $K_{18}^{(3)*}$ (see Lemma 3.5), $L_{6,12}^{(3)}$ (see Lemma 3.7), $L_{12,12}^{(3)}$ (see Lemma 3.7), $K_{6,12,12}^{(3)}$ (see Lemma 3.6), and $K_{12,12,12}^{(3)}$ (see Lemma 3.6). □

Acknowledgements and concluding remarks

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Participants in the ISU REU program have used the approach of the 2014 paper by Bryant et al. [6] to settle the spectrum problem for several 3-uniform hypergraphs of relatively small order. Since the work on tight cycle decompositions by Akin et al. [1] and by Bunge et al. [8], other ISU REU groups have settled the problem of decomposing the λ-fold complete 3-uniform hypergraph into tight 6-cycles [5] and nearly settled the corresponding λ-fold problem for tight 9-cycles. Another group [7] settled the problem of TC_6-decompositions of $L_{n,n}^{(3)}$ and about half of the corresponding problem for TC_9-decompositions.

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