Stirling permutations for partially ordered sets

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Abstract. We generalize the notion of a Stirling permutation of the multiset \( \{1,1,2,2,\ldots,n,n\} \) based on the usual linear order of the integers \( \{1,2,\ldots,n\} \) to any finite partially ordered set \( P \), a \( P \)-Stirling permutation. We give an algorithmic characterization of \( P \)-Stirling permutations. A partially ordered set determines a transitive directed graph, and a further extension of Stirling permutations to directed graphs is discussed.

1 Introduction

Let \( n \) be a positive integer and \( X_n = \{1,2,\ldots,n\} \). A Stirling permutation of the 2-multiset \( X_n^2 = \{1,1,2,2,\ldots,n,n\} \) is defined by the property:

\[
\text{(⋆) For each } k = 1,2,\ldots,n, \text{ between the two occurrences of } k \text{ only integers greater than } k \text{ occur.}
\]

For example, with \( n = 4 \), 23443211 is a Stirling permutation, but 13234421 is not. A Stirling permutation is a permutation of a specific multiset and so is a multipermutation. Stirling permutations have been generalized to arbitrary multisets using the same property (⋆).

In this paper we confine our attention to the multiset

\[
X_n^2 = \{1,1,2,2,\ldots,n,n\},
\]

that is, to the 2-permutations of \( \{1,2,\ldots,n\} \). Stirling permutations were introduced in [5] in connection with a study of Stirling numbers and Stirling polynomials. The total number of Stirling permutations of \( X_n^2 \) is the...
double factorial \((2n-1)!! = 1 \cdot 3 \cdots (2n-1)\). Stirling permutations have connections to other combinatorial objects. In [6] it is explained how Stirling permutations give rise to a combinatorial interpretation of the second-order Eulerian numbers. Moreover, Stirling permutations arise naturally for certain walks in plane trees [7], which we return to later. For some recent work on Stirling permutations, see [2, 3].

A Stirling permutation obtained from an ordinary permutation \(\pi\) of \(\{1, 2, \ldots, n\}\) by doubling each integer \(j\) in \(\pi\) is called a trivial-Stirling permutation. Thus, for instance, 221133 is a trivial Stirling permutation. Between the two occurrences of each integer \(k\) in a Stirling permutation, there is a Stirling permutation \(\sigma\) of the multiset \(\{l, l: k < l \leq n\}\), indeed \(k\sigma k\) is a Stirling permutation of \(\{k, k, k+1, k+1, \ldots, n, n\}\). For example, in the Stirling permutation 12344321, between the two 2’s there is a Stirling permutation 3443 of \(\{3, 3, 4, 4\}\) and between the two 1’s there is a Stirling permutation of \(\{2, 2, 3, 3, 4, 4\}\). Thus Stirling permutations of \(X_n^2\) can be constructed as follows: choose an integer \(k \leq n\) and a subset \(Y_k\) of \(\{k, k+1, \ldots, n\}\), and then choose a Stirling permutation \(\sigma_k\) of the 2-multiset \(Y_k^2\) with \(k\) as both the first and last integer. Now choose a new integer \(l\), a subset \(Y_l\) of new integers greater than or equal to \(l\), and put a Stirling permutation \(\sigma_l\) of \(Y_l^2\) with \(l\) as both the first and last integer on one of the two sides of \(\sigma_k\), giving a Stirling permutation of the 2-multiset \(Y_k^2 \cup Y_l^2\). Continue like this until all integers have been used.

A Stirling permutation of \(X_n^2 = \{1, 1, 2, 2, \ldots, n, n\}\) can be regarded as based on the reverse-permutation \(\zeta_n = n(n-1) \cdots 1\) of the set \(\{1, 2, \ldots, n\}\) in the sense that the order relation used in checking the Stirling property corresponds to the inversions of the permutation \(\zeta_n\). We can replace the permutation \(\zeta_n\) by an arbitrary permutation \(\pi_n = p_1 p_2 \cdots p_n\) of \(\{1, 2, \ldots, n\}\) to obtain a generalization of the classical \(\zeta_n\)-Stirling permutations. For a permutation \(\pi_n\) of \(\{1, 2, \ldots, n\}\), let \(I(\pi_n)\) be the set of inversions of \(\pi_n\) where an inversion is a pair \((\pi(i), \pi(j))\) where \(i < j\) and \(\pi(i) > \pi(j)\). Thus a \(\pi_n\)-Stirling permutation is a permutation of \(X_n^2\) such that the integers \(j\) between two occurrences of an integer \(k\) satisfy that \((j, k)\) is an inversion of \(\pi_n\); in particular, we have \(j > k\), but that in itself does not suffice, since \(j\) must precede \(k\) in \(\pi_n\). Any \(\pi_n\)-Stirling permutation is a \(\zeta_n\)-Stirling permutation, but the converse does not hold. For example, with the multiset \(X_3^2 = \{1, 1, 2, 2, 3, 3\}\) and \(\pi_3 = 231, 123321\) is not a \(\pi_3\)-Stirling permutation since 3, 2 is not an inversion of \(\pi_3\). In fact, the only nontrivial \(\pi_3\)-Stirling permutations are 122331 and 133221.

The weak Bruhat order \(\preceq_b\) on the set \(S_n\) of permutations of \(\{1, 2, \ldots, n\}\) is defined by: \(\sigma_n \preceq_b \pi_n\) provided that \(I(\sigma_n) \subseteq I(\pi_n)\). This is equivalent
to the property that $\sigma_n$ can be obtained from $\pi_n$ by a sequence of adjacent transpositions. On the other hand, the Bruhat order $\preceq_B$ is defined by $\sigma_n \preceq_B \pi_n$ provided that $\sigma_n$ can be obtained from $\pi_n$ by a sequence of transpositions each of which reduces the number of inversions by 1; thus $\mathcal{I}(\sigma_n)$ need not be a subset of $\mathcal{I}(\pi_n)$. It follows that if $\sigma_n \preceq_B \pi_n$, the set of $\sigma_n$-Stirling permutations need not be a subset of the set of $\pi_n$-Stirling permutations.

From the definitions we conclude that: If $\sigma_n \preceq_b \pi_n$, then a $\sigma_n$-Stirling permutation is also a $\pi_n$-Stirling permutation. In particular, as already remarked, any $\sigma_n$-Stirling permutation is also a $\zeta_n$-Stirling permutation. Denote by $S(\pi_n)$ the set of $\pi_n$-Stirling permutations. We thus have that

$$S(\pi_n) \subseteq S(\sigma_n) \text{ if and only if } \pi_n \preceq_b \sigma_n \text{ where equality holds if and only if } \pi_n = \sigma_n.$$  

**Example 1.1.** Let $n = 3$ and let $\pi_3$ be the permutation $312$. In this case, we have $\mathcal{I}(\pi_3) = \{(3, 1); (3, 2)\}$. Thus in a $\pi_3$-Stirling permutation between the two occurrences of 1, we cannot have a 2, since $(2, 1)$ is not an inversion of $\pi_3$. An example of a $\pi_3$-Stirling permutation is 112332, but 122331 is not. $\square$

The set $\mathcal{I}(\pi_n)$ of the inversions of a permutation $\pi_n$ determines a partially ordered set on $\{1, 2, \ldots, n\}$ whereby $i \preceq j$ if either $i = j$ or $(j, i)$ is an inversion of $\pi_n$ so that, in particular, $j > i$. In the classical case in which $\pi_n$ is the permutation $\zeta_n = n(n - 1) \cdots 21$, this reduces to $j > i$. This suggests a possible further generalization of Stirling permutations obtained by replacing a permutation $\pi_n$, its associated partially ordered set (poset), with an arbitrary finite poset $\mathcal{P} = (\mathcal{P}, \preceq)$. This concept of a $\mathcal{P}$-Stirling permutation, is introduced and explored in Section 2. A characterization of $\mathcal{P}$-Stirling permutations is given in Section 3, and it gives an algorithm for constructing all such objects. Finally, in Section 4 we discuss Stirling permutations for directed graphs.

## 2 Stirling permutations for a poset

Let $\mathcal{P} = (\mathcal{P}, \preceq)$ be a (finite) poset where $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$. A $\mathcal{P}$-Stirling permutation $\sigma$ is a permutation of the 2-multiset $\{p_1, p_1, p_2, p_2, \ldots, p_n, p_n\}$ such that, for $i = 1, 2, \ldots, n$, the following condition holds:
(I) For \(i = 1, 2, \ldots, n\), each element \(x \neq p_i\) that occurs between a pair of
\(p_i\)'s in \(\sigma\) satisfies \(p_i \prec x\).

In this definition (I), \(x\) cannot be incomparable to \(p_i\). This suggests a
modification of the definition of a Stirling permutation on a poset using the
condition:

(II) For \(i = 1, 2, \ldots, n\), each element \(x \neq p_i\) that occurs between a pair of
\(p_i\)'s in \(\sigma_n\) satisfies \(x \not\prec p_i\). So either \(p_i \prec x\) or \(x\) is incomparable to
\(p_i\).

We use \(\mathcal{P}\)-Stirling permutation to mean that (I) is satisfied and use weak
Stirling permutation to mean that (II) is satisfied. Both instances of each
maximal element of \(\mathcal{P} = (P, \prec)\) must be consecutive in \(\mathcal{P}\)-Stirling permuta-
tions. In weak \(\mathcal{P}\)-Stirling permutations between two maximal elements
\(p_i\) there can only be incomparable elements to \(p_i\). In Example 1.1, with
the multiset \(X_2^3 = \{1, 1, 2, 2, 3, 3\}\) and \(\pi_3 = 312, 112332\) is a \(\pi_3\)-Stirling
permutation but 132231 is not, but it is a weak \(\pi_3\)-Stirling permutation,
since \((2, 1)\) is not an inversion of \(\pi_3\) and thus 1 and 2 are incomparable in
this \(\mathcal{P}\).

Example 2.1. Consider \(\mathcal{P} = (P, \preceq)\), a totally unordered poset where
\(P = \{p_1, p_2, \ldots, p_n\}\)

has cardinality \(n\) (so no two elements are comparable). Then:

1. the number of \(\mathcal{P}\)-Stirling permutations is \(n!\), since each collection of
\(p_i\)'s has to be consecutive, and

2. the number of weak \(\mathcal{P}\)-Stirling permutations is \(\frac{(2n)!}{2^n}\) since now there
are no restrictions. (These are just the permutations of \(\{1, 1, 2, 2, \ldots, n, n\}\).)

Example 2.2. Consider the poset \(\mathcal{P}\) with elements \(\{p_1, p_2, p_3\}\) where only
\(p_1 \prec p_3\) and \(p_2 \prec p_3\). Examples of \(\mathcal{P}\)-Stirling permutations are
\(p_1p_1p_3p_3p_2p_2\) and \(p_1p_3p_3p_1p_2p_2\). We have that \(p_1p_2p_1p_2p_3\) is a weak-\(\mathcal{P}\)-Stirling permuta-
tion but not a \(\mathcal{P}\)-Stirling permutation, since there is a \(p_2\) between the
two \(p_1\)'s for which \(p_1 \not\prec p_2\).

Let \(Q_n = (X_n, \subseteq)\) denote the Boolean lattice of all subsets of \(X_n =
\{1, 2, \ldots, n\}\) partially ordered by inclusion. A \(Q_n\)-Stirling permutation is
a sequence of all the subsets of \(X_n\), each appearing twice, so that between
each pair of subsets \(A\) of \(X_n\) only supersets of \(A\) occur. We refer to such
\(Q_n\)-Stirling permutations as Boolean-Stirling permutations in general.
The $Q_n$-Stirling permutations can also be expressed in terms of $n$-tuples of 0’s and 1’s. Take the set of $2^n$ $n$-tuples of 0’s and 1’s (binary representations $a_1a_2\cdots a_n$ of the integers from 0 to $2^n - 1$) with partial order defined by

$$a_1a_2\cdots a_n \preceq b_1b_2\cdots b_n$$

if and only if $a_i = 1$ implies $b_i = 1$.

Between two equal integers in this sequence only larger integers can occur, but not all larger integers are possible. Geometrically, we have the vertices of an $n$-cube $Q_n$. A $(0,1)$ $n$-tuple $x$ having $k$ 1’s determines a face $F_x$ of $Q_n$ of dimension $n - k$ whose vertices are all $n$-tuples of 0’s and 1’s with 1’s in those $k$ places that $x$ has 1’s and possibly elsewhere. The Stirling property requires that between the two copies of the $n$-tuple $x$ with these $k$ 1’s only vertices on this $(n - k)$-dimensional face $F_x$ can occur (so they have 1’s in those $k$ places and possibly elsewhere). If $k = 0$, then there are no restrictions (the empty set is a subset of all sets).

**Example 2.3.** Take $n = 2$ so that we have the 2-tuples 00, 10, 01, 11. Then the following is a $Q_2$-Stirling permutation:

$$00, 10, 11, 11, 10, 01, 01, 00$$

or in terms of the corresponding integers 02332110. If $n = 3$, we have the example of a $Q_3$-Stirling permutation of $\{0, 1, 2, 3, 4, 5, 6, 7\}$ in terms of its binary representation:

$$000, 001, 010, 010, 001, 000, 101, 110, 111, 111, 110, 101, 011, 100, 100, 011,$$

or, in terms of the corresponding integers, 0122105677653443. But this is not an ordinary Stirling permutation of $\{0, 1, 2, 3, 4, 5, 6, 7\}$: between 101 and 101 (representing integers 5), we cannot have 110 (representing 6), even though 6 is larger that 5.

This example can be generalized to any integer $n \geq 2$. For example if $n = 4$,


Moreover, the parts separated by commas can be arbitrarily permuted. □

For this Boolean lattice $Q_n$, a weak $Q_n$-Stirling permutation is a listing of the subsets of $X_n$, each appearing twice, so that between two equal subsets $A$ there are only supersets or subsets incomparable to $A$.

Another partially ordered set that may be of interest is the partially ordered set on $X_n = \{1, 2, \ldots, n\}$ where the partial order is that of divisibility.

**Example 2.4.** Consider the partially ordered set $P_n = (X_n, \preceq)$ on the set $X_n = \{1, 2, \ldots, n\}$ where the partial order is that of divisibility. If $n = 6$, then a $P_6$-Stirling permutation is 144122366355. □
3 Characterization of \( \mathcal{P} \)-Stirling permutations

Let \( \mathcal{P} = (P, \preceq) \) be an arbitrary finite poset where \( P = \{p_1, p_2, \ldots, p_n\} \). A \( \mathcal{P} \)-Stirling permutation \( \sigma \) is a 2-permutation of \( P \) with the property that, for \( i = 1, 2, \ldots, n \), each element \( x \neq p_i \) that occurs between the two \( p_i \)’s in \( \sigma \) satisfies \( p_i \prec x \). Associated with \( \mathcal{P} \) we define the directed graph \( G(\mathcal{P}) \) with vertex set \( P = \{p_1, p_2, \ldots, p_n\} \) and edges \( p_i \rightarrow p_j \) provided \( p_i \prec p_j \). By the transitive law for posets, the directed graph \( G(\mathcal{P}) \) is transitive, that is, \( p_i \rightarrow p_j, p_j \rightarrow p_k \) imply \( p_i \rightarrow p_k \). So, in the usual Hasse diagram where an element \( p_i \) is below another element \( p_j \) in the diagram if \( p_i \prec p_j \), we have a directed edge \( p_i \rightarrow p_j \) from \( p_i \) to \( p_j \) in \( G(\mathcal{P}) \). Given a finite poset \( \mathcal{P} \), we want to characterize the \( \mathcal{P} \)-Stirling permutations and possibly find their number. In what follows, we will give a complete characterization of these permutations.

We now introduce a specific procedure for determining a walk \( W \) in \( G(\mathcal{P}) \) where its vertices, using some specified rules, give a 2-permutation \( \sigma \) of \( \mathcal{P} \). The walk is considered in the associated (undirected) graph of \( G(\mathcal{P}) \), so we can move forward or backward along edges of \( G(\mathcal{P}) \). We call this procedure a \( \mathcal{P} \)-depth-search, or \( \mathcal{P} \)-DS, for short (see [1, 4] for more on depth-first-search). The map \( \ell : P \rightarrow \{0, 1, 2\} \) gives a label to each element in \( P \) which counts the number of occurrences of each vertex in the walk as it progresses. Initially, \( \ell(p_i) = 0 \) (\( i = 1, 2, \ldots, n \)) and, when we terminate, \( \ell(p_i) = 2 \) for each vertex in the walk. We also define a predecessor function ‘prev’ for the vertices as they are visited in the walk.

\( \mathcal{P} \)-depth-search: \( (i) \) Choose some initial vertex \( p_i \) for the walk \( W \) as well as for \( \sigma \), and set \( \ell(p_i) = 1 \) and \( \text{prev}(p_i) = p_i \). For the general step, if \( u \) is the last vertex of \( W \) determined so far, where \( \ell(u) = 1 \), then the next vertex \( v \) of \( W \) is obtained by either a forward-step or a backward-step as follows:

(i) Forward-step: Choose a vertex \( v \) with \( \ell(v) = 0 \) such that \( u \rightarrow v \). Define \( \text{prev}(v) = u \) (the predecessor), relabel \( \ell(v) = 1 \), and add \( v \) onto both \( W \) and \( \sigma \);

(ii) Backward-step: Let \( v = \text{prev}(u) \) and relabel \( \ell(u) = 2 \). Add \( v \) onto \( W \) and add \( u \) (but not \( v \)) onto \( \sigma \).

Thus, in both types of steps we add to \( \sigma \) the head (terminal end vertex) of the directed edge. We terminate when the (initial) vertex \( p_i \) is met for
the second time in $W$, and we then add $p_{i_1}$ to $\sigma$. Associated with such a walk $W$ there is a tree $T_W$ consisting of the vertices and forward-step edges used in $W$. Note that some of the vertices of $G(P)$ may not be included in $T$ and $\sigma$, and that $W$ and $\sigma$ may have different lengths.

**Example 3.1.** Let $n = 9$ and consider the poset $P$ whose Hasse diagram is shown in Fig.1. A $P$-DS-search may give the following walk $W$ and corresponding 2-permutation $\sigma$

$W : \ p_1, p_3, p_6, p_9, p_6, p_3, p_5, p_8, p_5, p_3, p_1;$

$\sigma : \ p_1, p_3, p_6, p_9, p_6, p_5, p_8, p_5, p_3, p_1.$

The vertices obtained in a backward step are indicated in boldface in $W$. The associated tree $T_W$ is indicated by thick lines in the figure. Note that vertices $p_2, p_4, p_7$ are not included in $W$ and $\sigma$.

![Figure 1: Poset $P$ and a $P$-DS-search.](image)

**Lemma 3.2.** $P$-DS terminates and the constructed $\sigma$ is a 2-permutation of its vertex set, i.e., each vertex $v$ in $\sigma$ occurs twice. Moreover, each vertex $x$ that occurs between these two $v$’s satisfies $v \to x$, i.e., $v \prec x$, so $\sigma$ is a Stirling permutation of the corresponding 2-multiset.

**Proof.** Consider the general step in the construction of $W$ and $\sigma$ (as above) and let $u$ be the last vertex so far (in $W$). Let $P_W$ be the set of vertices in the current $W$ (so no repetition).

**Claim.** $P_W$ is the vertex set of a subtree $T_W$ in $G(P)$ with (directed) edges $(u, v)$ associated with each forward-step from $u$ to $v$. Moreover $\ell(v) \in \{1, 2\}$ for each $v \in P_W$, and the vertices in $T_W$ with $\ell(v) = 1$ is a directed subtree with root $p_{i_1}$.

**Proof of Claim.** The first statement follows directly from the fact that we start with a single vertex $p_{i_1}$ and in each forward step a new
vertex \( v \) is added to \( P_W \) and a new directed edge from an existing vertex \( u \) to \( v \). The new vertex is given the label 1. (This is a standard way to construct trees.) In a backward-step from \( u \) to its predecessor \( v = \text{prev}(u) \), no new vertex is added, so \( P_W \) is unchanged, and \( u \) is given label 2. The first time we do a backward step we leave a pendant vertex \( u \) of the subtree \( T_W \); let \( T'_W \) be the subtree obtained by deleting \( u \) (and the incident edge). Then \( T'_W \) is a directed subtree with root \( p_{i_1} \) where each of its vertices has label 1. The next backward-step has the same property: a pendant vertex \( u' \) gets the label 2 and the updated subtree \( T'_W \) obtained by deleting \( u' \) is a directed subtree with root \( p_{i_1} \) where each of its vertices has label 1. The second statement of the claim now follows by induction.

The process \( \mathcal{P} \)-DS has at most \( n - 1 \) forward-steps (as each such step leads to a new vertex not visited before). Thus, when the forward-steps are all done, the remaining steps are backward-steps and gradually the tree \( T_W \) shrinks to the single vertex \( p_{i_1} \). Each vertex in \( P_W \) is visited exactly twice (when its \( \ell \)-label is changed from 0 to 1, and later from 1 to 2). This proves the lemma because between the two occurrences of a vertex \( v \) in the generated sequence there are only vertices that are reachable from \( v \) by a directed path in the tree \( T_W \).

The Stirling 2-permutation \( \sigma \) as constructed in Lemma 3.2 will be called a \( \mathcal{P} \)-DS block, and we let \( P_\sigma \) denote its vertex set (which equals \( P_W \) for the corresponding walk \( W \)). We may now repeat this construction, and find a \( \mathcal{P}' \)-DS block in the subposet \( \mathcal{P}' \) induced by \( P \setminus P_\sigma \). This process may be repeated, for the remaining elements in \( P \), until we have found \( \mathcal{P} \)-DS blocks such that their vertex sets define an ordered partition of \( P \). The concatenation of these \( \mathcal{P} \)-DS blocks will be called a \( \mathcal{P} \)-DS sequence. Associated with each of the \( \mathcal{P} \)-DS blocks is a rooted directed tree, as described above.

We now state and prove a main result in this paper, a characterization of \( \mathcal{P} \)-Stirling permutations in a general poset \( \mathcal{P} \).

**Theorem 3.3.** Let \( \mathcal{P} = (P, \preceq) \) be a poset, and let \( \sigma \) be a 2-permutation of \( P \). Then \( \sigma \) is a \( \mathcal{P} \)-Stirling permutation if and only if \( \sigma \) is a \( \mathcal{P} \)-DS sequence.

**Proof.** Consider a \( \mathcal{P} \)-DS sequence \( \sigma \). Then each of its \( \mathcal{P} \)-DS blocks is a 2-permutation of its vertex set, by Lemma 3.2, and it follows that \( \sigma \) is a \( \mathcal{P} \)-Stirling permutation.

Conversely, let \( \sigma \) be a \( \mathcal{P} \)-Stirling permutation

\[ \sigma : v_1, v_2, v_3, \ldots, v_s. \]

(Thus, these elements are not distinct.)
Claim (Nestedness property). For each \( i, j \leq s, i \neq j \), between the two occurrences of \( v_i \) in \( \sigma \) the vertex \( v_j \) occurs either 0 or 2 times.

Proof of Claim. Assume \( v_j \) only occurs once between the two occurrences of \( v_i \) in \( \sigma \). Then the other \( v_j \) must be before the first \( v_i \) or after the second \( v_i \), so their internal order is e.g.

\[
v_i \cdots v_j \cdots v_i \cdots v_j.
\]

By the Stirling property this gives \( v_i \prec v_j \) and also \( v_j \prec v_i \), which contradicts the poset property (as \( v_i \neq v_j \)). The other case, when \( v_j \) before the first \( v_i \), is similar. This proves the Claim.

Let \( k \) be maximal such that the first \( k \) vertices in \( \sigma \) are distinct. Thus

\[
v_1 \prec v_2 \prec \cdots \prec v_k
\]

and \( v_{k+1} = v_i \) for some \( i \leq k \). Then \( i = k \), i.e., \( v_{k+1} = v_k \). This follows from the Nestedness property because if \( i < k \), then \( v_k \) would occur once between the two occurrences of \( v_i \). Moreover, the internal order in \( \sigma \) of the occurrences of \( v_1, v_2, \ldots, v_k \) is as follows:

\[
\sigma : v_1, v_2, v_3, \ldots, v_{k-1}, v_k, v_k, \ldots, v_{k-1}, \ldots, v_2, \ldots, v_1, \ldots \quad (2)
\]

Thus, the second occurrences of these \( v_i \)'s are in the opposite order. This is again due to the Nestedness property. Now we connect the structure of \( \sigma \) in (2) to a \( \mathcal{P} \)-DS sequence. Consider a walk \( W \) with vertices

\[
v_1, v_2, v_3, \ldots, v_{k-1}, v_k,
\]

these are \( k - 1 \) forward-steps. Next, do a backward-step from \( v_k \) to \( v_{k-1} \). This gives the following initial part of a \( \mathcal{P} \)-DS block \( \sigma^* \)

\[
\sigma^* : v_1, v_2, v_3, \ldots, v_{k-1}, v_k, v_k
\]

which coincides with the initial part of \( \sigma \). Next, consider the part \( \sigma' \) of \( \sigma \) that is between (the second) \( v_k \) and \( v_{k-1} \). There are two possibilities:

Case 1: \( \sigma' \) is empty. Then we perform a backward-step from \( v_{k-1} \) to \( v_{k-2} \), so \( v_{k-2} \) is added to \( W \) and \( v_{k-1} \) is added to \( \sigma^* \). Thus \( \sigma \) and \( \sigma^* \) coincide in the next position as well.

Case 2: \( \sigma' \) is nonempty. Since \( \sigma' \) is between the two occurrences of \( v_{k-1} \), see (2), the Stirling property means that every vertex \( v \) in \( \sigma' \) satisfies \( v_{k-1} \prec v \). Moreover, due to the Nestedness property, each such \( v \) occurs two times in \( \sigma' \). Let \( v' \) be the first vertex in \( \sigma' \). Then we perform a forward-step from \( v_{k-1} \) to \( v' \), so \( v' \) is added both to \( W \) and \( \sigma^* \). Thus \( \sigma \) and \( \sigma^* \) coincide in the next position as well.
In both cases we can repeat the argument to a smaller sequence of vertices. In Case 2 we will then construct a subtree with root \( v_{k-1} \). It is clear that by induction the final constructed \( \sigma^* \) equals \( \sigma \), as desired. Thus, every \( \mathcal{P} \)-Stirling permutation is also a \( \mathcal{P} \)-DS sequence, and the proof is complete. \( \square \)

**Example 3.4.** Consider again the poset \( \mathcal{P} \) whose Hasse diagram is shown in Fig.1. We have already discussed the \( \mathcal{P} \)-DS-block

\[
P_1, p_3, p_6, p_9, p_9, p_6, p_5, p_8, p_8, p_5, p_3, p_1.
\]

Another \( \mathcal{P} \)-DS-block is

\[
P_2, p_4, p_7, p_7, p_4, p_2.
\]

Concatenating these we get the \( \mathcal{P} \)-DS sequence and therefore \( \mathcal{P} \)-Stirling permutation

\[
P_1, p_3, p_6, p_9, p_9, p_6, p_5, p_8, p_8, p_5, p_3, p_1, p_2, p_4, p_7, p_7, p_4, p_2.
\]

We remark that our characterization Theorem 3.3 of \( \mathcal{P} \)-Stirling permutations is of a similar nature as the characterization of Stirling permutations via plane trees [7].

### 4 Stirling permutations for directed graphs

As discussed in Section 3, a poset \( \mathcal{P} = (P, \preceq) \) defines a directed graph \( G(\mathcal{P}) \) with vertex set \( P = \{p_1, p_2, \ldots, p_n\} \) and edges \( p_i \rightarrow p_j \) provided \( p_i \prec p_j \). By the transitive law for posets, the directed graph \( G(\mathcal{P}) \) is transitive: \( p_i \rightarrow p_j, p_j \rightarrow p_k \) imply \( p_i \rightarrow p_k \). Stirling permutations can be defined for any directed graph. The original definition of a Stirling permutation corresponds to the (linearly ordered) directed graph \( 1 \rightarrow 2 \rightarrow \cdots \rightarrow n \), extended with edges due to transitivity.

Consider an arbitrary directed graph \( G = (V, E) \). We can extend our definitions of \( \mathcal{P} \)-Stirling permutation and weak Stirling permutation as a 2-permutation \( \sigma \) of \( V \) in the obvious way:

(I) **\( G \)-Stirling permutation:** For \( v \in V \), each element \( x \neq v \) that occurs between a pair of \( v \)'s in \( \sigma \) satisfies \( v \rightarrow x \).

(II) **weak \( G \)-Stirling permutation:** For \( v \in V \), each element \( x \neq v \) that occurs between a pair of \( v \)'s in \( \sigma \) satisfies \( v \rightarrow x \) or there is no edge between \( x \) and \( v \) in either direction.
Example 4.1.
• Consider the directed graph $G$ of order 3 consisting of the 3-cycle $a \to b, b \to c, c \to a$ (so this does not result from a poset). Then the following are $G$-Stirling permutations:

$$aabbcc \text{ (6 of these); } ccabba \text{ (6 of these).}$$

In this case, every weak $G$-Stirling permutation is a $G$-Stirling permutation.

• Consider the directed graph $G$ with $V = \{x, a, b, c\}$, where only $a \to x, b \to x, c \to x$. Then e.g. $axxabbcc$ is a $G$-Stirling permutation and $caxxbabc$ is a weak $G$-Stirling permutation. □

Let $\sigma$ be a 2-permutation of $V$. For $v \in V$ let $\sigma^{(v-v)}$ denote the set of vertices occurring (at least once) between the two occurrences of $v$ in $\sigma$. Also, let $\Gamma^+_G(v)$ be the set of vertices $w$ with $v \to w$ in $G$. Then $\sigma$ is $G$-Stirling permutation if and only if

$$\sigma^{(v-v)} \subseteq \Gamma^+_G(v) \quad (v \in V). \quad (3)$$

We call a 2-permutation of $V$ a trivial 2-permutation provided that the two occurrences of $v$ are consecutive for each $v \in V$, i.e., $\sigma^{(v-v)} = \emptyset$. There are $n!$ trivial 2-permutations (when $n = |V|$), and each of these is clearly a $G$-Stirling permutation. The following proposition contains some basic properties of $G$-Stirling permutations.

Proposition 4.2.

(i) The set of $G$-Stirling permutations is the set of all trivial 2-permutations of $V$ if and only if $G$ has no edges.

(ii) If $G = (V, E)$ and $G' = (V, E')$ with $E \subseteq E'$, then every $G$-Stirling permutation is also a $G'$-Stirling permutation.

(iii) Let $G = (V, E)$ be the complete directed graph on $n$ vertices, i.e., $E = \{(i, j) : i, j \in V, i \neq j\}$. Then the $G$-Stirling permutations consists of all 2-permutations of $V$.

(iv) Let $G = (V, E)$ be a complete bipartite directed graph, i.e., $V$ consists of color classes $I$ and $J$ and all edges $(i, j)$ where $i \in I$ and $j \in J$. Then the $G$-Stirling permutations consists of all 2-permutations $\sigma$ of $V$ satisfying $\sigma^{(j-j)} = \emptyset \ (j \in J)$ and $\sigma^{(i-i)} \subseteq J \ (i \in I)$.
Proof.

(i) If $G$ has no edge, then, for a $G$-Stirling permutation $\sigma$, $\sigma^{(v-v)} = \emptyset$ for each $v \in V$, so $\sigma$ is a trivial 2-permutation. If $G$ has an edge, say $v_1 \to v_2$, then $\sigma = v_1v_2v_1v_3v_4 \cdots v_nv_n$ is a $G$-Stirling permutation which is not a trivial-permutation.

(ii) This is immediate from (3).

(iii) When $G$ is complete, $\Gamma^+_G(v) = V \setminus \{v\}$ so then (3) holds for any 2-permutations of $V$.

(iv) This also follows from (3).

Example 4.3. Let $T_n$ be the star with $V = \{1, 2, \ldots, n\}$ and edges $n \to 1, n \to 2, \ldots, n \to (n - 1)$. This is a special complete bipartite graph; see case (iv) in Proposition 4.2. Consider this star with $n = 4$. So we have only $4 \to i$ for $i = 1, 2, 3$. Let $\sigma$ be a $T_n$-Stirling permutation. Thus the two occurrences of $j$ have to be together ($j \leq 3$), and some examples of such $T_n$-Stirling permutations are 41122334, 22411334 and 33411422.

Corollary 4.4. The number of $T_n$-Stirling permutations when $T_n$ is the star with $n$ vertices in Example 4.3 is $n!(n - 1)/2$.

Proof. Let $N$ be the number to be computed. Let $\sigma$ be a $T_n$-Stirling permutation. Then for each $j \leq n - 1$ the two occurrences of $j$ in $\sigma$ must be consecutive. So, $N$ equals $(n - 1)!$ times the number of $T_n$-Stirling permutations with $1, 2, \ldots, n - 1$ occurring as $1, 1, 2, 2, \ldots, n - 1, n - 1$. We can place the two $n$’s in $\sigma$ in any of the $n$ positions labeled $x$ in $x, 1, 1, x, 2, 2, \ldots, x, n - 1, n - 1, x$. Thus

$$N = (n - 1)\binom{n}{2} = n!(n - 1)/2.$$ as desired. \[\square\]

For weak $T_n$-Stirling permutations there are additional possibilities since, for each $j \leq n - 1$, the two occurrences of $j$ need not be consecutive.

Proposition 4.5. The number of weak Stirling permutations for the star $T_n$ equals

$$(n - 1)! \sum_{a \geq 0, b \geq 0, c \geq 0, a + b + c = n - 1} \frac{(2a)!(2b)!(2c)!}{a!b!c!}$$

Proof. Now, for each $j \leq n - 1$ the two $j$’s must be either before the first $n$, or between the two $n$’s, or after the second $n$. Choosing $a, b,$ and $c$
of them before, between, and after the $n$’s and then taking an arbitrary permutation of both of the integers chosen, we get by direct computation

$$\sum_{a\geq 0, b\geq 0, c\geq 0, \quad a+b+c = n-1} \frac{(n-1)!}{a!b!c!} (2a)!(2b)!(2c)!$$

as desired.

Now let $T_n^*$ denote the digraph obtained from $T_n$ be reversing the direction for each edge, so the edges are now $i \to n$ $(i \leq n-1)$. This is also a complete bipartite graph, so we can again apply Proposition 4.2.

**Proposition 4.6.** The number of $T_n^*$-Stirling permutations is

$$(2n-1) \cdot (n-1)!.$$  

**Proof.** First we note that $1, 2, \ldots, (n-1)$ can be arbitrarily permuted in such a Stirling permutation and we cannot have $i,j,i$ $(1 \leq i,j \leq n-1, i \neq j)$ occurring as a subsequence; so the number is $(n-1)!$ times the number of those in which $1, 2, \ldots, (n-1)$ are in their natural order. The $n$’s have to be together and can be in any of the $(2n-1)$ places in-between $1, 1, 2, 2, \ldots, (n-1), (n-1)$. □

**Proposition 4.7.** The number of weak Stirling permutations for the star $T_n^*$ equals

$$\frac{(2n-1)!}{2^{n-1}}.$$  

**Proof.** Take an arbitrary permutation of $\{1, 1, 2, 2, \ldots, n-1, n-1\}$ and then insert the two $n$’s together in any of the resulting $2n-1$ places. □

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**References**


