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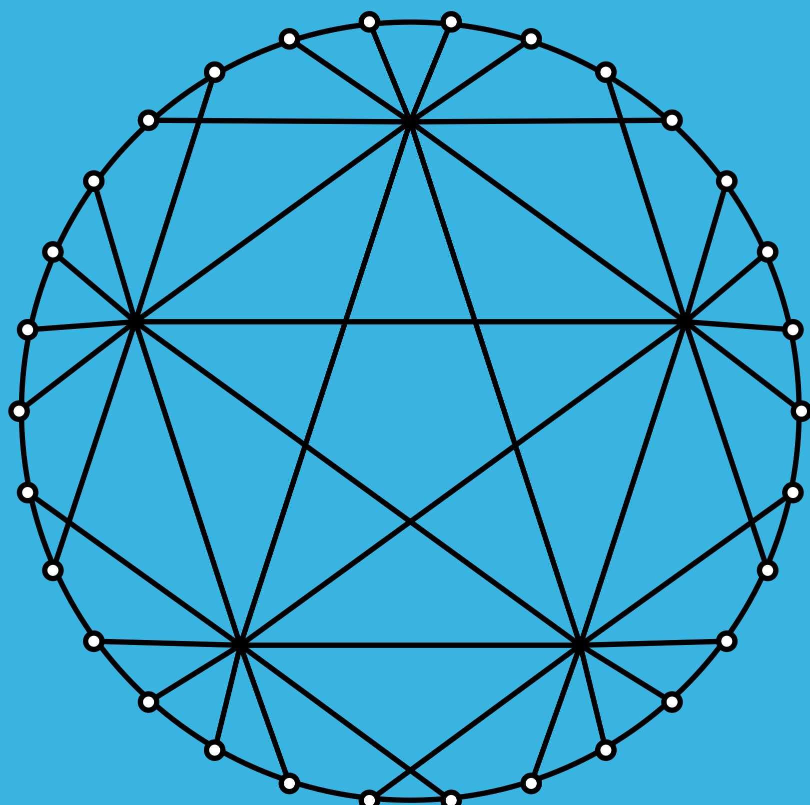

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Stirling permutations for partially ordered sets

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Abstract. We generalize the notion of a Stirling permutation of the multiset $\{1,1,2,2,\ldots,n,n\}$ based on the usual linear order of the integers $\{1,2,\ldots,n\}$ to any finite partially ordered set $\mathcal{P}$, a $\mathcal{P}$-Stirling permutation. We give an algorithmic characterization of $\mathcal{P}$-Stirling permutations. A partially ordered set determines a transitive directed graph, and a further extension of Stirling permutations to directed graphs is discussed.

1 Introduction

Let n be a positive integer and $X_n = \{1,2,\ldots,n\}$. A *Stirling permutation* of the 2-multiset $X_n^2 = \{1,1,2,2,\ldots,n,n\}$ is defined by the property:

($*$) For each $k = 1,2,\ldots,n$, between the two occurrences of k only integers greater than k occur.

For example, with $n = 4$, 23443211 is a Stirling permutation, but 13234421 is not. A Stirling permutation is a permutation of a specific multiset and so is a *multipermutation*. Stirling permutations have been generalized to arbitrary multisets using the same property ($*$).

In this paper we confine our attention to the multiset

$$X_n^2 = \{1,1,2,2,\ldots,n,n\},$$

that is, to the 2-*permutations* of $\{1,2,\ldots,n\}$. Stirling permutations were introduced in [5] in connection with a study of Stirling numbers and Stirling polynomials. The total number of Stirling permutations of X_n^2 is the

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double factorial $(2n-1)!! = 1\cdot 3\cdots(2n-1)$. Stirling permutations have connections to other combinatorial objects. In [6] it is explained how Stirling permutations give rise to a combinatorial interpretation of the second-order Eulerian numbers. Moreover, Stirling permutations arise naturally for certain walks in plane trees [7], which we return to later. For some recent work on Stirling permutations, see [2, 3].

A Stirling permutation obtained from an ordinary permutation π of $\{1,2,\ldots,n\}$ by doubling each integer i in π is called a *trivial-Stirling permutation.* Thus, for instance, 221133 is a trivial Stirling permutation. Between the two occurrences of each integer k in a Stirling permutation, there is a Stirling permutation σ of the multiset $\{l,l : k < l \leq n\}$, indeed $k\sigma k$ is a Stirling permutation of $\{k,k,k+1,k+1,\ldots,n,n\}$. For example, in the Stirling permutation 12344321, between the two 2's there is a Stirling permutation 3443 of $\{3,3,4,4\}$ and between the two 1's there is a Stirling permutation of $\{2,2,3,3,4,4\}$. Thus Stirling permutations of X_n^2 can be constructed as follows: choose an integer $k \leq n$ and a subset Y_k of $\{k,k+1,\ldots,n\}$, and then choose a Stirling permutation σ_k of the 2-multiset Y_k^2 with k as both the first and last integer. Now choose a new integer l, a subset Y_l of new integers greater than or equal to l, and put a Stirling permutation σ_l of Y_l^2 with l as both the first and last integer on one of the two sides of σ_k, giving a Stirling permutation of the 2-multiset $Y_k^2 \cup Y_l^2$. Continue like this until all integers have been used.

A Stirling permutation of $X_n^2 = \{1,1,2,2,\ldots,n,n\}$ can be regarded as based on the *reverse-permutation* $\zeta_n = n(n-1)\cdots 1$ of the set $\{1,2,\ldots,n\}$ in the sense that the order relation used in checking the Stirling property corresponds to the inversions of the permutation ζ_n. We can replace the permutation ζ_n by an arbitrary permutation $\pi_n = p_1p_2\cdots p_n$ of $\{1,2,\ldots,n\}$ to obtain a generalization of the classical ζ_n-Stirling permutations. For a permutation π_n of $\{1,2,\ldots,n\}$, let $\mathcal{I}(\pi_n)$ be the set of inversions of π_n where an *inversion* is a pair $(\pi(i),\pi(j))$ where $i < j$ and $\pi(i) > \pi(j)$. Thus a π_n*-Stirling permutation* is a permutation of X_n^2 such that the integers j between two occurrences of an integer k satisfy that (j,k) is an inversion of π_n; in particular, we have $j > k$, but that in itself does not suffice, since j must precede k in π_n. Any π_n-Stirling permutation is a ζ_n-Stirling permutation, but the converse does not hold. For example, with the multiset $X_3^2 = \{1,1,2,2,3,3\}$ and $\pi_3 = 231$, 123321 is not a π_3-Stirling permutation since $3,2$ is not an inversion of π_3. In fact, the only nontrivial π_3-Stirling permutations are 122331 and 133221.

The *weak Bruhat order* $\preceq_b$ on the set $\mathcal{S}_n$ of permutations of $\{1,2,\ldots,n\}$ is defined by: $\sigma_n \preceq_b \pi_n$ provided that $\mathcal{I}(\sigma_n) \subseteq \mathcal{I}(\pi_n)$. This is equivalent to the property that σ_n can be obtained from π_n by a sequence of adjacent transpositions. On the other hand, the *Bruhat order* $\preceq_B$ is defined by $\sigma_n \preceq_B \pi_n$ provided that σ_n can be obtained from π_n by a sequence of transpositions each of which reduces the *number* of inversions by 1; thus $\mathcal{I}(\sigma_n)$ need not be a subset of $\mathcal{I}(\pi_n)$. It follows that if $\sigma_n \preceq_B \pi_n$, the set of σ_n-Stirling permutations need not be a subset of the set of π_n-Stirling permutations.

From the definitions we conclude that: *If $\sigma_n \preceq_b \pi_n$, then a σ_n-Stirling permutation is also a π_n-Stirling permutation.* In particular, as already remarked, any σ_n-Stirling permutation is also a ζ_n-Stirling permutation. Denote by $\mathcal{S}(\pi_n)$ the set of π_n-Stirling permutations. We thus have that

$\mathcal{S}(\pi_n) \subseteq \mathcal{S}(\sigma_n)$ *if and only if* $\pi_n \preceq_b \sigma_n$ *where equality holds if and only if* $\pi_n = \sigma_n$.

Example 1.1. Let $n = 3$ and let π_3 be the permutation 312. In this case, we have $\mathcal{I}(\pi_3) = \{(3,1);(3,2)\}$. Thus in a π_3-Stirling permutation between the two occurrences of 1, we cannot have a 2, since $(2,1)$ is not an inversion of π_3. An example of a π_3-Stirling permutation is 112332, but 122331 is not. □

The set $\mathcal{I}(\pi_n)$ of the inversions of a permutation π_n determines a partially ordered set on $\{1,2,\ldots,n\}$ whereby $i \preceq j$ if either $i = j$ or (j,i) is an inversion of π_n so that, in particular, $j > i$. In the classical case in which π_n is the permutation $\zeta_n = n(n-1)\cdots 21$, this reduces to $j > i$. This suggests a possible further generalization of Stirling permutations obtained by replacing a permutation π_n, its associated partially ordered set (poset), with an arbitrary finite poset $\mathcal{P} = (P, \preceq)$. This concept of a $\mathcal{P}$-Stirling permutation, is introduced and explored in Section 2. A characterization of $\mathcal{P}$-Stirling permutations is given in Section 3, and it gives an algorithm for constructing all such objects. Finally, in Section 4 we discuss Stirling permutations for directed graphs.

2 Stirling permutations for a poset

Let $\mathcal{P} = (P, \preceq)$ be a (finite) poset where $P = \{p_1, p_2, \ldots, p_n\}$. A *$\mathcal{P}$-Stirling permutation* σ is a permutation of the 2-multiset $\{p_1, p_1, p_2, p_2, \ldots, p_n, p_n\}$ such that, for $i = 1, 2, \ldots, n$, the following condition holds:

(I) For $i = 1, 2, \ldots, n$, each element $x \neq p_i$ that occurs between a pair of p_i's in σ satisfies $p_i \prec x$.

In this definition (I), x cannot be incomparable to p_i. This suggests a modification of the definition of a Stirling permutation on a poset using the condition:

(II) For $i = 1, 2, \ldots, n$, each element $x \neq p_i$ that occurs between a pair of p_i's in σ_n satisfies $x \not\prec p_i$. So either $p_i \prec x$ or x is incomparable to p_i.

We use $\mathcal{P}$-Stirling permutation to mean that (I) is satisfied and use *weak Stirling permutation* to mean that (II) is satisfied. Both instances of each maximal element of $\mathcal{P} = (P, \preceq)$ must be consecutive in $\mathcal{P}$-Stirling permutations. In weak $\mathcal{P}$-Stirling permutations between two maximal elements p_i there can only be incomparable elements to p_i. In Example 1.1, with the multiset $X_3^2 = \{1, 1, 2, 2, 3, 3\}$ and $\pi_3 = 312$, 112332 is a π_3-Stirling permutation but 132231 is not, but it is a weak π_3-Stirling permutation, since $(2, 1)$ is not an inversion of π_3 and thus 1 and 2 are incomparable in this $\mathcal{P}$.

Example 2.1. Consider $\mathcal{P} = (P, \preceq)$, a totally unordered poset where

$$P = \{p_1, p_2, \ldots, p_n\}$$

has cardinality n (so no two elements are comparable). Then:

1. the number of $\mathcal{P}$-Stirling permutations is $n!$, since each collection of p_i's has to be consecutive, and

2. the number of weak $\mathcal{P}$-Stirling permutations is $\frac{(2n)!}{2^n}$ since now there are no restrictions. (These are just the permutations of $\{1, 1, 2, 2, \ldots, n, n\}$.) □

Example 2.2. Consider the poset $\mathcal{P}$ with elements $\{p_1, p_2, p_3\}$ where only $p_1 \prec p_3$ and $p_2 \prec p_3$. Examples of $\mathcal{P}$-Stirling permutations are $p_1p_1p_3p_3p_2p_2$ and $p_1p_3p_3p_1p_2p_2$. We have that $p_1p_2p_1p_2p_3p_3$ is a weak-$\mathcal{P}$-Stirling permutation but not a $\mathcal{P}$-Stirling permutation, since there is a p_2 between the two p_1's for which $p_1 \not\prec p_2$. □

Let $\mathcal{Q}_n = (X_n, \subseteq)$ denote the Boolean lattice of all subsets of $X_n = \{1, 2, \ldots, n\}$ partially ordered by inclusion. A $\mathcal{Q}_n$-Stirling permutation is a sequence of all the subsets of X_n, each appearing twice, so that between each pair of subsets A of X_n only supersets of A occur. We refer to such $\mathcal{Q}_n$-Stirling permutations as *Boolean-Stirling permutations* in general.

The $\mathcal{Q}_n$-Stirling permutations can also be expressed in terms of n-tuples of 0's and 1's. Take the set of 2^n n-tuples of 0's and 1's (binary representations $a_1a_2\cdots a_n$ of the integers from 0 to 2^n-1) with partial order defined by

$$a_1a_2\cdots a_n \preceq b_1b_2\cdots b_n \text{ if and only if } a_i = 1 \text{ implies } b_i = 1.$$

Between two equal integers in this sequence only larger integers can occur, but not all larger integers are possible. Geometrically, we have the vertices of an n-cube $\mathbf{Q}_n$. A $(0,1)$ n-tuple x having k 1's determines a face $\mathcal{F}_x$ of $\mathbf{Q}_n$ of dimension $n-k$ whose vertices are all n-tuples of 0's and 1's with 1's in those k places that x has 1's and possibly elsewhere. The Stirling property requires that between the two copies of the n-tuple x with these k 1's only vertices on this $(n-k)$-dimensional face $\mathcal{F}_x$ can occur (so they have 1's in those k places and possibly elsewhere). If $k=0$, then there are no restrictions (the empty set is a subset of all sets).

Example 2.3. Take $n=2$ so that we have the 2-tuples $00, 10, 01, 11$. Then the following is a $\mathcal{Q}_2$-Stirling permutation:

$$00, 10, 11, 11, 10, 01, 01, 00$$

or in terms of the corresponding integers 02332110. If $n=3$, we have the example of a $\mathcal{Q}_3$-Stirling permutation of $\{0,1,2,3,4,5,6,7\}$ in terms of its binary representation:

$$000, 001, 010, 010, 001, 000, 101, 110, 111, 111, 110, 101, 011, 100, 100, 011, \qquad (1)$$

or, in terms of the corresponding integers, 0122105677653443. But this is not an ordinary Stirling permutation of $\{0,1,2,3,4,5,6,7\}$: between 101 and 101 (representing integers 5), we cannot have 110 (representing 6), even though 6 is larger that 5.

This example can be generalized to any integer $n \geq 2$. For example if $n=4$,

$$01233210, 456665, 89(10)(11)(11)(10)98, (12)(13)(14)(15)(15)(14)(13)(12).$$

Moreover, the parts separated by commas can be arbitrarily permuted. □

For this Boolean lattice $\mathcal{Q}_n$, a weak $\mathcal{Q}_n$-Stirling permutation is a listing of the subsets of X_n, each appearing twice, so that between two equal subsets A there are only supersets or subsets incomparable to A.

Another partially ordered set that may be of interest is the partially ordered set on $X_n = \{1,2,\ldots,n\}$ where the partial order is that of divisibility.

Example 2.4. Consider the partially ordered set $\mathcal{P}_n = (X_n, \preceq)$ on the set $X_n = \{1,2,\ldots,n\}$ where the partial order is that of divisibility. If $n=6$, then a $\mathcal{P}_6$-Stirling permutation is 144122366355. □

3 Characterization of $\mathcal{P}$-Stirling permutations

Let $\mathcal{P} = (P, \preceq)$ be an arbitrary finite poset where $P = \{p_1, p_2, \ldots, p_n\}$. A $\mathcal{P}$-Stirling permutation σ is a 2-permutation of P with the property that, for $i = 1, 2, \ldots, n$, each element $x \neq p_i$ that occurs between the two p_i's in σ satisfies $p_i \prec x$. Associated with $\mathcal{P}$ we define the directed graph $G(\mathcal{P})$ with vertex set $P = \{p_1, p_2, \ldots, p_n\}$ and edges $p_i \to p_j$ provided $p_i \prec p_j$. By the transitive law for posets, the directed graph $G(\mathcal{P})$ is *transitive*, that is, $p_i \to p_j$, $p_j \to p_k$ imply $p_i \to p_k$. So, in the usual Hasse diagram where an element p_i is below another element p_j in the diagram if $p_i \prec p_j$, we have a directed edge $p_i \to p_j$ from p_i to p_j in $G(\mathcal{P})$. Given a finite poset $\mathcal{P}$, we want to characterize the $\mathcal{P}$-Stirling permutations and possibly find their number. In what follows, we will give a complete characterization of these permutations.

We now introduce a specific procedure for determining a walk W in $G(\mathcal{P})$ where its vertices, using some specified rules, give a 2-permutation σ of $\mathcal{P}$. The walk is considered in the associated (undirected) graph of $G(\mathcal{P})$, so we can move forward or backward along edges of $G(\mathcal{P})$. We call this procedure a *$\mathcal{P}$-depth-search*, or $\mathcal{P}$-DS, for short (see [1, 4] for more on depth-first-search). The map $\ell : P \to \{0, 1, 2\}$ gives a *label* to each element in P which counts the number of occurrences of each vertex in the walk as it progresses. Initially, $\ell(p_i) = 0$ $(i = 1, 2, \ldots, n)$ and, when we terminate, $\ell(p_i) = 2$ for each vertex in the walk. We also define a *predecessor function* 'prev' for the vertices as they are visited in the walk.

$\mathcal{P}$-depth-search: Choose some initial vertex p_{i_1} for the walk W as well as for σ, and set $\ell(p_{i_1}) = 1$ and $\text{prev}(p_{i_1}) = p_{i_1}$. For the general step, if u is the last vertex of W determined so far, where $\ell(u) = 1$, then the next vertex v of W is obtained by either a forward-step or a backward-step as follows:

(i) *Forward-step*: Choose a vertex v with $\ell(v) = 0$ such that $u \to v$. Define $\text{prev}(v) = u$ (the predecessor), relabel $\ell(v) = 1$, and add v onto both W and σ;

(ii) *Backward-step*: Let $v = \text{prev}(u)$ and relabel $\ell(u) = 2$. Add v onto W and add u (but not v) onto σ.

Thus, in both types of steps we add to σ the head (terminal end vertex) of the directed edge. We terminate when the (initial) vertex p_{i_1} is met for the second time in W, and we then add p_{i_1} to σ. Associated with such a walk W there is a tree T_W consisting of the vertices and forward-step edges used in W. Note that some of the vertices of $G(\mathcal{P})$ may not be included in T and σ, and that W and σ may have different lengths.

Example 3.1. Let $n = 9$ and consider the poset $\mathcal{P}$ whose Hasse diagram is shown in Fig.1. A $\mathcal{P}$-DS-search may give the following walk W and corresponding 2-permutation σ

$$\begin{aligned} W:&\quad p_1, p_3, p_6, p_9, \mathbf{p_6}, \mathbf{p_3}, p_5, p_8, \mathbf{p_5}, \mathbf{p_3}, \mathbf{p_1};\\ \sigma:&\quad p_1, p_3, p_6, p_9, p_9, p_6, p_5, p_8, p_8, p_5, p_3, p_1.\end{aligned}$$

The vertices obtained in a backward step are indicated in boldface in W. The associated tree T_W is indicated by thick lines in the figure. Note that vertices p_2, p_4, p_7 are not included in W and σ.

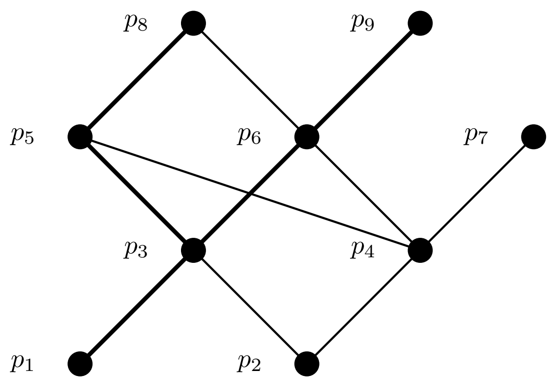

Figure 1: Poset $\mathcal{P}$ and a $\mathcal{P}$-DS-search.

Lemma 3.2. *$\mathcal{P}$-DS terminates and the constructed σ is a 2-permutation of its vertex set, i.e., each vertex v in σ occurs twice. Moreover, each vertex x that occurs between these two v's satisfies $v \to x$, i.e., $v \prec x$, so σ is a Stirling permutation of the corresponding 2-multiset.*

Proof. Consider the general step in the construction of W and σ (as above) and let u be the last vertex so far (in W). Let P_W be the set of vertices in the current W (so no repetition).

> **Claim.** *P_W is the vertex set of a subtree T_W in $G(\mathcal{P})$ with (directed) edges (u, v) associated with each forward-step from u to v. Moreover $\ell(v) \in \{1, 2\}$ for each $v \in P_W$, and the vertices in T_W with $\ell(v) = 1$ is a directed subtree with root p_{i_1}.*
>
> *Proof of Claim.* The first statement follows directly from the fact that we start with a single vertex p_{i_1} and in each forward step a new

vertex v is added to P_W and a new directed edge from an existing vertex u to v. The new vertex is given the label 1. (This is a standard way to construct trees.) In a backward-step from u to its predecessor $v = \text{prev}(u)$, no new vertex is added, so P_W is unchanged, and u is given label 2. The first time we do a backward step we leave a pendant vertex u of the subtree T_W; let T'_W be the subtree obtained by deleting u (and the incident edge). Then T'_W is a directed subtree with root p_{i_1} where each of its vertices has label 1. The next backward-step has the same property: a pendant vertex u' gets the label 2 and the updated subtree T'_W obtained by deleting u' is a directed subtree with root p_{i_1} where each of its vertices has label 1. The second statement of the claim now follows by induction.

The process $\mathcal{P}$-DS has at most $n-1$ forward-steps (as each such step leads to a new vertex not visited before). Thus, when the forward-steps are all done, the remaining steps are backward-steps and gradually the tree T'_W shrinks to the single vertex p_{i_1}. Each vertex in P_W is visited exactly twice (when its ℓ-label is changed from 0 to 1, and later from 1 to 2). This proves the lemma because between the two occurrences of a vertex v in the generated sequence there are only vertices that are reachable from v by a directed path in the tree T_W. □

The Stirling 2-permutation σ as constructed in Lemma 3.2 will be called a *$\mathcal{P}$-DS block*, and we let P_σ denote its vertex set (which equals P_W for the corresponding walk W). We may now repeat this construction, and find a *$\mathcal{P}'$-DS block* in the subposet $\mathcal{P}'$ induced by $P \setminus P_\sigma$. This process may be repeated, for the remaining elements in P, until we have found $\mathcal{P}$-DS blocks such that their vertex sets define an ordered partition of P. The concatenation of these $\mathcal{P}$-DS blocks will be called a *$\mathcal{P}$-DS sequence*. Associated with each of the $\mathcal{P}$-DS blocks is a rooted directed tree, as described above.

We now state and prove a main result in this paper, a characterization of $\mathcal{P}$-Stirling permutations in a general poset $\mathcal{P}$.

Theorem 3.3. *Let $\mathcal{P} = (P, \preceq)$ be a poset, and let σ be a 2-permutation of P. Then σ is a $\mathcal{P}$-Stirling permutation if and only if σ is a $\mathcal{P}$-DS sequence.*

Proof. Consider a $\mathcal{P}$-DS sequence σ. Then each of its $\mathcal{P}$-DS blocks is a 2-permutation of its vertex set, by Lemma 3.2, and it follows that σ is a $\mathcal{P}$-Stirling permutation.

Conversely, let σ be a $\mathcal{P}$-Stirling permutation

$$\sigma : v_1, v_2, v_3, \ldots, v_s.$$

(Thus, these elements are not distinct.)

Claim (Nestedness property). *For each $i, j \leq s$, $i \neq j$, between the two occurrences of v_i in σ the vertex v_j occurs either* 0 *or* 2 *times.*

Proof of Claim. Assume v_j only occurs once between the two occurrences of v_i in σ. Then the other v_j must be before the first v_i or after the second v_i, so their internal order is e.g.

$$v_i \cdots v_j \cdots v_i \cdots v_j.$$

By the Stirling property this gives $v_i \prec v_j$ and also $v_j \prec v_i$ which contradicts the poset property (as $v_i \neq v_j$). The other case, when v_j before the first v_i, is similar. This proves the Claim.

Let k be maximal such that the first k vertices in σ are distinct. Thus

$$v_1 \prec v_2 \prec \cdots \prec v_k$$

and $v_{k+1} = v_i$ for some $i \leq k$. Then $i = k$, i.e., $v_{k+1} = v_k$. This follows from the Nestedness property because if $i < k$, then v_k would occur once between the two occurrences of v_i. Moreover, the internal order in σ of the occurrences of $v_1, v_2, \ldots, v_k$ is as follows:

$$\sigma : v_1, v_2, v_3, \ldots, v_{k-1}, v_k, v_k, \ldots, v_{k-1}, \ldots, v_2, \ldots, v_1, \ldots \tag{2}$$

Thus, the second occurrences of these v_i's are in the opposite order. This is again due to the Nestedness property. Now we connect the structure of σ in (2) to a $\mathcal{P}$-DS sequence. Consider a walk W with vertices

$$v_1, v_2, v_3, \ldots, v_{k-1}, v_k,$$

these are $k-1$ forward-steps. Next, do a backward-step from v_k to v_{k-1}. This gives the following initial part of a $\mathcal{P}$-DS block σ^*

$$\sigma^* : v_1, v_2, v_3, \ldots, v_{k-1}, v_k, v_k$$

which coincides with the initial part of σ. Next, consider the part σ' of σ that is between (the second) v_k and v_{k-1}. There are two possibilities:

Case 1: $\boldsymbol{\sigma'}$ is empty. Then we perform a backward-step from v_{k-1} to v_{k-2}, so v_{k-2} is added to W and v_{k-1} is added to σ^*. Thus σ and σ^* coincide in the next position as well.

Case 2: $\boldsymbol{\sigma'}$ is nonempty. Since σ' is between the two occurrences of v_{k-1}, see (2), the Stirling property means that every vertex v in σ' satisfies $v_{k-1} \prec v$. Moreover, due to the Nestedness property, each such v occurs two times in σ'. Let v' be the first vertex in σ'. Then we perform a forward-step from v_{k-1} to v', so v' is added both to W and σ^*. Thus σ and σ^* coincide in the next position as well.

In both cases we can repeat the argument to a smaller sequence of vertices. In Case 2 we will then construct a subtree with root v_{k-1}. It is clear that by induction the final constructed σ^* equals σ, as desired. Thus, every $\mathcal{P}$-Stirling permutation is also a $\mathcal{P}$-DS sequence, and the proof is complete. $\square$

Example 3.4. Consider again the poset $\mathcal{P}$ whose Hasse diagram is shown in Fig.1. We have already discussed the $\mathcal{P}$-DS-block

$$p_1, p_3, p_6, p_9, p_9, p_6, p_5, p_8, p_8, p_5, p_3, p_1.$$

Another $\mathcal{P}$-DS-block is

$$p_2, p_4, p_7, p_7, p_4, p_2.$$

Concatenating these we get the $\mathcal{P}$-DS sequence and therefore $\mathcal{P}$-Stirling permutation

$$p_1, p_3, p_6, p_9, p_9, p_6, p_5, p_8, p_8, p_5, p_3, p_1, p_2, p_4, p_7, p_7, p_4, p_2.$$

We remark that our characterization Theorem 3.3 of $\mathcal{P}$-Stirling permutations is of a similar nature as the characterization of Stirling permutations via plane trees [7].

4 Stirling permutations for directed graphs

As discussed in Section 3, a poset $\mathcal{P} = (P, \preceq)$ defines a directed graph $G(\mathcal{P})$ with vertex set $P = \{p_1, p_2, \ldots, p_n\}$ and edges $p_i \to p_j$ provided $p_i \prec p_j$. By the transitive law for posets, the directed graph $G(\mathcal{P})$ is transitive: $p_i \to p_j$, $p_j \to p_k$ imply $p_i \to p_k$. Stirling permutations can be defined for any directed graph. The original definition of a Stirling permutation corresponds to the (linearly ordered) directed graph $1 \to 2 \to \cdots \to n$, extended with edges due to transitivity.

Consider an arbitrary directed graph $G = (V, E)$. We can extend our definitions of $\mathcal{P}$-Stirling permutation and weak Stirling permutation as a 2-permutation σ of V in the obvious way:

(I) *G-Stirling permutation*: For $v \in V$, each element $x \neq v$ that occurs between a pair of v's in σ satisfies $v \to x$.

(II) *weak G-Stirling permutation*: For $v \in V$, each element $x \neq v$ that occurs between a pair of v's in σ satisfies $v \to x$ or there is no edge between x and v in either direction.

Example 4.1.

- Consider the directed graph G of order 3 consisting of the 3-cycle $a \to b, b \to c, c \to a$ (so this does not result from a poset). Then the following are G-Stirling permutations:

  $$aabbcc \text{ (6 of these)}; ccabba \text{ (6 of these)}.$$

  In this case, every weak G-Stirling permutation is a G-Stirling permutation.

- Consider the directed graph G with $V = \{x, a, b, c\}$, where only $a \to x,\ b \to x,\ c \to x$. Then e.g. $axxabbcc$ is a G-Stirling permutation and $caxxbabc$ is a weak G-Stirling permutation. □

Let σ be a 2-permutation of V. For $v \in V$ let $\sigma^{(v-v)}$ denote the set of vertices occurring (at least once) between the two occurrences of v in σ. Also, let $\Gamma^+_G(v)$ be the set of vertices w with $v \to w$ in G. Then σ is G-Stirling permutation if and only if

$$\sigma^{(v-v)} \subseteq \Gamma^+_G(v) \quad (v \in V). \tag{3}$$

We call a 2-permutation of V a *trivial* 2*-permutation* provided that the two occurrences of v are consecutive for each $v \in V$, i.e., $\sigma^{(v-v)} = \emptyset$. There are $n!$ trivial 2-permutations (when $n = |V|$), and each of these is clearly a G-Stirling permutation. The following proposition contains some basic properties of G-Stirling permutations.

Proposition 4.2.

(i) *The set of G-Stirling permutations is the set of all trivial* 2*-permutations of V if and only if G has no edges.*

(ii) *If $G = (V, E)$ and $G' = (V, E')$ with $E \subseteq E'$, then every G-Stirling permutation is also a G'-Stirling permutation.*

(iii) *Let $G = (V, E)$ be the complete directed graph on n vertices, i.e., $E = \{(i, j) : i, j \in V, i \neq j\}$. Then the G-Stirling permutations consists of all* 2*-permutations of V.*

(iv) *Let $G = (V, E)$ be a complete bipartite directed graph, i.e., V consists of color classes I and J and all edges (i, j) where $i \in I$ and $j \in J$. Then the G-Stirling permutations consists of all* 2*-permutations σ of V satisfying $\sigma^{(j-j)} = \emptyset$ $(j \in J)$ and $\sigma^{(i-i)} \subseteq J$ $(i \in I)$.*

Proof.

(i) If G has no edge, then, for a G-Stirling permutation σ, $\sigma^{(v-v)} = \emptyset$ for each $v \in V$, so σ is a trivial 2-permutation. If G has an edge, say $v_1 \to v_2$, then $\sigma = v_1 v_2 v_2 v_1 v_3 v_3 \cdots v_n v_n$ is a G-Stirling permutation which is not a trivial-permutation.

(ii) This is immediate from (3).

(iii) When G is complete, $\Gamma_G^+(v) = V \setminus \{v\}$ so then (3) holds for any 2-permutations of V.

(iv) This also follows from (3).

□

Example 4.3. Let T_n be the star with $V = \{1, 2, \dots, n\}$ and edges $n \to 1, n \to 2, \dots, n \to (n-1)$. This is a special complete bipartite graph; see case (iv) in Proposition 4.2. Consider this star with $n = 4$. So we have only $4 \to i$ for $i = 1, 2, 3$. Let σ be a T_n-Stirling permutation. Thus the two occurrences of j have to be together ($j \leq 3$), and some examples of such T_n-Stirling permutations are 41122334, 22411334 and 33411422.

Corollary 4.4. *The number of T_n-Stirling permutations when T_n is the star with n vertices in Example* 4.3 *is* $n!(n-1)/2$.

Proof. Let N be the number to be computed. Let σ be a T_n-Stirling permutation. Then for each $j \leq n-1$ the two occurrences of j in σ must be consecutive. So, N equals $(n-1)!$ times the number of T_n-Stirling permutations with $1, 2, \dots, n-1$ occuring as $1, 1, 2, 2, \dots, n-1, n-1$. We can place the two n's in σ in any of the n positions labeled x in $x, 1, 1, x, 2, 2, \dots, x, n-1, n-1, x$. Thus

$$N = (n-1)!\binom{n}{2} = n!(n-1)/2.$$

as desired. □

For weak T_n-Stirling permutations there are additional possibilities since, for each $j \leq n-1$, the two occurrences of j need not be consecutive.

Proposition 4.5. *The number of weak Stirling permutations for the star T_n equals*

$$(n-1)! \sum_{\substack{a \geq 0,\, b \geq 0,\, c \geq 0, \\ a+b+c = n-1}} \frac{(2a)!(2b)!(2c)!}{a!b!c!}$$

Proof. Now, for each $j \leq n-1$ the two j's must be either before the first n, or between the two n's, or after the second n. Choosing a, b, and c

of them before, between, and after the n's and then taking an arbitrary permutation of both of the integers chosen, we get by direct computation

$$\sum_{\substack{a\geq 0,\, b\geq 0,\, c\geq 0,\\ a+b+c\,=\,n-1}} \frac{(n-1)!}{a!b!c!}(2a)!(2b)!(2c)!$$

as desired. □

Now let T_n^* denote the digraph obtained from T_n be reversing the direction for each edge, so the edges are now $i \to n$ $(i \leq n-1)$. This is also a complete bipartite graph, so we can again apply Proposition 4.2.

Proposition 4.6. *The number of T_n^*-Stirling permutations is*

$$(2n-1)\cdot(n-1)!.$$

Proof. First we note that $1, 2, \ldots, (n-1)$ can be arbitrarily permuted in such a Stirling permutation and we cannot have i, j, i $(1 \leq i, j \leq n-1, i \neq j)$ occurring as a subsequence; so the number is $(n-1)!$ times the number of those in which $1, 2, \ldots, (n-1)$ are in their natural order. The n's have to be together and can be in any of the $(2n-1)$ places in-between $1, 1, 2, 2, \ldots, (n-1), (n-1)$. □

Proposition 4.7. *The number of weak Stirling permutations for the star T_n^* equals*

$$\frac{(2n-1)!}{2^{n-1}}.$$

Proof. Take an arbitrary permutation of $\{1, 1, 2, 2, \ldots, n-1, n-1\}$ and then insert the two n's together in any of the resulting $2n-1$ places. □

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