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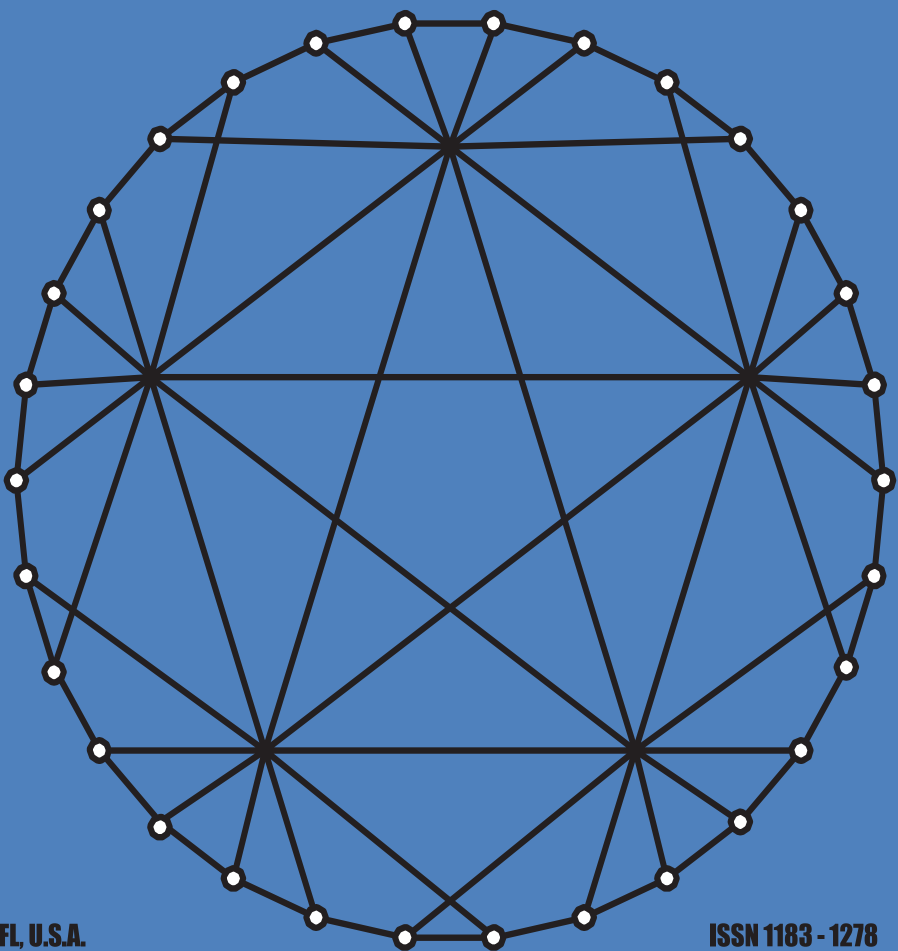

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Proper Hamiltonian-Connected Graphs

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Abstract

An edge coloring of a graph G is proper if every two adjacent edges of G have different colors. A graph G is Hamiltonian-connected if every two vertices of G are connected by a Hamiltonian path. An edge coloring of a Hamiltonian-connected graph G is a proper Hamiltonian-path coloring if every two vertices of G are connected by a properly colored Hamiltonian path. The minimum number of colors in a proper Hamiltonian-path coloring of G is the proper Hamiltonian-connection number of G. Proper Hamiltonian-connection numbers are determined for several classes of Hamiltonian-connected graphs.

Key Words: Hamiltonian-connected graph, proper Hamiltonian-path coloring , proper Hamiltonian-connection number.

AMS Subject Classification: 05C15, 05C45.

1 Introduction

A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G and a graph having a Hamiltonian cycle is a *Hamiltonian graph.* A *Hamiltonian path* in a graph G is a path containing every vertex of G. A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u-v$ path for every pair u, v of distinct vertices of G. For a graph G, let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively, and for a nontrivial graph G, let $\sigma_2(G) = \min\{\deg u + \deg v : uv \notin E(G)\}$ where $\deg w$ is the degree of a vertex w in G. Ore [14] proved the following results in 1963.

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Theorem 1.1. *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$, then G is Hamiltonian-connected.*

Corollary 1.2. *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.*

During 1960-1980, there was a great deal of research activity involving Hamiltonian properties of powers of graphs. For a connected graph G and a positive integer k, the *kth power* G^k of G is that graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \leq d_G(u,v) \leq k$ where $d_G(u,v)$ is the distance between two vertices u and v in G (or the length of a shortest $u-v$ path in G). The graph G^2 is called the *square* of G and G^3 the *cube* of G. In 1960, Sekanina [13] proved the following result.

Theorem 1.3. *If G is a nontrivial connected graph, then the cube of G is Hamiltonian-connected.*

In the 1960s, it was conjectured independently by Nash-Williams [12] and Plummer (see [6, p.139]) that the square of every 2-connected graph is Hamiltonian. In 1974, Fleischner [7] verified this conjecture. Also, in 1974 and using Fleischner's result, Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [4] proved the following.

Theorem 1.4. *If G is a 2-connected graph, then the square of the graph G is Hamiltonian-connected. In particular, the square of every Hamiltonian graph is Hamiltonian-connected.*

A *proper edge coloring* c of a nonempty graph G is a function c on $E(G)$ with the property that $c(e) \neq c(f)$ for every two adjacent edges e and f of G. If the colors are chosen from a set of k colors, then c is called a *k-edge coloring* of G. The minimum positive integer k for which G has a proper k-edge coloring is called the *chromatic index* of G and is denoted by $\chi'(G)$. It is immediate for every nonempty graph G that $\chi'(G) \geq \Delta(G)$. The most important theorem dealing with chromatic index is one obtained by Vizing [15].

Theorem 1.5. (Vizing's Theorem) *For every nonempty graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

As a result of Vizing's theorem, the chromatic index of every nonempty graph G is one of two numbers, namely $\Delta(G)$ or $\Delta(G)+1$.

A *rainbow coloring* of a connected graph G is an edge coloring c of G with the property that for every two vertices u and v of G, there exists a $u-v$ *rainbow path* (no two edges of the path are colored the same). In this case, G is *rainbow-connected* (*with respect to c*). The minimum number of colors needed for a rainbow coloring of G is referred to as the *rainbow connection number* of G. These concepts were introduced and studied by Chartrand, Johns, McKeon and Zhang in 2006. The first paper [5] on this topic was published in 2008. In recent years, this topic has been studied by many and there is a book [10] on rainbow colorings, published in 2012. In 2016, Hamiltonian-connected rainbow colorings were introduced by Chartrand and studied by Bi, Byers and Zhang [1]. An edge coloring of a Hamiltonian-connected graph G is a *Hamiltonian-connected rainbow coloring* if every two vertices of G are connected by a rainbow Hamiltonian path. The minimum number of colors required of a Hamiltonian-connected rainbow coloring of G is the *rainbow Hamiltonian-connection number* of G. Here we study the corresponding concept for proper edge colorings.

Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is *properly colored* or, more simply, P is a *proper path* in G if no two adjacent edges of P are colored the same. An edge coloring c is a *proper-path coloring* of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u-v$ path in G. If k colors are used, then c is referred to as a *proper-path k-coloring.* The minimum k for which G has a proper-path k-coloring is called the *proper connection number* of G. Recently, this topic has been studied by many (see [2, 3] for example). In fact, there is a dynamic survey of this topic due to Li and Magnant [9].

2 Proper Hamiltonian-Path Colorings

If G is a Hamiltonian-connected graph with a proper edge coloring, then for every two vertices u and v of G, there is a proper Hamiltonian $u - v$ path in G. However, if our primary interest concerns edge colorings of graphs G with the property that for every two vertices u and v of G, there exists a proper Hamiltonian $u - v$ path in G, then this may very well be possible using fewer than $\chi'(G)$ colors. Of course, graphs possessing such edge colorings are necessarily Hamiltonian-connected. For a Hamiltonian-connected graph G, an edge coloring $c : E(G) \to [k]$ is a *proper Hamiltonian-path k-coloring* if every two vertices of G are connected by a proper Hamiltonian path in G. An edge coloring c is a *proper Hamiltonian-path coloring* if c is a proper Hamiltonian-path k-coloring for some positive integer k. The minimum number of colors in a proper Hamiltonian-path coloring of G is the *proper Hamiltonian-connection number* of G, denoted by $\mathrm{hpc}(G)$. Since every proper edge coloring of a Hamiltonian-connected graph G is a proper Hamiltonian-path coloring of G and there is no proper Hamiltonian-path 1-coloring of G, it follows that

$$2 \le \mathrm{hpc}(G) \le \chi'(G). \tag{1}$$

To illustrate these concepts, consider the graph $G = C_6^2$. Since $\Delta(G) = 4$ and the edge coloring of G in Figure 1(a) is a proper 4-edge coloring, it follows that $\chi'(G) = \Delta(G) = 4$. Next, consider the 2-edge coloring c of G shown in Figure 1(b).

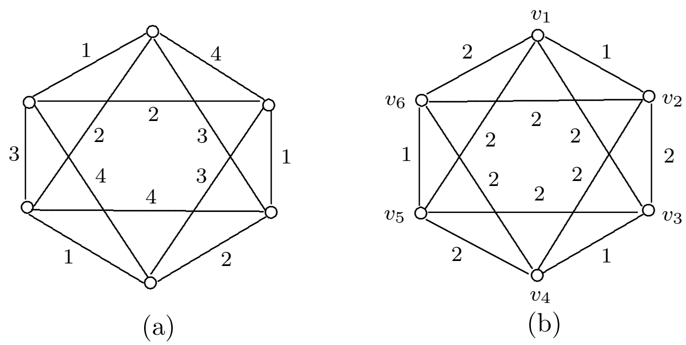


Figure 1: A proper 4-edge coloring and a proper Hamiltonian-path 2-coloring of C_6^2

We show that c is a proper Hamiltonian-path coloring of G; that is, every two vertices u and v of G are connected by a proper Hamiltonian $u - v$ path P in G. If $\{u, v\} = \{v_1, v_2\}$ or $\{u, v\} = \{v_1, v_6\}$, say the former, let $P = (v_1, v_6, v_5, v_4, v_3, v_2)$; if $\{u, v\} = \{v_1, v_3\}$ or $\{u, v\} = \{v_1, v_5\}$, say the former, let $P = (v_1, v_2, v_6, v_5, v_4, v_3)$; while if $\{u, v\} = \{v_1, v_4\}$, let $P = (v_1, v_2, v_6, v_5, v_3, v_4)$. By the symmetry of this edge coloring, c is proper a Hamiltonian-path 2-coloring and so $\text{hpc}(G) = 2$. Thus, $\text{hpc}(G) < \chi'(G)$.

Next, we give an example of a graph G with $\text{hpc}(G) = \chi'(G)$. Let $G = K_3 \,\square\, K_2$, where the two triangles K_3 in G are (u, x, w, u) and (v, y, z, v) and $uv, xy, wz \in E(G)$. Since there is a proper 3-edge coloring of G shown in Figure 2 and $\Delta(G) = 3$, it follows that $\chi'(G) = 3$. Hence, $\text{hpc}(G) \leq 3$.

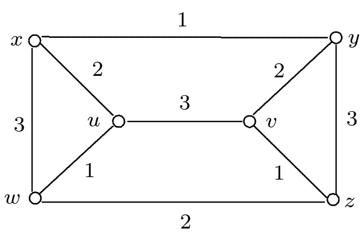


Figure 2: A proper 3-edge coloring of $K_3 \,\square\, K_2$

We now show that $\text{hpc}(G) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring c of G using the colors red (color 1) and blue (color 2). There are only two Hamiltonian $u-v$ paths, namely (u, w, x, y, z, v) and (u, x, w, z, y, v). Because of the symmetry of these paths, we may assume that the first path is a proper Hamiltonian $u-v$ path and whose edges are colored as $c(uw) = c(xy) = c(zv) = 1$ and $c(wx) = c(yz) = 2$. Next, we consider a proper Hamiltonian $x-z$ path. There are only two Hamiltonian $x-z$ paths in G, namely, $Q_1 = (x, w, u, v, y, z)$ and $Q_2 = (x, y, v, u, w, z)$. Since the path $Q = (w, u, v, y)$ lies on both Q_1 and Q_2, it follows that Q must be proper. This implies that $c(uv) = 2$ and $c(vy) = 1$. Similarly, there are only two Hamiltonian $w-y$ paths in G, each of which contains the path (x, u, v, z), and so this path must be proper. This implies that $c(ux) = 1$. We now consider a proper Hamiltonian $x-v$ path. There are only two Hamiltonian $x-v$ paths in G, namely, $R_1 = (x, u, w, z, y, v)$ and $R_2 = (x, y, z, w, u, v)$. Since the path $R = (y, z, w, u)$ lies on both R_1 and R_2, it follows that R must be properly colored by the colors 1 and 2. Since $c(yz) = 2$ and $c(wu) = 1$, this is impossible. Thus, there is no proper Hamiltonian $x-v$ path in G, which is a contradiction. Therefore, $\text{hpc}(G) \geq 3$ and so $\text{hpc}(G) = 3$.

We now consider some well-known Hamiltonian-connected graphs, beginning with complete graphs, which are supergraphs of all Hamiltonian-connected graphs. It is easy to see that $\text{hpc}(K_3) = 3$. When $n \geq 4$, $\text{hpc}(K_n) = 2$, however, which we verify next.

Theorem 2.1. *For every integer $n \geq 4$,* $\text{hpc}(K_n) = 2$.

Proof. We consider two cases, according to whether n is even or n is odd.

Case 1. *n is even.* The complete graph $G = K_n$ contains a 1-factor F. Define an edge coloring c of G by assigning the color red to each edge of F and the color blue to the remaining edges of G. We show that c is a proper Hamiltonian-path 2-coloring of G; that is, for every two vertices u and v of G, there is a proper Hamiltonian $u-v$ path in G. Let $n = 2k$ and let $V(G) = \{v_1, v_2, \ldots, v_{2k}\}$. Suppose that $E(F) = \{v_{2i-1}v_{2i} : 1 \leq i \leq k\}$. There are two possibilities, depending on whether uv is a blue edge or uv is a red edge. Thus, we may assume that either (1) $u = v_1$ and $v = v_{2k}$ or (2) $u = v_2$ and $v = v_1$. Consider the properly colored Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{2k}, v_1)$ of G. If (1) occurs, then $(u = v_1, v_2, \ldots, v_{2k} = v)$ is a proper Hamiltonian $u-v$ path in G; while if (2) occurs, then $(u = v_2, v_3, \ldots, v_{2k}, v_1 = v)$ is a proper Hamiltonian $u-v$ path in G. Therefore, $\text{hpc}(K_n) = 2$.

Case 2. *$n \geq 5$ is odd.* Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in $G = K_n$. Define a coloring c of G by assigning the color red to each edge of C and the color blue to the remaining edges of G. We show that c is a proper Hamiltonian-path 2-coloring of G; that is, for every two vertices u and v of G, there is a proper Hamiltonian $u-v$ path in G. We may assume that $v = v_n$ and $u = v_i$ for some integer i with $1 \le i \le (n-1)/2$.

First, suppose that $u = v_1$. If $n \equiv 1 \pmod 4$, then

$$(u = v_1, v_2, v_4, v_3, v_5, v_6, v_8, v_7, v_9, \ldots, v_{n-3}, v_{n-1}, v_{n-2}, v_n = v)$$

is a proper Hamiltonian $u-v$ path in G; while if $n \equiv 3 \pmod 4$, then

$$(u = v_1, v_2, v_4, v_3, v_5, v_6, v_8, v_7, v_9, \ldots, v_{n-5}, v_{n-3}, v_{n-4}, v_{n-1}, v_{n-2}, v_n = v)$$

is a proper Hamiltonian $u-v$ path in G.

Next, suppose that $u = v_j$ where $2 \le j \le (n-1)/2$. If $n = 5$, then $u = v_2$ and $(v_5, v_3, v_4, v_1, v_2)$ is a proper Hamiltonian $u-v$ path in G. Thus, we may assume that $n \ge 7$ is odd. Let $A = \{v_1, v_2, \ldots, v_{j-1}\}$ and $B = \{v_{j+1}, v_{j+2}, \ldots, v_{n-1}\}$. Let $|A| = a$ and $|B| = b$. Since $n \ge 7$ is odd, it follows that (1) $b \ge 3$ and (2) $a + b = n - 2$ is odd and so a and b are of opposite parity. We consider two subcases, according to whether a is even or a is odd.

Subcase 2.1. *a is even.* Then

$$Q = (u = v_j, v_{j-2}, v_{j-1}, v_{j-4}, u_{j-3}, v_{j-6}, v_{j-5}, \ldots, v_1, v_2, v_{j+2})$$

is a proper $u - v_{j+2}$ path in G with $V(Q) = \{v_1, v_2, \ldots, v_j\} \cup \{v_{j+2}\}$ and

$$\begin{aligned} Q' \quad = \quad & (u = v_{j+2}, v_{j+1}, v_{j+4}, v_{j+3}, u_{j+6}, v_{j+5}, v_{j+8}, u_{j+7}, \ldots, \\ & v_{n-2}, v_{n-3}, v_{n-1}, v_n = v) \end{aligned}$$

is a proper $v_{j+2} - v$ path in G with $V(Q') = \{v_{j+1}, v_{j+2}, \ldots, v_n\}$. Thus, $V(Q) \cup V(Q') = V(G)$, $V(Q) \cap V(Q') = \{v_{j+1}\}$ and $v_2 v_{j+2}$ and $v_{j+1} v_{j+2}$ have distinct colors (namely, $v_2 v_{j+2}$ is blue and $v_{j+1} v_{j+2}$ is red). Therefore, the path Q followed by Q' produces a proper Hamiltonian $u - v$ path in G.

Subcase 2.2. *a is odd.* If $a \equiv 3 \pmod 4$, then

$$Q = (u = v_j, v_{j-1}, v_{j-3}, v_{j-2}, u_{j-4}, v_{j-5}, v_{j-7}, u_{j-6}, \ldots, v_1, v_2, v_{j+1})$$

is a proper $u - v_{j+1}$ path in G; while if $a \equiv 1 \pmod 4$, then

$$Q = (u = v_j, v_{j-1}, v_{j-3}, v_{j-2}, u_{j-4}, v_{j-5}, v_{j-7}, u_{j-6}, \ldots, v_3, v_4, v_1, v_2, v_{j+1})$$

is a proper $u - v_{j+1}$ path in G. We now show that Q can be extended to a proper Hamiltonian $u - v$ path in G. If $b \equiv 0 \pmod 4$, then

$$\begin{aligned} Q' \quad = \quad & (v_{j+1}, v_{j+2}, v_{j+4}, v_{j+3}, u_{j+5}, v_{j+6}, v_{j+8}, u_{j+7}, \ldots, \\ & v_{n-3}, v_{n-1}, v_{n-2}, v_n = v) \end{aligned}$$

is a proper $v_{j+1} - v$ path in G; while if $b \equiv 2 \pmod 4$, then $b \geq 6$ (since $b \geq 3$) and

$$\begin{aligned} Q' &= (v_{j+1}, v_{j+2}, v_{j+4}, v_{j+3}, u_{j+5}, v_{j+6}, v_{j+8}, u_{j+7}, \ldots, \\ &\quad v_{n-4}, v_{n-1}, v_{n-2}, v_n = v) \end{aligned}$$

is a proper $v_{j+1} - v$ path in G. Thus, as in Case 1, the path Q followed by Q' produces a proper Hamiltonian $u - v$ path in G. $\square$

We saw that if G is a Hamiltonian-connected graph of order at least 4, then $\delta(G) \geq 3$. There are infinitely many Hamiltonian-connected cubic graphs. For each odd integer $n \geq 3$, the prism $C_n \,\square\, K_2$ is cubic and Hamiltonian-connected (see [8]). We saw that $\mathrm{hpc}(C_3 \,\square\, K_2) = 3$. In fact, $\mathrm{hpc}(C_n \,\square\, K_2) = 3$ for all odd integers $n \geq 3$.

Theorem 2.2. *For each odd integer $n \geq 3$,* $\mathrm{hpc}(C_n \,\square\, K_2) = 3$.

Proof. For an odd integer $n \geq 3$, let $G = C_n \,\square\, K_2$, which is constructed from the two n-cycles $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ by adding the n edges $u_i v_i$ for $1 \leq i \leq n$. Since $\chi'(G) = 3$, it follows by (1) that $\mathrm{hpc}(G) \leq 3$. It remains to show that $\mathrm{hpc}(G) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring c of G using the colors 1 and 2.

First, consider a proper Hamiltonian $u_1 - u_3$ path P in G. Observe that either P begins with u_1, u_2 or P ends with u_2, u_3. Suppose first that P begins with u_1, u_2. Hence, P must begin with u_1, u_2, v_2 and so $u_1u_n, u_1v_1 \notin E(P)$. Since each vertex in $V(G) - \{u_1, u_3\}$ has degree 2 in P, it follows that $v_1v_n, v_1v_2 \in E(P)$ and so P begins with the subpath $(u_1, u_2, v_2, v_1, v_n)$. Since $u_nu_1 \notin E(P)$ and u_n has degree 2 in P, it follows that $u_nv_n, u_nu_{n-1} \in E(P)$ and so P contains the subpath $(u_1, u_2, v_2, v_1, v_n, u_n, u_{n-1})$. Similarly, $v_nv_{n-1} \notin E(P)$ and $u_{n-1}v_{n-1}, v_{n-1}v_{n-2} \in E(P)$. Continuing in this way, we see that P is the following path

$$P_1 = (u_1, u_2, v_2, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_4, v_4, v_3, u_3). \quad (2)$$

Next, suppose that P ends with u_2, u_3. This implies that $u_1u_2, u_3v_3, u_3u_4 \notin E(P)$ and so $u_2v_2, v_2v_3, v_3v_4 \in E(P)$. Hence, P ends at the subpath $(v_4, v_3, v_2, u_2, u_3)$. An argument similar to the one above shows that P is the following path

$$P_2 = (u_1, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_4, v_4, v_3, v_2, u_2, u_3).$$

In either case, P must contain the subpath

$$P' = (v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_4, v_4, v_3).$$

By the symmetry of the graph G, we may assume, without loss of generality, that $P = P_1$, described in (2). Since c is a proper Hamiltonian-path 2-coloring of G using the colors 1 and 2, we may assume, without loss of generality, that $c(u_1u_2) = 1$. Since P_1 is a proper path and $c(u_1u_2) = 1$, it follows that $c(u_2v_2) = 2$ and $c(v_1v_2) = 1$. For the remaining edges e of P_1, it follows that $c(e) = 1$ if $e = u_iv_i$ and $c(e) = 2$ if e belongs to one of the two n-cycles. In particular, $c(v_1v_n) = 2$. Next, consider a proper Hamiltonian $u_3 - u_5$ path Q in G. An argument above shows that there are two possibilities for Q. Furthermore, Q must contain the subpath

$$Q' = (v_3, v_2, u_2, u_1, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_6, v_6, v_5).$$

Since Q' is proper and $c(u_2v_2) = 2$, it follows that $c(v_3v_2) = 1$ and so the colors of Q' are alternately colored by 1 and 2, beginning with 1. In particular, $c(v_1v_n) = 1$, which contradicts the fact that $c(v_1v_n) = 2$. $\square$

3 Minimum Hamiltonian-Connected Graphs

We start this section with a useful observation.

Observation 3.1. *If H is a Hamiltonian-connected spanning subgraph of a graph G, then* $\mathrm{hpc}(G) \le \mathrm{hpc}(H)$.

If G is a Hamiltonian-connected graph that is not complete and u and v are nonadjacent vertices of G, then $G + uv$ is also Hamiltonian-connected and $\mathrm{hpc}(G+uv) \le \mathrm{hpc}(G)$ by Observation 3.1. This suggests that Hamiltonian-connected graphs having the greatest proper Hamiltonian connection numbers are minimal Hamiltonian-connected graphs. This leads us to consider Hamiltonian-connected graphs of order n and minimum size. Every Hamiltonian-connected graph of order at least 4 is 3-connected. Therefore, if G is a Hamiltonian-connected graph of $n \ge 4$, then $\delta(G) \ge 3$, which implies that the minimum possible size of a Hamiltonian-connected graph of order n is $\lfloor \frac{3n+1}{2} \rfloor$. The following result is due to Moon [11].

Theorem 3.2. *For each integer $n \ge 4$, there exists a Hamiltonian-connected graph of order n and size $\lfloor \frac{3n+1}{2} \rfloor$.*

We now determine the proper Hamiltonian connection numbers of graphs belonging to two classes of Hamiltonian-connected graphs of order n and size $\lfloor \frac{3n+1}{2} \rfloor$, one class for n even and the other class for n odd, beginning with the case when n is even.

For each integer $k \ge 2$, let $P_k \square K_2$ be the grid of order $2k$ in which two paths of order k are $P_k = (x_1, x_2, \ldots, x_k)$ and $P'_k = (y_1, y_2, \ldots, y_k)$ such that $x_iy_i \in E(P_k \square K_2)$ for $1 \le i \le k$. Now, let H_k be the cubic graph of order $2k + 2$ obtained by adding two adjacent vertices u and v to the grid $P_k \,\square\, K_2$ and joining (1) the vertex u to x_1 and y_1 and (2) the vertex v to x_k and y_k in $P_k \,\square\, K_2$. Each graph H_k has the property that it is Hamiltonian-connected (see [11]) and $\mathrm{hpc}(H_k) = \chi'(H_k) = \Delta(H_k) = 3$. We verify this now.

Theorem 3.3. *For each integer $k \geq 2$, $\mathrm{hpc}(H_k) = 3$.*

Proof. Let $C = (u, x_1, x_2, \ldots, x_k, v, y_k, y_{k-1}, \ldots, y_3, y_2, y_1, u)$ be a Hamiltonian cycle of H_k. Define a proper 3-edge coloring of H_k by alternately assigning the colors 1 and 3 to the edges of C and assigning the color 2 to the remaining edges of H_k. Thus, $\mathrm{hpc}(H_k) \leq \chi'(H_k) = 3$. Figure 3(a) shows this edge coloring for the case when k is odd and Figure 3(b) shows this edge coloring for the case when k is even.

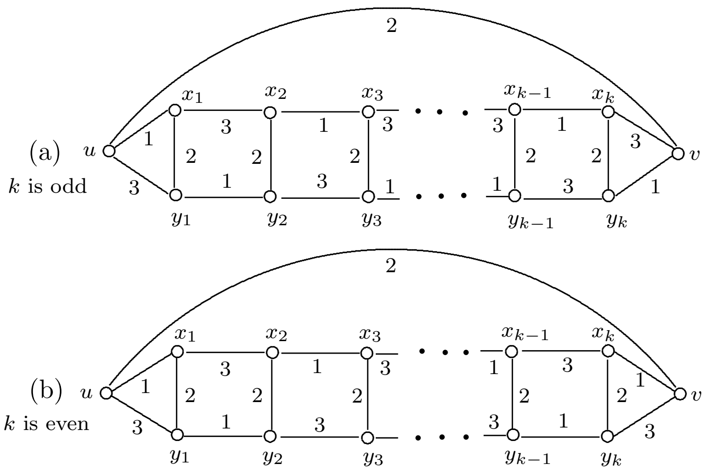


Figure 3: Edge colorings of H_k

It therefore remains to show that $\mathrm{hpc}(H_k) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring c of H_k using the colors 1 and 2. First, consider a proper Hamiltonian $u - v$ path. There are only two Hamiltonian $u - v$ paths in G. Because of the symmetry of these paths, we consider the path $(u, x_1, y_1, y_2, x_2, x_3, y_3, \ldots, x_k, y_k, v)$ if k is odd and $(u, x_1, y_1, y_2, x_2, x_3, y_3, \ldots, y_k, x_k, v)$ if k is even. Choosing $c(ux_1) = 1$, the colors of the remaining edges on the path are determined as shown in Figure 4 when k is odd.

Next, consider a proper Hamiltonian $u - x_2$ path P in H_k. If P begins with u, y_1, then P cannot contain x_1, which is impossible. Suppose that P begins with u, v. Then P must end as $x_3, y_3, y_2, y_1, x_1, x_2$. Since $c(x_3y_3) = 2$, it follows that $c(y_2y_3) = 1$, which is impossible as $c(y_1y_2) = 1$. Hence, P must

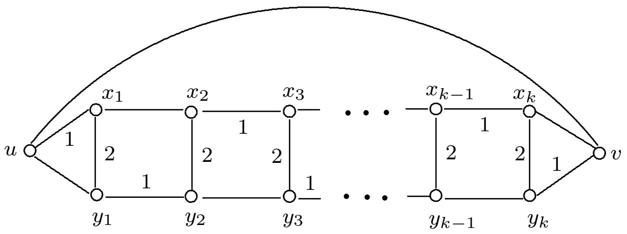


Figure 4: A step in the proof of Proposition 3.3 when k is odd

begin with u, x_1 and so

$$P = (u, x_1, y_1, y_2, \ldots, y_k, v, x_k, x_{k-1}, \ldots, x_2).$$

Furthermore,

$$\text{the edges of } P \text{ are alternately colored 1 and 2.} \tag{3}$$

We now consider the Hamiltonian $x_1 - x_2$ paths in G. There are only two Hamiltonian $x_1 - x_2$ paths Q and Q' in G, where

$$Q = (x_1, u, y_1, y_2, \ldots, y_k, v, x_k, x_{k-1}, \ldots, x_2)$$

and

$$Q' = \begin{cases} (x_1, y_1, u, v, y_k, x_k, x_{k-1}, y_{k-1}, \ldots, y_2, x_2) & \text{if } k \text{ is even} \\ (x_1, y_1, u, v, x_k, y_k, y_{k-1}, x_{k-1}, \ldots, y_2, x_2) & \text{if } k \text{ is odd.} \end{cases}$$

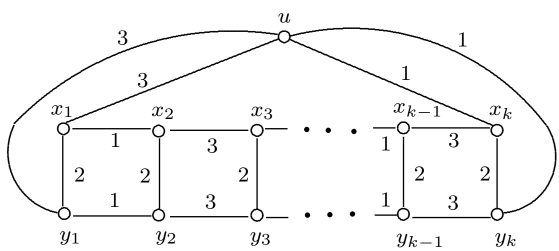
If $c(uy_1) = 1$, then Q is not proper and so Q' must be proper. However then, $c(x_i x_{i+1}) = 1$ for each integer i with $1 \le i \le k-1$, which contradicts (3). Hence, the edges of the Hamiltonian $x_1 - x_2$ path Q are alternately colored 1 and 2, beginning and ending with 1. Now, consider a Hamiltonian $u - y_2$ path Q. Proceeding as above with the path P, we see that Q must contain x_1y_1, x_1x_2, x_2x_3 as consecutive edges on Q. Since $c(x_1y_1) = 2$, it follows that $c(x_1x_2) = 1$. However, $c(x_2x_3) = 1$, which is impossible. Thus, no such proper Hamiltonian $u - y_2$ path exists. Therefore, $\text{hpc}(H_k) \ge 3$ and so $\text{hpc}(H_k) = 3$. $\square$

For each integer $k \ge 3$, recall that $P_k \square K_2$ is the grid of order $2k$ in which two paths of order k are $P_k = (x_1, x_2, \ldots, x_k)$ and $P'_k = (y_1, y_2, \ldots, y_k)$ such that $x_i y_i \in E(P_k \square K_2)$ for $1 \le i \le k$. The graph F_k of order $2k+1$ is constructed from $P_k \square K_2$ by adding a new vertex u and joining u to each vertex in $\{x_1, x_k, y_1, y_k\}$. Thus, F_k has $2k$ vertices of degree 3 and one vertex of degree 4. It is known [11] that F_k is a Hamiltonian-connected graph of odd order and has the minimum size of a Hamiltonian-connected graph of order $2k+1$ for each integer $k \ge 3$. Furthermore, $\chi'(F_k) = \Delta(F_k) = 4$. We show that $\text{hpc}(F_k) = 3$.

Theorem 3.4. *For each integer $k \geq 3$,* $\mathrm{hpc}(F_k) = 3$.

Proof. For each integer $k \geq 3$, let $P_k \,\square\, K_2$ be the grid of order $2k$ in which two paths of order k are $P_k = (x_1, x_2, \ldots, x_k)$ and $P'_k = (y_1, y_2, \ldots, y_k)$ such that $x_i y_i \in E(P_k \,\square\, K_2)$ for $1 \leq i \leq k$. The graph F_k of order $2k+1$ is constructed from $P_k \,\square\, K_2$ by adding a new vertex u and joining u to each vertex in $\{x_1, x_k, y_1, y_k\}$. Define an edge coloring $c : E(F_k) \to \{1, 2, 3\}$ of F_k by alternately assigning the colors 1 and 3 to the edges of P_k and P'_k beginning with 1 and assigning the color 2 to the remaining edges of $P_k \,\square\, K_2$. Furthermore, if $k \geq 3$ is odd, then let $c(ux_1) = c(uy_1) = 3$ and $c(ux_k) = c(uy_k) = 1$ and if $k \geq 4$ is even, then let $c(ux_1) = c(uy_1) = 3$ and $c(ux_k) = c(uy_k) = 2$. Figure 5(a) shows this edge coloring for the case when k is odd and Figure 5(b) shows this edge coloring for the case when k is even.

(a)
k is odd



(b)
k is even

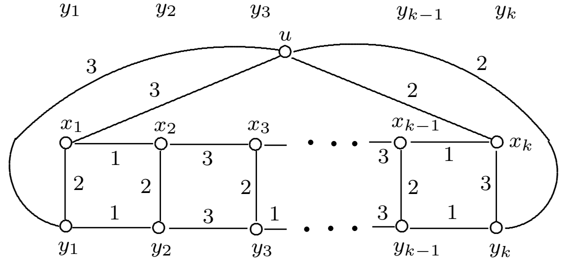


Figure 5: Edge colorings of F_k

Next, we show that the 3-edge coloring of F_k described in Figure 5 is a proper Hamiltonian-path 3-coloring of F_k; that is, we show that F_k contains a proper Hamiltonian $w - z$ path for each pair w, z of distinct vertices of F_k. First, observe that every Hamiltonian path P of F_k is proper unless P contains both ux_1 and uy_1 or contains both ux_k and uy_k. Hence, if either w or z is u, then F_k contains a proper Hamiltonian $w - z$ path with initial vertex u. Therefore, we may assume that neither w nor z is u. We consider the following cases.

Case 1. *$\{w, z\} = \{x_i, x_j\}$ or $\{w, z\} = \{y_i, y_j\}$, where $i < j$, say the former.*

If i is even, then consider the $x_i - u$ path

$$\begin{aligned} P' &= (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, \\ &\quad y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, y_1, x_1, u); \end{aligned}$$

while if i is odd, then consider the $x_i - u$ path

$$\begin{aligned} P' &= (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, \\ &\quad y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, x_1, y_1, u). \end{aligned}$$

Next, if $k - j$ is even, then consider the $u - x_j$ path

$$P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, \ldots, y_j, x_j);$$

while if $k - j$ is odd, then consider the $u - x_j$ path

$$P'' = (u, x_k, y_k, y_{k-1}, x_{k-1}, x_{k-2}, y_{k-2}, \ldots, y_j, x_j).$$

Then, P' followed by P'' is a proper Hamiltonian $x_i - x_j$ path.

Case 2. $\{w, z\} = \{x_i, y_j\}$. We may assume that $i \leq j$. There are two subcases.

Subcase 2.1. $i = j$. If i is even, then consider the $x_i - u$ path

$$P' = (x_i, x_{i-1}, y_{i-1}, y_{i-2}, x_{i-2}, x_{i-3}, \ldots, x_1, y_1, u);$$

while if i is odd, then consider the $x_i - u$ path

$$P' = (x_i, x_{i-1}, y_{i-1}, y_{i-2}, x_{i-2}, x_{i-3}, \ldots, y_1, x_1, u).$$

Next, if $k - i$ is even, then consider the $u - y_i$ path

$$P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, \ldots, x_{i+1}, y_{i+1}, y_i);$$

while if $k - i$ is odd, then consider the $u - y_i$ path

$$P'' = (u, x_k, y_k, y_{k-1}, x_{k-1}, x_{k-2}, \ldots, x_{i+1}, y_{i+1}, y_i).$$

Then, P' followed by P'' is a proper Hamiltonian $x_i - y_i$ path.

Subcase 2.2. $i < j$. If i is even, then consider the $x_i - u$ path

$$\begin{aligned} P' &= (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, \\ &\quad y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, y_1, x_1, u); \end{aligned}$$

while if i is odd, then consider the $x_i - u$ path

$$\begin{aligned} P' &= (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, \\ &\quad y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, x_1, y_1, u). \end{aligned}$$

If $k - j$ is even, then consider the $u - y_j$ path

$$P'' = (u, x_k, y_k, y_{k-1}, x_{k-1}, x_{k-2}, y_{k-2}, \ldots, x_j, y_j);$$

while if $k - j$ is odd, then consider the $u - y_j$ path

$$P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, x_{k-2}, \ldots, x_j, y_j).$$

Then, P' followed by P'' is a proper Hamiltonian $x_i - y_j$ path.

It therefore remains to show that $\mathrm{hpc}(F_k) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring c of F_k using the colors 1 and 2. First, consider a proper Hamiltonian $u - v$ path. We consider two cases, according to whether k is odd or k is even.

Case 1. *$k \geq 3$ is odd.* Let $k = 2t + 1$ for some positive integer t. First, consider the vertices x_{t+1} and u. Let P be a proper Hamiltonian $x_{t+1} - u$ path in F_k. First, observe that P cannot start with x_{t+1}, y_{t+1}. Thus, either P starts with x_{t+1}, x_t or starts with x_{t+1}, x_{t+2}. Suppose, without loss of generality, that P starts with x_{t+1}, x_t. Since $x_{t+1}x_{t+2}, x_{t+1}y_{t+1} \notin E(P)$ and y_{t+1} and x_{t+2} have degree 2 on P, it follows that

$$(y_t, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}) \text{ is a subpath of } P. \tag{4}$$

If $t \geq 2$, then $x_t y_t \notin E(P)$ (for otherwise, y_{t-1} cannot belong to P). Similarly, $x_i y_i \notin E(P)$ for $2 \leq i \leq t$. Hence, P contains the subpath $(x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2})$. By (4), if t is odd, then

$$P = (x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}, y_{t+3}, \ldots, y_k, x_k, u);$$

while if t is even, then

$$P = (x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}, y_{t+3}, \ldots, x_k, y_k, u).$$

Since c is a proper Hamiltonian-path 2-coloring of F_k using the colors 1 and 2, we may assume that P is alternately colored 1 and 2, beginning with 1 and ending with 2. Thus, the colors of some edges of $P_k \,\square\, K_2$ are determined. This is shown for $k \in \{5, 7\}$ in Figure 6 where each bold edge belongs to the path P. In particular, $\{c(y_1y_2), c(x_2x_3)\} = \{c(x_{t+1}x_t), c(x_{t+2}x_{t+3})\} = \{1, 2\}$.

Next, consider the vertices x_1 and u. Let Q be a proper Hamiltonian $x_1 - u$ path in F_k. Since Q cannot begin with x_1, u, exactly one of x_1x_2 and x_1y_1 is an edge of Q. We consider these two subcases.

Subcase 1.1. *$x_1x_2 \in E(Q)$ and $x_1y_1 \notin E(Q)$.* Then

$$Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_1, u).$$

Since $c(x_tx_{t+1}) = 1$, it follows that $c(x_{t+1}x_{t+2}) = 2$ and $c(x_{t+2}x_{t+3}) = 1$, which is a contradiction.

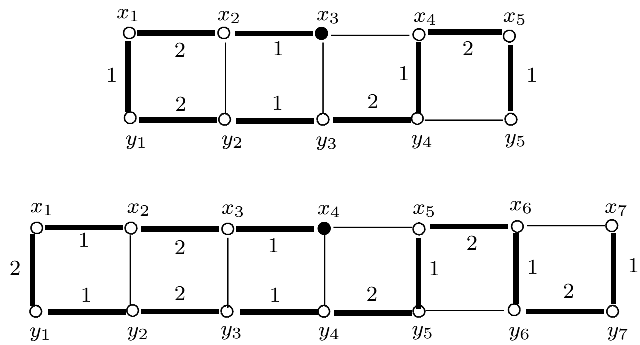


Figure 6: The colors of some edges of $P_k \,\square\, K_2$ in Case 1 for $k \in \{5, 7\}$

Subcase 1.2. $x_1x_2 \notin E(Q)$ *and* $x_1y_1 \in E(Q)$. Here,

$$Q = (x_1, y_1, y_2, x_2, x_3, \ldots, y_{k-2}, y_{k-1}, x_{k-1}, x_k, y_k, u).$$

Since $\{c(y_1y_2), c(x_2x_3)\} = \{1, 2\}$, there is no color for y_2x_2 and so Q is not proper.

Case 2. $k \geq 4$ *is even.* Let $k = 2t$ for some integer $t \geq 2$. First, consider the vertices x_t and u. Let P be a proper Hamiltonian $x_t - u$ path in F_k. As in Case 1, the path P cannot start with x_t, y_t. Thus, either P starts with x_t, x_{t-1} or x_t, x_{t+1}. We consider these two subcases.

Subcase 2.1. P *starts with* x_t, x_{t-1}. Since y_t and x_{t+1} have degree 2 in P, it follows that

$$(y_{t-1}, y_t, y_{t+1}, x_{t+1}, x_{t+2}) \text{ is a subpath of } P. \tag{5}$$

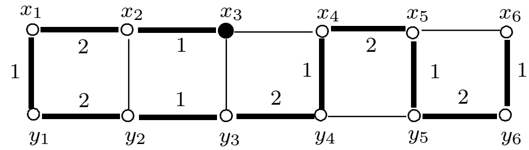
If $t \geq 3$, then $x_{t-1}y_{t-1} \notin E(P)$ (for otherwise, y_{t-2} cannot belong to P). Hence, P begins with the subpath $(x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t)$. Because of (5), if $t \geq 3$ is odd, then

$$P = (x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t, y_{t+1}, x_{t+1}, x_{t+2}, \ldots, y_{k-1}, y_k, x_k, u);$$

while if $t \geq 2$ is even, then

$$P = (x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t, y_{t+1}, x_{t+1}, x_{t+2}, \ldots, x_{k-1}, x_k, y_k, u).$$

Since c is a proper Hamiltonian-path 2-coloring of F_k using the colors 1 and 2, we may assume that P is alternately colored 1 and 2, beginning with 1 which is shown in Figure 7. In particular, $c(x_{t-1}x_t) = 1$ and $c(x_{t+1}x_{t+2}) = 2$ whether t is odd or even.



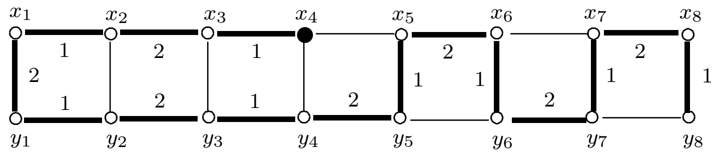


Figure 7: The colors of some edges of $P_k \,\square\, K_2$ in Subcase 2.1 for $k \in \{6, 8\}$

Next, consider the vertices x_1 and u. Let Q be a proper Hamiltonian $x_1 - u$ path in F_k. Since Q cannot begin with x_1, u, exactly one of x_1x_2 and x_1y_1 is an edge of Q.

- ⋆ First, suppose that x_1x_2 is an edge of Q and x_1y_1 is not an edge of Q. Since each of x_2 and y_1 has degree 2 in Q, it follows that Q starts with (x_1, x_2, x_3) and ends at (y_2, y_1, u). This forces that Q is the following path

$$Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_2, y_1, u).$$

  Since $c(x_{t-1}x_t) = 1$ and $c(x_{t+1}x_{t+2}) = 2$, regardless of the color of x_tx_{t+1}, it follows that Q is not proper.

- ⋆ Next, suppose that x_1y_1 is an edge of Q and x_1x_2 is not an edge of Q. Since each of x_2 and y_1 has degree 2 in Q, it follows that Q must start with $(x_1, y_1, y_2, x_2, x_3)$. This forces that Q is the following path

$$Q = (x_1, y_1, y_2, x_2, x_3, y_3, y_4, \ldots, x_{k-2}, x_{k-1}, y_{k-1}, y_k, x_k, u).$$

  Since $\{c(y_1y_2), c(x_2x_3)\} = \{1, 2\}$ (see Figure 7), regardless of the color of x_2y_2, it follows that Q is not proper.

Subcase 2.2. *P starts with x_t, x_{t+1}.* Since $x_tx_{t-1}, x_ty_t \notin E(P)$, it follows that $(x_{t-2}, x_{t-1}, y_{t-1}, y_t, y_{t+1})$ is a subpath of P. Thus, if $t \geq 3$ is odd, then

$$P = (x_t, x_{t+1}, \ldots, x_k, y_k, y_{k-1}, \ldots, y_t, y_{t-1}, x_{t-1}, x_{t-2}, \ldots, x_2, x_1, y_1, u)$$

and if $t \geq 2$ is even, then

$$P = (x_t, x_{t+1}, \ldots, x_k, y_k, y_{k-1}, \ldots, y_t, y_{t-1}, x_{t-1}, x_{t-2}, \ldots, y_2, y_1, x_1, u).$$

Since c is a proper Hamiltonian-path 2-coloring of F_k using the colors 1 and 2, we may assume that P is alternately colored 1 and 2, beginning with 1 which is shown in Figure 8.

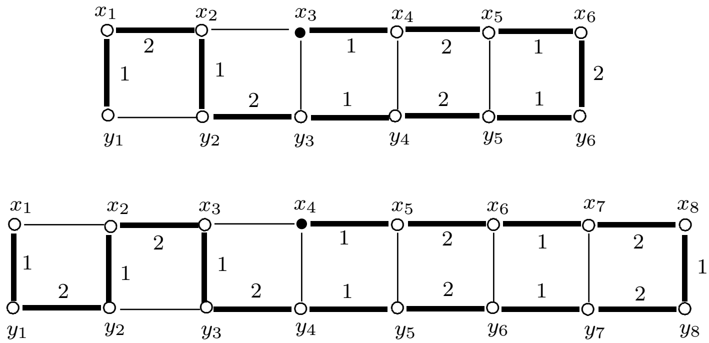


Figure 8: The colors of some edges of $P_k \,\square\, K_2$ in Subcase 2.2

Next, consider the vertices x_1 and u. Let Q be a proper Hamiltonian $x_1 - u$ path in F_k. Since Q cannot begin with x_1, u, exactly one of x_1x_2 and x_1y_1 is an edge of Q.

- ⋆ First, suppose that x_1x_2 is an edge of Q and x_1y_1 is not an edge of Q. Since y_1 has degree 2 in Q, it follows that Q ends at (y_2, y_1, u). Furthermore, $x_2y_2 \notin E(Q)$ and so $x_2x_3, y_2y_3 \in E(Q)$. This forces that Q is the following path

$$Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_2, y_1, u).$$

  Since $\{c(x_tx_{t+1}), c(x_{t-2}x_{t-1})\} = \{1, 2\}$, there is no color for $x_{t-1}x_t$ and so Q is not proper.

- ⋆ Next, suppose that x_1y_1 is an edge of Q and x_1x_2 is not an edge of Q. Since each of x_2 and y_1 has degree 2 in Q and $y_1u \notin E(Q)$, it follows that

$$Q = (x_1, y_1, y_2, x_2, x_3, y_3, \ldots, y_{t-1}, y_t, x_t, x_{t+1}, \ldots, y_k, x_k, u).$$

  Since $\{c(y_{t-1}y_t), c(x_tx_{t+1})\} = \{1, 2\}$, there there is no color for $c(x_ty_t)$ and so Q is not proper. □

It has been shown in [3] that if G is a 2-connected graph, then the proper connection number of G is at most 3. Since every Hamiltonian-connected graph G of order at least 4 is 2-connected (in fact, 3-connected), $\mathrm{pc}(G) \le 3$. We have seen no Hamiltonian-connected graph G where $\mathrm{hpc}(G) > 3$, which leads to the following cojecture.

Conjecture 3.5. *If G is a Hamiltonian-connected graph, then* $\mathrm{hpc}(G) \le 3$.

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