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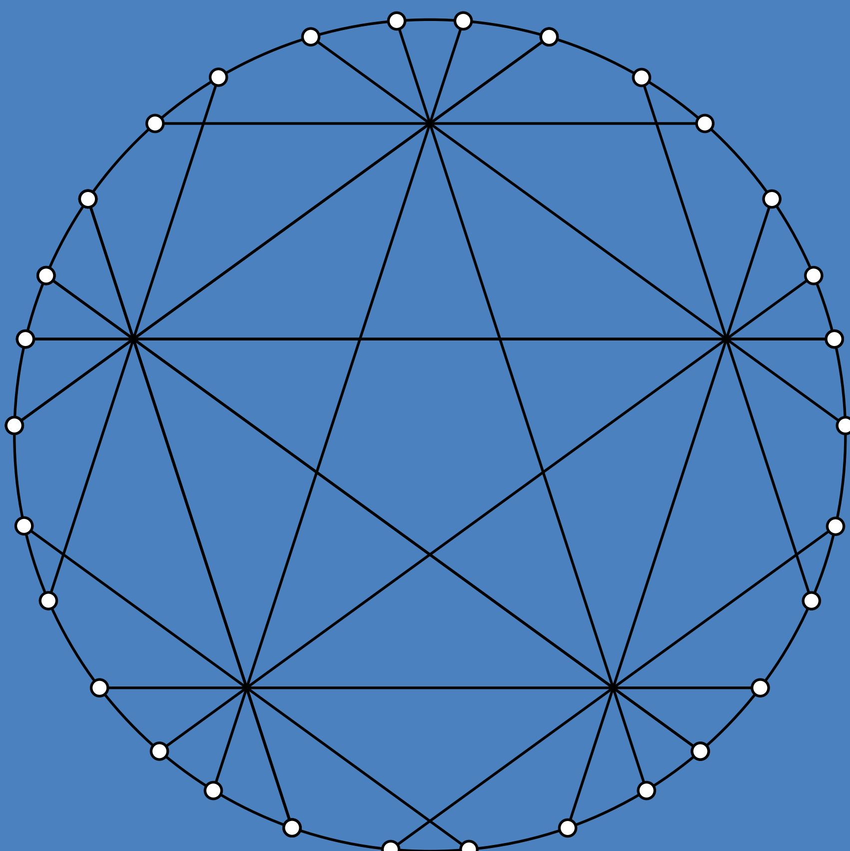

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An Intersection/Union Theorem for Several Families of Finite Sets

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Abstract

Given u families $\mathcal{A}_1, ..., \mathcal{A}_u$ of subsets of the finite set $\{1, ..., m\}$, suppose that the intersection of any s subsets drawn from different families is non-empty, and the union of any t subsets drawn from different families is not equal to $\{1, ..., m\}$, how big can $|\mathcal{A}_1|+...+|\mathcal{A}_u|$ be? This question is answered in this paper; the answers depend on s, t, u and m, and are all best possible. Special cases of this problem were considered in an earlier paper by the present author in 1978.

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1 Introduction

In [6] the present author gave some intersection and union theorems for several families of subsets of a finite set X. Here we give a more general theorem which includes all the earlier theorems as special cases.

Very roughly, we impose two kinds of condition, a union condition which says that the union of t sets drawn from distinct families is never the set X, and an intersection condition which says that the intersection of s sets drawn from distinct families is never empty. The question we ask is: if the families are $\mathcal{A}_1, ..., \mathcal{A}_u$, then how large can $|\mathcal{A}_1| + ... + |\mathcal{A}_u|$ be? The answer depends on the values of s, t, u and $|X|$.

The answers are all analogues of one of the following statements (here a family $\mathcal{A}$ of subsets of $\{1, ..., m\}$ is *intersecting* if $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \bigcap A_2 \neq \phi$ and is *non-union* if $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \bigcup A_2 \neq \{1, ..., m\}$);

1. A set of m elements has 2^m subsets.

2. A maximal intersecting family $\mathcal{A}$ of subsets of a set of m elements has 2^{m-1} subsets (for each pair $(A, X \backslash A)$, exactly one is in A).

3. A maximal intersecting, non-union family of subsets of a set of m elements has 2^{m-2} subsets.

The result 3 was conjectured by Brace and Daykin [2] in 1972 and different proofs were found by Anderson [1], Daykin and Lovász [3], Greene and Kleitman [4], Schönheim [7], Seymour [8] and Hilton [5]. The proofs in our main theorem were inspired by Schönheim's proof and Seymour's proof.

2 The main theorem

We prove

Theorem 1. *Let* $\mathcal{A}_1, \ldots, \mathcal{A}_u$ *be* u *families of subsets of the finite set* $\{1, \ldots, m\} = X$. *Let*

$$A_{i_1} \bigcap A_{i_2} \bigcap \cdots \bigcap A_{i_s} \neq \phi$$

and

$$A_{j_1} \bigcup A_{j_2} \bigcup \cdots \bigcup A_{j_t} \neq X$$

whenever $\{i_1, i_2, \ldots, i_s\}$ *and* $\{j_1, j_2, \ldots, j_t\}$ *are two subsets of* $\{1, ..., u\}$ *with* $1 \leq i_1 < i_2 < ...i_s \leq u$ *and* $1 \leq j_1 < j_2 < ...j_t \leq u$ *and with* $A_{i_k} \in \mathcal{A}_{i_k}$ *and* $A_{j_l} \in \mathcal{A}_{j_l}$ *for* $1 \leq k \leq s$ *and* $1 \leq l \leq t$. *Let* $2 \leq s \leq t$. *Then*

I. If $s \leq t \leq 2s - 1$ *then*

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \leq \begin{cases} u2^m & \text{for } 1 \leq u \leq s-1, \\ (s-1)2^m & \text{for } s-1 \leq u \leq 4(s-1), \\ u2^{m-2} & \text{for } u \geq 4(s-1). \end{cases}$$

II. If $2s - 1 \leq t$ *then*

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \leq \begin{cases} u2^m & \text{for } 1 \leq u \leq s-1, \\ (s-1)2^m & \text{for } s-1 \leq u \leq 2(s-1), \\ u2^{m-1} & \text{for } 2(s-1) \leq u \leq t-1, \\ (t-1)2^{m-1} & \text{for } t-1 \leq u \leq 2(t-1), \\ u2^{m-2} & \text{for } 2(t-1) \leq u. \end{cases}$$

The bound $u2^m$ is obtained by letting $\mathcal{A}_1, ..., \mathcal{A}_u$ each consist of all 2^m subsets of $\{1, ..., m\}$. The bound $(s-1)2^m$ is obtained by letting $\mathcal{A}_1, ..., \mathcal{A}_{s-1}$ each consist of all 2^m subsets of $\{1, ..., m\}$, and letting $|\mathcal{A}_s| = |\mathcal{A}_{s+1}| = \cdots = |\mathcal{A}_u| = 0$. The bound $u2^{m-2}$ is obtained by letting $\mathcal{A}_1, ..., \mathcal{A}_u$ each consist of all 2^{m-2} subsets of $\{1, ..., m\}$ which contain $\{1\}$ and do not contain $\{m\}$. These are the bounds in I. For the bounds in II, the bound $u2^{m-1}$ for $2(s-1) \leq u \leq t-1$ is achieved by letting $\mathcal{A}_1, \ldots, \mathcal{A}_u$ each consist of all 2^{m-1} subsets of $\{1, \ldots, m\}$ containing $\{1\}$. The bound $(t-1)2^{m-1}$ for $t-1 \leq u \leq 2(t-1)$ is achieved by letting $\mathcal{A}_1, \ldots, \mathcal{A}_{t-1}$ each consist of all 2^{m-1} subsets of $\{1, \ldots, m\}$ containing $\{1\}$, and letting $\mathcal{A}_t = \ldots = \mathcal{A}_u = \phi$.

In the earlier paper, we proved the following special cases of our Theorem 1. Theorem 1 of the earlier paper was the special case when $s = t$. Theorem 2 was the special case when $s = 2$ and $u \geq t - 1 \geq 1$. Theorem 3 was in effect the special case when $t = 1$, $u \geq s - 1 \geq 1$.

3 Three useful lemmas

We shall need the following lemmas. The first was proved in [6], but we include the proof here to make this account self-contained.

Lemma 2. *If* $2 \leq s$ *and* $0 \leq r \leq 3 \cdot 2^{m-2}$ *then*

$$(4s-5)\{2^{m-2} - (2^{\frac{m}{2}} - (2^{m-2}+r)^{\frac{1}{2}})^2\} - r \geq 0. \tag{1}$$

Proof. The left hand side of (1) equals

$$\begin{aligned}
&(4s-5)2^{m-2} - (4s-5)2^m - (4s-5)(2^{m-2}+r) + \\
&+2(4s-5)2^{m/2}(2^{m-2}+r)^{1/2} - r \\
&= -(4s-5)2^m - 4(s-1)r + 2(4s-5)2^{m/2}(2^{m-2}+r)^{1/2} \\
&\geq 0
\end{aligned}$$

since

$$\begin{aligned}
& \{2(4s-5)2^{m/2}(2^{m-2}+r)^{1/2}\}^2 - \{(4s-5)2^m + 4(s-1)r\}^2 \\
= \;& 4(4s-5)^2 \cdot 2^m \cdot (2^{m-2}+r) - (4s-5)^2 2^{2m} - 16(s-1)^2 r^2 \\
- \;& 2(4s-5)(s-1) \cdot 4 \cdot 2^m \cdot r \\
= \;& 4r\{(4s-5)^2 \cdot 2^m - 4(s-1)^2 r - 2(4s-5)(s-1)2^m\} \\
= \;& 4r\{(4s-5)(2s-3)2^m - 4(s-1)^2 r\} \\
\geq \;& 4r\{(8s^2 - 22s + 15)2^m - (3s^2 - 6s + 3)2^m\} \\
= \;& 2^{m+2} r(5s-6)(s-2) \\
\geq \;& 0.
\end{aligned}$$

□

Lemma 3. *If* $2 \leq 2(s-1) < t$ *and* $0 \leq r \leq 3 \cdot 2^{m-2}$ *then*

$$(2t-3)\{2^{m-2} - (2^{m/2} - (2^{m-2}+r)^{1/2})^2\} - r \geq 0.$$

Proof. Since $2(s-1) < t$ it follows that $2t-3 \geq 2(2s-1)-3 = 4s-5$, so by Lemma 2,

$$\begin{aligned} & (2t-3)\{2^{m-2} - (2^{m/2} - (2^{m-2}+r)^{1/2})^2\} - r \\ \geq\ & (4s-5)\{2^{m-2} - (2^{m/2} - (2^{m-2}+r)^{1/2})^2\} - r \\ \geq\ & 0. \end{aligned}$$

□

The third lemma is a theorem of Seymour [8].

Lemma 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of $\{1,\ldots,m\}$, and let $\mathcal{A}$ and $\mathcal{B}$ be incomparable (that is, for no $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is it true that $A \supseteq B$ or $B \supseteq A$). Then*

$$|\mathcal{A}|^{\frac{1}{2}} + |\mathcal{B}|^{\frac{1}{2}} \leq 2^{\frac{m}{2}}\,.$$

4 Proof of the main theorem

Proof of Theorem 1.

We first suppose that $s \leq t \leq 2s-1$.

If $u \leq s-1$ then the intersection condition and the non-union condition are both vacuous, so it is obvious that the maximum value of $|\mathcal{A}_1| + ... + |\mathcal{A}_u|$ is achieved when each of $\mathcal{A}_1, ..., \mathcal{A}_u$ consists of all subsets of 2^m. Then

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| = u2^m.$$

Now suppose that $u \geq s-1$. Suppose that $t \leq u \leq 4(s-1)$. We may suppose that $|\mathcal{A}_1| \geq |\mathcal{A}_2| \geq \cdots \geq |\mathcal{A}_u|$. Since $A_{i_1} \bigcap A_{i_2} \bigcap \ldots \bigcap A_{i_s} \neq \phi$ and $A_{j_1} \bigcup A_{j_2} \bigcup \ldots \bigcup A_{j_t} \neq X$ whenever $1 \leq i_1 < i_2 < \cdots < i_s \leq u$ and $1 \leq j_1 < j_2 < \cdots < j_t \leq u$ it follows that $\mathcal{A}_1$ and $\overline{\mathcal{A}_i}$ are incomparable whenever $i \geq 2$ where $\overline{\mathcal{A}_i} = \{\{1,\ldots,m\}\backslash A_i : A_i \in \mathcal{A}_i\}$. That is, no set in $\mathcal{A}_1$ contains or is contained by any set in $\overline{\mathcal{A}_i}$. Therefore by Seymour's inequality (Lemma 4),

$$|\mathcal{A}_1|^{1/2} + |\mathcal{A}_i|^{1/2} = |\mathcal{A}_1|^{1/2} + |\overline{\mathcal{A}_i}|^{1/2} \leq 2^{m/2}\,.$$

If $|\mathcal{A}_1| \le 2^{m-2}$ then

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \le 4(s-1)2^{m-2} = (s-1)2^m$$

as asserted. So suppose that

$$|\mathcal{A}_1| = 2^{m-2} + r$$

where $0 \le r \le 3 \cdot 2^{m-2}$. If $s \le u \le 4(s-1)$, then

$$\begin{aligned}
& (s-1)2^m - (|\mathcal{A}_1| + \cdots + |\mathcal{A}_u|) \\
\ge\ & (s-1)2^m - \{2^{m-2} + r + (u-1)(2^{m/2} - (2^{m-2}+r)^{1/2})^2\} \\
=\ & (4s-5)2^{m-2} - (u-1)(2^{m/2} - (2^{m-2}+r)^{1/2})^2 - r \\
\ge\ & (4s-5)\{2^{m-2} - (2^{m/2} - (2^{m-2}+r)^{1/2})^2\} - r \\
\ge\ & 0, \text{ by Lemma 2.}
\end{aligned}$$

We know from the above that if $u = s-1$ or $u = t$ then $|\mathcal{A}_1| + \ldots + |\mathcal{A}_u| \le (s-1)2^m$. Suppose now that $s-1 \le u \le t-1 (\le 2(s-1))$, and suppose for a contradiction that $|\mathcal{A}_1| + \ldots + |\mathcal{A}_u| > (s-1)2^m$. Then, by adjoining families $\mathcal{A}_{u+1}, \ldots, \mathcal{A}_t$ with $\mathcal{A}_{u+1} = \ldots = \mathcal{A}_t = \phi$, we would obtain a set of families $\mathcal{A}_1, \ldots, \mathcal{A}_t$ satisfying the intersection and non-union rules, and with $|\mathcal{A}_1| + \ldots + |\mathcal{A}_t| > (s-1)2^m$, contradicting our result above for $u = t$.

Finally, suppose that $u \ge 4(s-1)$. Then, as above, $\mathcal{A}_1$ and $\overline{\mathcal{A}}_i$ are incomparable for $i \ge 2$. Therefore, if $|\mathcal{A}_1| = 2^{m-2} + r$, we have

$$\begin{aligned}
& u2^{m-2} - (|\mathcal{A}_1| + \cdots + |\mathcal{A}_u|) \\
\ge\ & u2^{m-2} - \{2^{m-2} + r + (u-1)(2^{m/2} - (2^{m-2}+r)^{1/2})^2\} \\
=\ & 2^{m-2} + (u-1)\{2^{m-2} - (2^{m/2} - (2^{m-2}+r)^{1/2})^2\} - 2^{m-2} - r \\
\ge\ & (4s-5)\{(2^{m-2} - (2^{m/2} - (2^{m-2}+r)^{1/2})^2\} - r \\
\ge\ & 0, \text{ by Lemma 2.}
\end{aligned}$$

This completes the proof of I.

Next suppose that $2s-1 \le t$. The argument to show that $|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \le u2^m$ when $u \le s-1$ is the same in this case as in the previous case I. Now suppose that $s-1 \le u \le 2(s-1)$. We may again suppose that $|\mathcal{A}_1| \ge \cdots \ge |\mathcal{A}_u|$. Since $A_{i_1} \bigcap A_{i_2} \bigcap \ldots \bigcap A_{i_s} \ne \phi$ when $\{i_1, .., i_s\}$ is an s-subset of $\{1, ..., u\}$, it follows that $A \in \mathcal{A}_{s-i} \Rightarrow \overline{A} \notin \mathcal{A}_{s+i-1}$ whenever $s+i-1 \le 2(s-1)$. Then at most two of the statements

$$A \in \mathcal{A}_{s-i}, \overline{A} \in \mathcal{A}_{s-i}, A \in \mathcal{A}_{s+i-1}, \overline{A} \in \mathcal{A}_{s+i-1}$$

can be true for $1 \le i \le u-s+1$, so

$$|\mathcal{A}_{s-i}| + |\mathcal{A}_{s+i-1}| \le 2^m.$$

Therefore

$$|\mathcal{A}_{2s-u-1}| + \cdots + |\mathcal{A}_u| \le (u-s+1)2^m.$$

But $|\mathcal{A}_j| \le 2^m$ for $1 \le i \le 2s-u-2$, so we have

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \le (u-s+1)2^m + (2s-u-2)2^m = (s-1)2^m,$$

as asserted.

Next suppose that $2(s-1) \le u \le t-1$. If $A \in \mathcal{A}_1$ then $\overline{A} \notin \mathcal{A}_j$ for $j > 1$, so $|\mathcal{A}_1| + |\mathcal{A}_j| \le 2^m$. Suppose that $|\mathcal{A}_1| = 2^{m-1} + r$, where $0 \le r \le 2^{m-1}$. Then

$$\begin{aligned} |\mathcal{A}_1| + \cdots + |\mathcal{A}_u| &\le 2^{m-1} + r + (u-1)(2^{m-1} - r) \\ &\le u2^{m-1} - (u-2)r \\ &\le u2^{m-1}. \end{aligned}$$

Next suppose that $(t-1) \le u \le 2(t-1)$. If $u = t-1$ then, as just above,

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \le u2^{m-1} = (t-1)2^{m-1}.$$

It is convenient to consider the case when $t-1 \le u \le 2(t-1)$ in further detail after the next case (but note that there is no circularity of argument since we do not use the result for $t-1 \le u \le 2(t-1)$ in proving the next case).

Next suppose that $u \ge 2(t-1)$. If $|\mathcal{A}_1| \le 2^{m-2}$ then $|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \le u2^{m-2}$ as asserted. So suppose that $|\mathcal{A}_1| = 2^{m-2} + r$ for some r, $0 \le r \le 3 \cdot 2^{m-2}$. Then, as above, $\mathcal{A}_1$ and $\overline{\mathcal{A}}_i$ are incomparable and we have

$$|\mathcal{A}_1|^{1/2} + |\mathcal{A}_i|^{1/2} = |\mathcal{A}_1|^{1/2} + |\overline{\mathcal{A}}_i|^{1/2} \le 2^{m/2},$$

so

$$|A_i| \le (2^{m/2} - (2^{m-2} + r)^{1/2})^2.$$

Therefore

$$\begin{aligned} & u2^{m-2} - \{|\mathcal{A}_1| + \cdots + |\mathcal{A}_u|\} \\ \ge\ & u2^{m-2} - \{2^{m-2} + r + (u-1)(2^{m/2} - (2^{m-2} + r)^{1/2})^2\} \\ =\ & 2^{m-2} + (u-1)\{2^{m-2} - (2^{m/2} - (2^{m-2} + r)^{\frac{1}{2}})^2\} - 2^{m-2} - r \\ \ge\ & (2t-3)\{2^{m-2} - (2^{m/2} - (2^{m-2} + r)^{\frac{1}{2}})^2)\} - r \\ \ge\ & 0, \text{ by Lemma 3.} \end{aligned}$$

Finally suppose that $(t-1) \leq u \leq 2(t-1)$. Note that, from just above, if $u = 2(t-1)$ then $|\mathcal{A}_1| + \cdots + |\mathcal{A}_{2(t-1)}| \leq 2(t-1)2^{m-2} = (t-1)2^{m-1}$. If for some u, $t-1 \leq u \leq 2(t-1)$ we had $\mathcal{A}_1, ..., \mathcal{A}_u$ satisfing the intersection rule but with $|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| > (t-1)2^{m-1} = (t-1)2^{m-1}$, then, by adjoining $\mathcal{A}_{u+1}, ..., \mathcal{A}_{2(t-1)}$ with $|\mathcal{A}_{u+1}| = \cdots = |\mathcal{A}_{2(t-1)}| = 0$, we would have $2(t-1)$ families satisfying the intersection rule but with $|\mathcal{A}_1| + \cdots + |\mathcal{A}_{2(t-1)}| > (t-1)2^{m-1}$, a contradiction. Therefore

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \leq (t-1)2^{m-1}$$

in this case.

This completes the proof of II. □

What happens in the case not considered in Theorem 1 where $t < s$? This is easy to find by taking complements. We obtain:

Theorem 5. *With m, u, s and t defined as in Theorem 1, suppose that $2 \leq t \leq s$. Then*

I. If $t \leq s \leq 2t-1$, then

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \leq \begin{cases} u2^m & \text{for } 1 \leq u \leq t-1, \\ (t-1)2^m & \text{for } t-1 \leq u \leq 4(t-1), \\ u2^{m-2} & \text{for } u \geq 4(t-1). \end{cases}$$

II. If $2t-1 \leq s$ then

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \leq \begin{cases} u2^m & \text{for } 1 \leq u \leq t-1, \\ (t-1)2^m & \text{for } t-1 \leq u \leq 2(t-1), \\ u2^{m-1} & \text{for } 2(t-1) \leq u \leq s-1, \\ (s-1)2^{m-1} & \text{for } s-1 \leq u \leq 2(s-1), \\ u2^{m-2} & \text{for } 2(s-1) \leq u. \end{cases}$$

Proof. Suppose that $2 \leq t \leq s$. Since

$$A_{i_1} \bigcap A_{i_2} \bigcap \ldots \bigcap A_{i_s} \neq \phi$$

it follows that

$$\overline{A}_{i_1} \bigcup \overline{A}_{i_2} \bigcup \ldots \bigcup \overline{A}_{i_s} \neq X$$

and since

$$A_{j_1} \bigcup A_{j_2} \bigcup \ldots \bigcup A_{j_t} \neq X$$

it follows that $\overline{\mathcal{A}}_{j_1} \bigcap \overline{\mathcal{A}}_{j_2} \bigcap \ldots \bigcap \overline{\mathcal{A}}_{j_t} \neq \phi$. Since

$$|\mathcal{A}_i| = |\{A : A \in \mathcal{A}_i\}| = |\{\overline{\mathcal{A}} : A \in \mathcal{A}_i\}|,$$

we can interchange t and s in the bounds found in Theorem 1 to find the correct bounds in this theorem. □

5 Further remarks

We could extend this study to a more general extremal problem. Suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_u$ are families of distinct subsets of $\{1, 2, \ldots, m\}$. Let $g(u, h, k, s, t, m)$ be the maximum value of

$$|\mathcal{A}_1| + |\mathcal{A}_2| + \ldots + |\mathcal{A}_u|$$

in the case where

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_s}| \geq h$$

and

$$|A_{j_1} \cap A_{j_2} \cap \ldots \cap A_{j_t}| \leq m - k$$

whenever $i_1, i_2, \ldots, i_s$ are distinct subscripts from $\{1, 2, \ldots, u\}$ and $A_{i_l} \in \mathcal{A}_{i_l}$ $(1 \leq l \leq s)$, and similarly $j_1, j_2, \ldots, j_t$ are distinct subscripts from $\{1, 2, \ldots, u\}$ and $A_{j_l} \in \mathcal{A}_{j_l}$ $(1 \leq l \leq t)$.

Theorem 1 of [6] is the special case of our Theorem 1 when $s = t$. We would like to suggest the following generalization of Theorem 1 of [6]. Theorem 1 of [6] is the special case of the conjecture when $h = 1$.

Conjecture 6. *Let m, h, u and s be positive integers with $u \geq s \geq 1$. Let $\mathcal{A}_1, ..., \mathcal{A}_u$ be u families of distinct subsets of $\{1, ..., m\}$ such that*

$$|A_{i_1} \bigcap ... \bigcap A_{i_s}| \geq h$$

and

$$|A_{i_1} \bigcup ... \bigcup A_{i_s}| \leq m - h$$

whenever $i_1, ..., i_s$ are distinct subscripts from $\{1, ..., u\}$ and $A_{i_j} \in \mathcal{A}_{i_j}$ $(1 \leq i \leq s)$. Then

$$|\mathcal{A}_1| + \cdots + |\mathcal{A}_u| \leq \begin{cases} u2^m & for \quad u \leq s-1\,, \\ (s-1)2^m & for \quad s-1 \leq u \leq 2^{2h}(s-1)\,, \\ u2^{m-2h} & for \quad u \geq 2^{2h}(s-1)\,. \end{cases}$$

In other words, we conjecture that

$$g(u,h,h,s,s,m) = \begin{cases} u2^m & for \quad u \le s-1\,, \\ (s-1)2^m & for \quad s-1 \le u \le 2^{2h}(s-1)\,, \\ u2^{m-2h} & for \quad u \ge 2^{2h}(s-1)\,. \end{cases}$$

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