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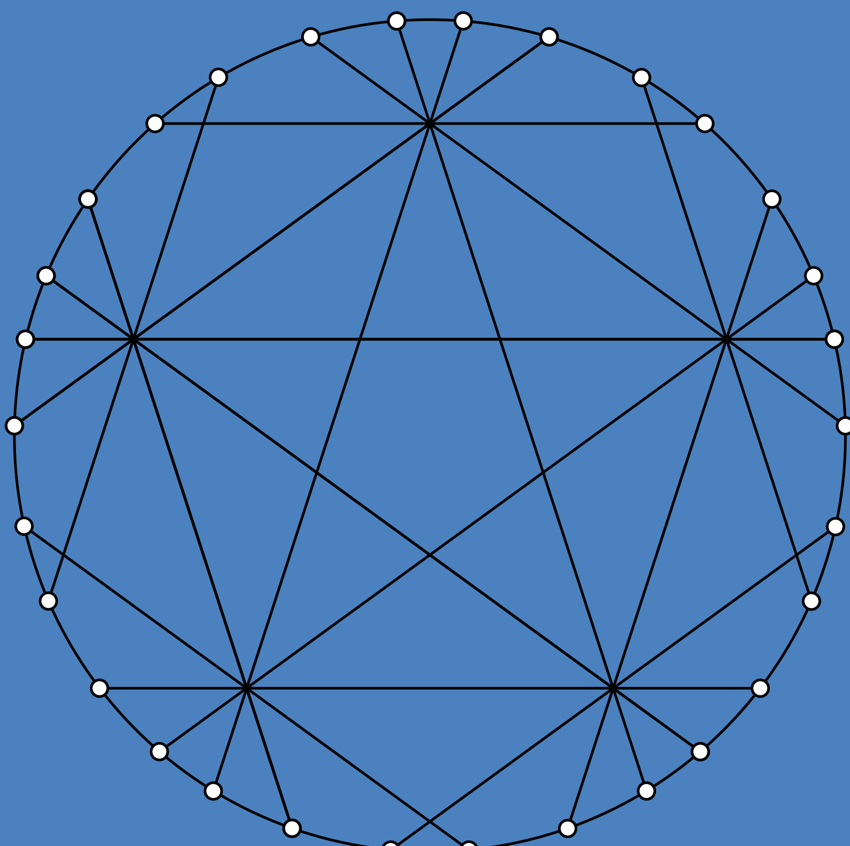

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On Power-Sequence and Matryoshka Terraces for $\mathbb{Z}_n$

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Abstract: Some narcissistic power-sequence terraces for $\mathbb{Z}_n$ are obtained, where n is the product of three distinct odd primes p, q and r such that $\mathrm{lcm}(p-1, q-1, r-1) = (p-1)(q-1)(r-1)/4$ and $p = 3$. The paper's emphasis is on the strategy used to find such terraces, not on a formal presentation of the underlying mathematics. Special attention is paid to matryoshka terraces. Further, a combinatorial method is given for the construction of matryoshka $\mathbb{Z}_n$ terraces for values n with any number of prime factors.

Keywords: 2-sequencings; Latin squares; narcissistic terraces; power-sequence terraces; primitive lambda-roots.

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1 Introduction

Terraces for groups were introduced by Bailey [10] as a tool to construct quasi-complete latin squares, although here we restrict our attention to cyclic groups where they were implicitly used earlier by Williams [18]. Terraces have since been used in many other combinatorial constructions, see [14].

There has also been interest in elegant properties that terraces might have in their own right. This includes the notion of being a power-sequence terrace, an idea developed in a series of papers by the first and third authors [3, 4, 5, 6, 7, 8, 9]. This is the main topic that we explore in this paper. In addition we look at a particular type that arose in the study of power-sequence terraces—matryoshka, or Russian doll, terraces—and give a combinatorial construction that demonstrates their existence in a much wider class of cyclic groups than previously known.

The necessary definitions are given in the next section.

Authors' Note: The third author, Donald Preece, died in January 2014 after a most productive life in mathematics and other endeavours; see [11] for an extended obituary. At a memorial day* held at Queen Mary, University of London in September 2015 there were conversations about some of Donald's unpublished results and fragments of papers. One such has been completed [17].

Shortly afterwards, the second author re-discovered a draft Donald had written on the topic of power-sequence terraces for $\mathbb{Z}_n$ where n is a product of three distinct primes with various restrictions. The following two sections and much of Section 4 is essentially that draft with only the alterations required to make the paper as a whole coherent.

2 Basic definitions

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an arrangement of the elements of $\mathbb{Z}_n$, and then let $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$ be the ordered sequence given by $b_i = a_{i+1} - a_i$ $(i = 1, 2, \ldots, n-1)$. On a definition given by Bailey [10, p. 325], the

*http://www.maths.qmul.ac.uk/~pjc/dapday.html arrangement $\mathbf{a}$ is a *terrace* for $\mathbb{Z}_n$ (or, in short, a $\mathbb{Z}_n$ terrace), if the sequences $\mathbf{b}$ and $-\mathbf{b}$ between them contain exactly two occurrences of each element from $\mathbb{Z}_n \setminus \{0\}$. The sequence $\mathbf{b}$ is the *2-sequencing* associated with $\mathbf{a}$. A terrace for $\mathbb{Z}_p$ is *narcissistic* [4, 5] if $b_i = b_{n-i}$ $(i = 1, 2, \ldots, n-1)$.

For convenience, when we assign a numerical value to the difference between any two successive elements a_i and a_{i+1} we take that value to be the integer δ which is congruent, modulo n, to either b_i or $-b_i$ and which satisfies $\delta \in (0, (n-1)/2)$.

Anderson and Preece [3, 4, 5, 6, 7, 8, 9] gave constructions for "power-sequence terraces" for $\mathbb{Z}_n$ where n is odd. A terrace is called *power-sequence* if it can be partitioned into segments one of which contains merely the zero element of $\mathbb{Z}_n$, whereas each other segment is either (a) a sequence of successive powers of an element of $\mathbb{Z}_n$, or (b) such a sequence multiplied through by a constant. Many of the sequences $x^0, x^1, \ldots, x^{s-1}$ are *full-cycle* sequences such that $x^s = x^0$, but half cycles containing s elements are often used when $x^s = -x^0$. Within a single terrace there may be full-cycle sequences of different lengths, and half-cycle sequences of different lengths.

Some narcissistic power-sequence terraces for $\mathbb{Z}_n$ with $n = pq^t$ were given in [5]. The constructions in that paper are based on *primitive λ-roots* of n, *i.e.* on a unit x from $\mathbb{Z}_n$ such that the order of x is a large as is theoretically possible. The present paper carries the work of [5] forward to $\mathbb{Z}_n$ where $n = pqr$, $p = 3$ and

$$\operatorname{lcm}(p-1, q-1, r-1) = (p-1)(q-1)(r-1)/\xi(n)$$

with $\xi(n) = 4$.

Again our constructions are based on primitive λ-roots of n, which here have order $(p-1)(q-1)(r-1)/\xi(n)$. As in previous papers we use *strong* primitive λ-roots, which are those primitive λ-roots x for which $x - 1$ is also a unit and -1 is not a power of x. However, we omit details of this, as we choose now to place our emphasis on the strategy used to find terraces, not on a formal presentation of the underlying mathematics.

3 Construction procedures

How then are power-sequence terraces for $\mathbb{Z}_{pqr}$ to be constructed? We restrict ourselves here to narcissistic terraces with 0 in the middle; such terraces are *centred*. For these we effectively have only **half**-terraces to construct, the second half of each terrace being merely the reverse of the negative of the first half.

Consider first the narcissistic terrace

$$3 \quad 9 \quad 1 \quad 2 \quad 4 \quad 8 \quad 5 \quad 0 \quad 10 \quad 7 \quad 11 \quad 13 \quad 14 \quad 6 \quad 12 \tag{1}$$

for $\mathbb{Z}_{pq}$ with $p = 3$ and $q = 5$. The elements following the zero are merely the negatives of those before, in reverse order. So we need consider only the sequence formed by the first 8 elements, none of which is the negative of any other. These elements are

$$3^1 \quad 3^2 \;\mid\; 2^0 \quad 2^1 \quad 2^2 \quad 2^3 \;\mid\; 5^1 \;\mid\; 0 \qquad \pmod{15}$$

where the fences | divide the sequence into 4 segments, and the successive differences between adjoining elements are 6, 7, 1, 2, 4, 3, 5, *i.e.* each of $1, 2, \ldots, 7$ exactly once, as required for a terrace for $\mathbb{Z}_{15}$. The first two elements of the terrace comprise the members of half of the cycle $3 \cdot 3^0 \;\; 3 \cdot 3^1 \;\; 3 \cdot 3^2 \;\; 3 \cdot 3^3 \pmod{15}$ or $3^1 \;\; 3^2 \;\; 3^3 \;\; 3^4 \pmod{15}$ where $3^3 = -3^1$ and $3^4 = -3^2$; the next four elements are the members of the entire cycle $2^0 \;\; 2^1 \;\; 2^2 \;\; 2^3 \pmod{15}$; and the next solitary element represents (trivially) half of the cycle $5^1 \;\; 5^2$ where $5^2 = -5^1$. Breaking off the cycle $3^1 \;\; 3^2 \;\; 3^3 \;\; 3^4$ half-way through, after 3^2, means that we lose the difference $3^3 - 3^2 = 12 - 9 = 3$, which must be provided elsewhere in the half-terrace; it appears at the second fence as $2^3 - 5^1$. Breaking the cycle of successive powers of 2 between 2^3 and 2^4 $(= 2^0)$ means that we lose the difference $2^3 - 2^0 = 7$; the loss is compensated for by $2^0 - 3^2$ at the first fence. Finally, breaking off the cycle $5^1 \;\; 5^2$ after 5^1 means that we lose the difference $5^2 - 5^1 = 10 - 5 = 5$; this is made good as $5^1 - 0$ at the third fence. The art of constructing power-sequence terraces for $\mathbb{Z}_n$ where n is composite is thus the art of stitching segments together so that the differences at the seams/fences compensate for the differences lost by the breaking of complete cycles or by taking half-cycles.

Constructional and checking errors are easily made if we forget to distinguish between complete cycles and half-cycles, so for security we place a colon at the start and end of a half-cycle segment. Our $\mathbb{Z}_{15}$ terrace (1) can thus be written

$$: \; 3 \quad 9 \; : \;\mid\; 1 \quad 2 \quad 4 \quad 8 \;\mid\; : \; 5 \; : \;\mid\; 0 \;\mid\; \text{negatives} \, .$$

For longer terraces we may wish, for conciseness, to give only the first element of a segment, and an indication of how subsequent elements in that segment are to be obtained from the first. We use $\xrightarrow{x}$ to indicate that each successive element in a segment is obtained from the previous one by multiplication by x; similarly $\xleftarrow{x}$ indicates successive multiplications by x when we work from right to left. Thus our $\mathbb{Z}_{15}$ terrace can be written

$$: \ 3 \ \xleftarrow{2} \ : \ | \ 1 \ \xrightarrow{2} \ | : \ 5 \ : \ | \ 0 \ | \quad \text{negatives} \ .$$

To represent the type of construction used here, we let (p) denote a segment containing multiples of p, and (u) denote a segment containing units of $\mathbb{Z}_{pq}$. Likewise $|_p$ denotes a fence where the difference is a multiple of p, and so on. Accordingly, with $p = 3$ and $q = 5$, the construction-type of our $\mathbb{Z}_{15}$ terrace is

$$(p) \ |_u \ (u) \ |_p \ (q) \ |_q \ 0 \ | \quad \text{negatives} \ .$$

We readily see that a sub-sequence such as $(p) \ |_p \ (q)$ would be impossible.

Moving on now to terraces for $\mathbb{Z}_{pqr}$ where p, q and r are distinct odd primes, we use (p) to denote a segment whose elements are multiples of p but not of either q or r, also (pq) to denote a segment whose elements are multiples of both p and q but not of r, and so on. As we here restrict ourselves to situations where $p = 3$ and

$$\mathrm{lcm}\,(p-1, q-1, r-1) = (p-1)(q-1)(r-1)/4 \ ,$$

our $\mathbb{Z}_{pqr}$ terraces can be constructed with just 4 segments for the units, the length of each of these segments being $(p-1)(q-1)(r-1)/4$. We restrict ourselves to narcissistic terraces with 0 in the middle and two segments of units to the left of 0; we use (u) and (v) to denote these two segments. On the left of 0 our terraces also have just one segment for each of (p), (q), (r), (pq), (pr) and (qr), so each of the terraces has 17 segments in total. No segment is allowed to have a repeated value amongst its differences. We must also ensure that the difference between any two adjacent elements in (u) or (v) is itself a unit, that the difference between any two adjacent elements in (p) is itself a multiple of p but not of q or r, and so on. The differences for (u) must be distinct from those for (v). We use $|_u$ where the fence difference is the difference missing from the segment (u), and so on. If a segment of units has $\xrightarrow{x}$ or $\xleftarrow{x}$, then x is a primitive λ-root of n.

For $n = pqr = 105$, 165 or 231, with $p = 3$, a possible construction-type for a narcissistic $\mathbb{Z}_n$ terrace is

$$(u) \ |_u \ (p) \ |_v \ (v) \ |_{pr} \ (q) \ |_p \ (r) \ |_{qr} \ (pr) \ |_r \ (qr) \ |_q \ (pq) \ |_{pq} \ 0 \ | \quad \text{negatives} \ . \qquad (2)$$

(The reader should check that no subsequence is included here that is *prima facie* impossible.) Still with $p = 3$, this type can be achieved with either $q > r$ or $r > q$.

For $n = 105$, perhaps the easiest way to start an attempt to obtain a $\mathbb{Z}_n$ terrace of type (2) is to prepare a worksheet as in Table 1. This gives details of the possibilities for segments that use $\xrightarrow{2}$, and mentions other possibilities that may need to be written out more fully if ever $\xrightarrow{2}$ or $\xleftarrow{2}$ fails.

The worksheet indicates that an easy start can be made by writing down the sequence

$$\ldots\ 67\ |_{(42)}\ \overbrace{25\ \ 50\ \ 100\ \ 95\ \ 85\ \ 65}^{40}\ |_{(12)}\ \overbrace{77\ \ 91\ \ 98\ \ 49}^{28}\ |_{(35)}$$

$$:\ \overbrace{84\ \ 63}^{42}\ :\ |_{(28)}\ :\ \overbrace{35}^{35}\ :\ |_{(40)}\ \overbrace{75\ \ 45\ \ 90}^{15}\ |_{(15)}\ 0$$

where the bracketed number at the foot of each fence is the difference at the fence, and the number above each segment is the difference lost by breaking the corresponding cycle or half-cycle. Here, the upper and lower numbers are the same, save that the extra fence difference 12 indicates the need for the segment (p), not yet included, to have 12 as its missing difference, This can be achieved in different ways, including taking the segment as

$$3 \cdot 2^{60} \quad 3 \cdot 2^{55} \quad 3 \cdot 2^{50} \quad \ldots \quad 3 \cdot 2^{5} \quad = \quad 3 \quad 69 \quad 12 \quad \ldots \quad 96\ .$$

A narcissistic terrace can then be completed as

$$1 \xleftarrow{2} |\,3 \xrightarrow{23} |\,29 \xrightarrow{2} |\,25 \xrightarrow{2} |\,77 \xleftarrow{2} |: 84 \xrightarrow{2} : |: 35 : |\,75 \xrightarrow{2} |\,0\,|\,\text{negatives}, \tag{3}$$

which is

$$1\ 53\ 79\ 92\ 46\ 23\ 64\ 32\ 16\ 8\ 4\ 2\ |\ 3\ 69\ 12\ 66\ 48\ 54\ 87\ 6\ 33\ 24\ 27\ 96\ |$$

$$29\ 58\ 11\ 22\ 44\ 88\ 71\ 37\ 74\ 43\ 86\ 67\ |\ 25\ 50\ 100\ 95\ 85\ 65\ |\ 77\ 91\ 98\ 49\ |$$

$$:\ 84\ 63\ :\ |:\ 35\ :\ |\ 75\ 45\ 90\ |\ 0\ | \quad \text{negatives}\ .$$

This is of the type (2) above, with $p = 3$, $q = 5$, $r = 7$.

When we find a terrace by such a trial-and-error approach, it will not necessarily have 1 at the start of the first segment of units. However, if that segment starts with the element c, the whole terrace can then be multiplied through by c^{-1} to put it into the standardised form used in the rest of this paper.

Table 1: Worksheet for $n = 105 = 3 \times 5 \times 7$

Cycles for the units:

(i) 1 2 4 8 16 32 64 23 46 92 79 53
(ii) 11 22 44 88 71 37 74 43 86 67 29 58
and their negatives

Or use the cycle $2^0 \quad 2^5 \quad 2^{10} \quad \ldots \quad 2^{55}$ and its cosets.
Or use the cycle $17^0 \quad 17^1 \quad 17^2 \quad \ldots \quad 17^{11}$
or $17^0 \quad 17^5 \quad 17^{10} \quad \ldots \quad 17^{55}$ and its cosets.

Cycles for the multiples of 15

15 30 60 and its negative.
(Or use half of 30 90 60 75 15 45)

Cycle for the multiples of 21

21 42 84 63 (use half)

Cycle for the multiples of 35

35 70 (use half)

Cycles for the remaining multiples of 3

3 6 12 24 48 96 87 69 33 66 27 54
and its negative

Or use $3 \cdot 2^0 \quad 3 \cdot 2^5 \quad 3 \cdot 2^{10} \quad \ldots \quad 3 \cdot 2^{55}$
or $3 \cdot 3^0 \quad 3 \cdot 3^1 \quad 3 \cdot 3^2 \quad \ldots \quad 3 \cdot 3^{11}$
or $3 \cdot 3^0 \quad 3 \cdot 3^5 \quad 3 \cdot 3^{10} \quad \ldots \quad 3 \cdot 3^{55}$

Cycles for the remaining multiples of 5

5 10 20 40 80 55 and its negative, *viz.*
100 95 85 65 25 50

Cycles for the remaining multiples of 7

7 14 28 56 and its negative. *viz.*
98 91 77 49

Table 2: Construction-types of terraces for $\mathbb{Z}_{105}$, $\mathbb{Z}_{165}$ and $\mathbb{Z}_{231}$ $(p = 3)$

A:	$(u) \,|_{pq}\, (v) \,|_{v}\, (pq) \,|_{q}\, (q) \,|_{u}\, (p) \,|_{p}\, (pr) \,|_{r}\, (r) \,|_{pr}\, (qr) \,|_{qr}\, 0 \,|$ negatives
B:	$(u) \,|_{u}\, (p) \,|_{v}\, (v) \,|_{pr}\, (q) \,|_{p}\, (r) \,|_{qr}\, (pr) \,|_{r}\, (qr) \,|_{q}\, (pq) \,|_{pq}\, 0 \,|$ negatives
C:	$(q) \,|_{pr}\, (u) \,|_{u}\, (p) \,|_{v}\, (v) \,|_{p}\, (r) \,|_{qr}\, (pr) \,|_{r}\, (qr) \,|_{q}\, (pq) \,|_{pq}\, 0 \,|$ negatives
D:	$(qr) \,|_{r}\, (pr) \,|_{qr}\, (r) \,|_{p}\, (u) \,|_{u}\, (p) \,|_{v}\, (v) \,|_{pr}\, (q) \,|_{q}\, (pq) \,|_{pq}\, 0 \,|$ negatives
E:	$(u) \,|_{u}\, (pr) \,|_{v}\, (v) \,|_{qr}\, (p) \,|_{p}\, (pq) \,|_{q}\, (q) \,|_{pq}\, (qr) \,|_{pr}\, (r) \,|_{r}\, 0 \,|$ negatives
F:	$(q) \,|_{pq}\, (qr) \,|_{pr}\, (r) \,|_{r}\, (pr) \,|_{q}\, (u) \,|_{u}\, (pq) \,|_{v}\, (v) \,|_{qr}\, (p) \,|_{p}\, 0 \,|$ negatives

For $n = pqr = 105$, 165 and 231, we have obtained narcissistic $\mathbb{Z}_n$ terraces for each of the construction-types given in Table 2, where type B is the type (2) given above.

Specimen terraces are obtainable from Tables 3, 4 and 5 where the first type B entry for $n = 105$ yields the terrace (3) derived above.

Throughout the examples of types B–F in these tables, a segment $1 \xleftarrow{x}$ is followed, two segments later, by a segment $d \xrightarrow{x}$ for some unit d. The restriction that the two segments of units should have the same multiplier working in opposite directions is however unnecessary. The reader may wish to gain practice by finding similar terraces where this restriction does not hold. Readers may also care to make their own searches for terraces that are of types B–F save that $(u) \,|_{u} \ldots |_{v}\, (v)$ is replaced by $(u) \,|_{v} \ldots |_{u}\, (v)$.

Any readers who have deduced the simple reason for the two entries "impossible" for type E in Tables 4 and 5 should be able to construct some of their own terraces for $\mathbb{Z}_{pqr}$. The "impossible" entries for type A arise from circumstances where the sequence comprising the first three segments of a terrace cannot be constructed.

Table 3: First 52 entries of some narcissistic $\mathbb{Z}_{105}$ terraces (types A–F as in Table 2)

$\mathbb{Z}_{105}$; $(p,q,r) = (3,5,7)$

A. $1 \xrightarrow{32} |\, 68 \xrightarrow{32} |\, 15 \xrightarrow{2} |\, 5 \xrightarrow{2} |\, 33 \xleftarrow{32} |{:}\ 84 \xleftarrow{2} {:} |\, 28 \xrightarrow{2} |{:}\ 35\ {:} |$

B. $1 \xleftarrow{2} |\, 3 \xrightarrow{23} |\, 29 \xrightarrow{2} |\, 25 \xrightarrow{2} |\, 77 \xleftarrow{2} |{:}\ 84 \xrightarrow{2} {:} |{:}\ 35\ {:} |\, 75 \xrightarrow{2} |$

C. $95 \xleftarrow{2} |\, 1 \xleftarrow{17} |\, 33 \xleftarrow{17} |\, 44 \xrightarrow{17} |\, 49 \xleftarrow{2} |{:}\ 63 \xrightarrow{2} {:} |{:}\ 70\ {:} |\, 60 \xrightarrow{2} |$

D. ${:}\ 70\ {:} |{:}\ 63 \xleftarrow{2} {:} |\, 14 \xrightarrow{2} |\, 1 \xleftarrow{47} |\, 93 \xrightarrow{2} |\, 76 \xrightarrow{47} |\, 95 \xleftarrow{2} |\, 75 \xrightarrow{2} |$

E. $1 \xleftarrow{32} |{:}\ 63 \xrightarrow{2} {:} |\, 29 \xrightarrow{32} |\, 72 \xrightarrow{32} |{:}\ 90 \xrightarrow{3} {:} |\, 50 \xrightarrow{2} |{:}\ 70\ {:} |\, 91 \xrightarrow{2} |$

F. $85 \xleftarrow{2} |{:}\ 35\ {:} |\, 56 \xleftarrow{2} |{:}\ 63 \xrightarrow{2} {:} |\, 1 \xleftarrow{23} |\, 45 \xrightarrow{2} |\, 43 \xrightarrow{23} |\, 81 \xrightarrow{2} |$

$\mathbb{Z}_{105}$; $(p,q,r) = (3,7,5)$

A. $1 \xrightarrow{23} |\, 11 \xrightarrow{23} |{:}\ 63 \xrightarrow{2} {:} |\, 14 \xrightarrow{2} |\, 81 \xleftarrow{23} |{:}\ 75 \xrightarrow{3} {:} |\, 65 \xrightarrow{2} |{:}\ 70\ {:} |$

B. $1 \xleftarrow{2} |\, 3 \xrightarrow{32} |\, 11 \xrightarrow{2} |\, 28 \xrightarrow{2} |\, 80 \xleftarrow{2} |{:}\ 90 \xrightarrow{5} {:} |{:}\ 70\ {:} |{:}\ 84 \xrightarrow{2} {:} |$

C. $98 \xleftarrow{2} |\, 1 \xleftarrow{2} |\, 3 \xleftarrow{2} |\, 74 \xrightarrow{2} |\, 40 \xleftarrow{2} |{:}\ 45 \xleftarrow{3} {:} |{:}\ 35\ {:} |{:}\ 42 \xrightarrow{2} {:} |$

D. ${:}\ 35\ {:} |{:}\ 90 \xrightarrow{3} {:} |\, 5 \xrightarrow{2} |\, 1 \xleftarrow{2} |\, 3 \xrightarrow{2} |\, 71 \xrightarrow{2} |\, 28 \xleftarrow{2} |{:}\ 84 \xrightarrow{2} {:} |$

E. $1 \xleftarrow{23} |\, 45 \xleftarrow{2} |\, 67 \xrightarrow{23} |\, 9 \xrightarrow{23} |{:}\ 42 \xleftarrow{2} {:} |\, 14 \xrightarrow{2} |{:}\ 70\ {:} |\, 10 \xrightarrow{2} |$

F. $91 \xleftarrow{2} |{:}\ 35\ {:} |\, 80 \xleftarrow{2} |{:}\ 30 \xrightarrow{5} {:} |\, 1 \xleftarrow{32} |{:}\ 63 \xleftarrow{2} {:} |\, 86 \xrightarrow{32} |\, 18 \xrightarrow{2} |$

Table 4: First 82 entries of some narcissistic $\mathbb{Z}_{165}$ terraces (types A–F as in Table 2)

$\mathbb{Z}_{165}$; $(p,q,r) = (3,5,11)$

A. $1 \xrightarrow{62} |\, 53 \xrightarrow{62} |\, 135 \xleftarrow{5} |\, 100 \xrightarrow{5} |\, 27 \xrightarrow{13} |{:}\ 66 \xleftarrow{2} {:} |\, 77 \xrightarrow{2} |{:}\ 55\ {:} |$

B. $1 \xleftarrow{62} |\, 123 \xrightarrow{13} |\, 119 \xrightarrow{62} |\, 160 \xrightarrow{5} |\, 77 \xleftarrow{2} |{:}\ 99 \xrightarrow{2} {:} |{:}\ 110\ {:} |{:}\ 15 \xrightarrow{2} {:} |$

C. $80 \xrightarrow{14} |\, 1 \xleftarrow{2} |\, 3 \xrightarrow{8} |\, 14 \xrightarrow{2} |\, 154 \xleftarrow{2} |{:}\ 33 \xrightarrow{2} {:} |{:}\ 55\ {:} |{:}\ 75 \xrightarrow{2} {:} |$

D. ${:}\ 110\ {:} |{:}\ 99 \xrightarrow{2} {:} |\, 143 \xrightarrow{2} |\, 1 \xleftarrow{2} |\, 3 \xrightarrow{18} |\, 104 \xrightarrow{2} |\, 85 \xleftarrow{5} |{:}\ 105 \xrightarrow{2} {:} |$

E. $1 \xleftarrow{17} |{:}\ 33 \xrightarrow{2} {:} |\, 89 \xrightarrow{17} |\, 57 \xleftarrow{18} |\, 15 \xrightarrow{4} |\, 65 \xleftarrow{14} |{:}\ 55\ {:} |\, 121 \xrightarrow{2} |$

F. $25 \xleftarrow{5} |{:}\ 110\ {:} |\, 11 \xleftarrow{2} |{:}\ 33\ {:} |\, 1 \xleftarrow{8} |{:}\ 15 \xleftarrow{2} {:} |\, 28 \xrightarrow{8} |\, 141 \xrightarrow{2} |$

$\mathbb{Z}_{165}$; $(p,q,r) = (3,11,5)$

A. impossible

B. $1 \xleftarrow{62} |\, 123 \xleftarrow{13} |\, 146 \xrightarrow{62} |\, 88 \xrightarrow{2} |\, 35 \xleftarrow{5} |\, 120 \xleftarrow{3} |{:}\ 55\ {:} |{:}\ 99 \xrightarrow{2} {:} |$

C. $77 \xrightarrow{2} |\, 1 \xleftarrow{2} |\, 3 \xrightarrow{13} |\, 89 \xrightarrow{2} |\, 10 \xleftarrow{5} |\, 105 \xrightarrow{4} |{:}\ 110\ {:} |{:}\ 66 \xrightarrow{2} {:} |$

D. ${:}\ 110\ {:} |\, 135 \xrightarrow{3} |\, 155 \xrightarrow{5} |\, 1 \xleftarrow{2} |\, 3 \xrightarrow{17} |\, 26 \xrightarrow{2} |\, 88 \xleftarrow{2} |{:}\ 99 \xrightarrow{2} {:} |$

E. impossible

F. $44 \xleftarrow{2} |{:}\ 55\ {:} |\, 145 \xrightarrow{14} |\, 135 \xrightarrow{3} |\, 1 \xleftarrow{17} |{:}\ 33 \xleftarrow{2} {:} |\, 146 \xrightarrow{17} |\, 138 \xrightarrow{2} |$

Table 5: First 115 entries of some narcissistic $\mathbb{Z}_{231}$ terraces (types A–F as in Table 2)

$\mathbb{Z}_{231}$; $(p,q,r) = (3,7,11)$
A. $1 \xrightarrow{170} \mid 32 \xrightarrow{170} \mid 126 \xrightarrow{4} \mid 133 \xrightarrow{5} \mid 171 \xrightarrow{46} \mid 198 \xrightarrow{3} : \mid 187 \xrightarrow{2} \mid{:}\ 77\ : \mid$
B. $1 \xleftarrow{2} \mid 3 \xrightarrow{46} \mid 221 \xrightarrow{2} \mid 28 \xleftarrow{14} \mid 143 \xleftarrow{2} \mid{:}\ 132 \xleftarrow{3} : \mid{:}\ 154\ : \mid{:}\ 21 \xrightarrow{2} : \mid$
C. $119 \xleftarrow{5} \mid 1 \xleftarrow{2} \mid 3 \xrightarrow{30} \mid 113 \xrightarrow{2} \mid 121 \xleftarrow{2} \mid{:}\ 165 \xleftarrow{3} : \mid{:}\ 77\ : \mid{:}\ 63 \xrightarrow{2} : \mid$
D. $:\ 77\ : \mid{:}\ 165 \xrightarrow{3} : \mid 176 \xrightarrow{2} \mid 1 \xleftarrow{2} \mid 3 \xrightarrow{18} \mid 137 \xrightarrow{2} \mid 217 \xleftarrow{5} \mid{:}\ 105 \xrightarrow{2} : \mid$
E. $1 \xleftarrow{149} \mid{:}\ 66 \xrightarrow{5} : \mid 127 \xrightarrow{149} \mid 144 \xrightarrow{72} \mid 168 \xleftarrow{4} \mid 161 \xleftarrow{5} \mid{:}\ 154\ : \mid 55 \xrightarrow{2} \mid$
F. $203 \xrightarrow{26} \mid{:}\ 154\ : \mid 220 \xleftarrow{2} \mid{:}\ 198 \xleftarrow{3} : \mid 1 \xleftarrow{74} \mid{:}\ 147 \xleftarrow{2} : \mid 61 \xrightarrow{74} \mid 108 \xrightarrow{2} \mid$

$\mathbb{Z}_{231}$; $(p,q,r) = (3,11,7)$
A. impossible
B. $1 \xleftarrow{2} \mid 3 \xleftarrow{3} \mid 160 \xrightarrow{2} \mid 143 \xleftarrow{2} \mid 49 \xrightarrow{5} \mid 210 \xrightarrow{3} \mid{:}\ 154\ : \mid 66 \xrightarrow{2} \mid$
C. $44 \xrightarrow{2} \mid 1 \xleftarrow{47} \mid 93 \xleftarrow{3} \mid 218 \xrightarrow{47} \mid 112 \xrightarrow{26} \mid 105 \xleftarrow{3} \mid{:}\ 154\ : \mid 132 \xrightarrow{2} \mid$
D. $:\ 154\ : \mid 84 \xrightarrow{4} \mid 98 \xrightarrow{20} \mid 1 \xleftarrow{26} \mid 51 \xrightarrow{51} \mid 76 \xrightarrow{26} \mid 11 \xleftarrow{2} \mid 33 \xrightarrow{2} \mid$
E. impossible
F. $209 \xleftarrow{2} \mid{:}\ 154\ : \mid 91 \xrightarrow{26} \mid 147 \xrightarrow{4} \mid 1 \xleftarrow{149} \mid{:}\ 66 \xrightarrow{3} : \mid 193 \xrightarrow{149} \mid 177 \xrightarrow{2} \mid$

4 Matryoshka terraces

When we examine the 17 segments of a terrace of type A (see again Table 2), we find that the 7 segments in the middle (*i.e.* the zero segment, the 3 segments on its left, and the 3 segments on its right) cover **all** multiples of r in $\mathbb{Z}_n$. Furthermore, if we divide all the elements in these 7 segments by r we obtain a terrace for $\mathbb{Z}_{pq}$. In this sense, we have a terrace for $\mathbb{Z}_{pq}$ *nested* in a terrace for $\mathbb{Z}_{pqr}$. In the same sense, a terrace for $\mathbb{Z}_p$ is nested in the terrace for $\mathbb{Z}_{pq}$ and thus in the terrace for $\mathbb{Z}_{pqr}$. Thus the terraces of type A are *matryoshka* (Russian doll) terraces as defined in [5, §6]. The idea behind the terminology of matryoshka goes a long way back in the theory of designs: in 1847 Kirkman [12] showed how to construct a Steiner triple system of order $2n+1$ containing a Steiner triple system of order n, whenever such a system of order n exists.

In the obvious notation used in [5], the terraces of type A are

$$(pqr \supset pq \supset p) \text{ matryoshka terraces.}$$

Similarly, the terraces of types B, C, and D are

$$(pqr \supset r) \text{ matryoshka terraces.}$$

Table 6: Numbers of starting segments for type A matryoshka terraces

n	factors	number	constraint
105	$3 \cdot 5 \cdot 7$	$8,325$	$a = 1$
165	$3 \cdot 5 \cdot 11$	$201,600$	$a = 1$
231	$3 \cdot 7 \cdot 11$	$100,107$	$a = 1, \alpha = \beta$
285	$3 \cdot 5 \cdot 19$	$188,100$	$a = 1, \alpha = \beta$
345	$3 \cdot 5 \cdot 23$	$5,085$	$a = 1, \alpha = \beta, \gamma = \delta$

There also exist $(pqr \supset pq \supset p)$ matryoshka terraces in which the units segments are not at the ends. Enthusiastic readers may wish to find examples of these.

All these matryoshka terraces, as well as those previously appearing in the literature have the successive internal terraces exactly at their centre. We call such matryoshka terraces *perfect*. In the next section we construct both perfect and imperfect matryoshka terraces.

Returning to type A perfect matryoshka terraces, observe that once we have specified the first five segments the problem reduces to looking for matryoshka terraces for $\mathbb{Z}_{pq}$ of which many are available, for example in [5]. Those first five segments are defined by ten numbers:

$$a \overset{\alpha}{\to} \mid b \overset{\beta}{\to} \mid c \overset{\gamma}{\to} \mid d \overset{\delta}{\to} \mid e \overset{\epsilon}{\to} \quad .$$

Without loss of generality, we can assume that $a = 1$.

A computer search reveals that there are usually many valid ways to build these first five segments for admissable values of n up to 450. Table 6 records the result of the search, where the constraint column indicates conditions imposed beyond the arithmetic ones that are required by the matryoshka terrace definition and Type A structure. Table 7 gives an example for each order. There are no type A examples for $n \in \{357, 429\}$.

5 A combinatorial construction

Our combinatorial method for constructing matryoshka terraces is similar in style to one introduced by Bailey [10] to find terraces for abelian groups of odd order that has already seen variations that construct terraces for

Table 7: Examples of starting segments for type A matryoshka terraces

n	factors	segments
105	$3 \cdot 5 \cdot 7$	$1 \xrightarrow{17} \mid 23 \xrightarrow{17} \mid 60 \xrightarrow{4} \mid 10 \xrightarrow{2} \mid 72 \xrightarrow{23}$
165	$3 \cdot 5 \cdot 11$	$1 \xrightarrow{17} \mid 23 \xrightarrow{17} \mid 135 \xrightarrow{9} \mid 20 \xrightarrow{14} \mid 123 \xrightarrow{13}$
231	$3 \cdot 7 \cdot 11$	$1 \xrightarrow{86} \mid 32 \xrightarrow{86} \mid 189 \xrightarrow{3} \mid 7 \xrightarrow{14} \mid 87 \xrightarrow{46}$
285	$3 \cdot 5 \cdot 19$	$1 \xrightarrow{17} \mid 53 \xrightarrow{17} \mid 255 \xrightarrow{4} \mid 10 \xrightarrow{47} \mid 102 \xrightarrow{78}$
345	$3 \cdot 5 \cdot 23$	$1 \xrightarrow{17} \mid 98 \xrightarrow{17} \mid 15 \xrightarrow{3} \mid 10 \xrightarrow{62} \mid 102 \xrightarrow{48}$

non-abelian groups [1] and for narcissistic terraces [16]. In each of these applications, it was essential to be able to find "starter-translate" terraces in order to use the method. The variation we present here requires a slightly different property.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_s)$ be a terrace for $\mathbb{Z}_s$, where s is odd, with 2-sequencing $\mathbf{b} = (b_1, b_2, \ldots, b_{s-1})$. Then $\mathbf{a}$ is *starter-translate* if for each $x \in \mathbb{Z}_s \setminus \{0\}$ we have exactly one occurrence from the set $\{x, -x\}$ among the 2-sequencing elements

$$b_1, b_3, b_5, \ldots, b_{s-2}.$$

That is, each element, up to sign, appears once in the odd positions and hence also once in the even positions.

Here is our variation, which has two cases depending on the parity of $\frac{s-1}{2}$. Let $\mathbf{a}$ be a terrace with the notation of the previous paragraph. If $s \equiv 1 \pmod 4$ then $\mathbf{a}$ is *starter-translate with a jump* or, more briefly, a *jumpstarter terrace* if for each $x \in \mathbb{Z}_s \setminus \{0\}$ we have exactly one occurrence from the set $\{x, -x\}$ among the 2-sequencing elements

$$b_1, b_3, b_5, \ldots, b_{(s-3)/2}, b_{(s+3)/2}, b_{(s+7)/2}, \ldots, b_{s-1}.$$

If $s \equiv 3 \pmod 4$ then $\mathbf{a}$ is a *jumpstarter terrace* if for each $x \in \mathbb{Z}_s \setminus \{0\}$ we have exactly one occurrence from the set $\{x, -x\}$ among the 2-sequencing elements

$$b_1, b_3, b_5, \ldots, b_{(s-1)/2}, b_{(s+5)/2}, b_{(s+9)/2}, \ldots, b_{s-1}.$$

Roughly, in both cases each element, up to sign, appears once among the odd positions in the first half of the 2-sequencing and once among the even positions in the second half. Hence they also each occur once among the even positions in the first half and odd positions in the second half.

Example 5.1 *The Owens terrace, as described in [3], is a jumpstarter terrace for* $\mathbb{Z}_s$*. When* $s \equiv 1 \pmod 4$ *it is given by*

$$(0, 1, s-2, 3, s-4, \ldots, (s-3)/2, (s+1)/2;\\ (s+3)/2, (s-1)/2, (s+7)/2, (s-5)/2, \ldots, s-1, 2)$$

where the semi-colon indicates a switch in the pattern. The corresponding 2*-sequencing is*

$$(1, -3, 5, -7, \ldots, -4, 2; 1; -2, 4, -6, 8, \ldots, -5, 3).$$

When $s \equiv 3 \pmod 4$ *It is given by*

$$(0, 1, s-2, 3, s-4, \ldots, (s+3)/2, (s-1)/2;\\ (s+1)/2, (s+5)/2, (s-3)/2, (s+7)/2, \ldots, s-1, 2)$$

with 2*-sequencing*

$$(1, -3, 5, -7, \ldots, 4, -2; 1; 2, -4, 6, -8, \ldots, -5, 3).$$

For instance, the Owens terrace for $\mathbb{Z}_{13}$ *is*

$$(0, 1, 11, 3, 9, 5, 7, 8, 6, 10, 4, 12, 2)$$

with 2-sequencing

$$(1, 10, 5, 6, 9, 2, 1, 11, 4, 7, 8, 3).$$

The Owens terrace for $\mathbb{Z}_{15}$ *is*

$$(0, 1, 13, 3, 11, 5, 9, 7, 8, 10, 6, 12, 4, 14, 2)$$

with 2-sequencing

$$(1, 12, 5, 8, 9, 4, 13, 1, 2, 11, 6, 7, 10, 3).$$

Although jumpstarter terraces are new, in the case $s \equiv 1 \pmod 4$ they generalise an existing terrace property introduced in [2] and named in [3]. A terrace for $\mathbb{Z}_s$, where s is odd, is *echoing* if its 2-sequencing is of the form

$$(b_1, b_2, \ldots, b_{(s-1)/2}, \pm b_1, \pm b_2, \ldots, \pm b_{(s-1)/2}).$$

When $s \equiv 1 \pmod 4$ an echoing terrace for $\mathbb{Z}_s$ is a jumpstarter terrace.

At which orders s the cyclic group $\mathbb{Z}_s$ admits an echoing terrace is an open question. When s is a prime that has 2 as a primitive root we have the following echoing power-sequence terrace [3, 15]:

$$(0, (s+1)/2, (s+1)/4, \ldots, 4, 2, 1).$$

A power-sequence construction is also known for $s \equiv 5 \pmod 8$ when $s-2$ is a prime with 2 as a primitive root [8].

We can now give the main constructions. Unlike those in earlier sections and the broader literature, the matryoshka terraces it produces are not narcissistic and not always perfect. For clarity, we give a simpler direct product construction first and then generalise this in Theorem 5.4.

Theorem 5.2 *Let s and t be odd with* $\gcd(s,t) = 1$. *Given a jumpstarter terrace* $\mathbf{a}$ *for* $\mathbb{Z}_s$ *and a terrace* $\mathbf{c}$ *for* $\mathbb{Z}_t$ *we can construct a* $(st \supset t)$ *matryoshka terrace for* $\mathbb{Z}_{st}$ *with* $\mathbf{c}$ *as the inner terrace. If* $s \equiv 1 \pmod 4$ *then the constructed matryoshka terrace is perfect.*

Proof. Let $\mathbf{a} = (a_1, a_2, \ldots a_s)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_t)$ and let their 2-sequencings be $\mathbf{b} = (b_1, b_2, \ldots b_{s-1})$ and $\mathbf{d} = (d_1, d_2, \ldots, d_{t-1})$ respectively. We construct a terrace with the desired properties for $\mathbb{Z}_s \times \mathbb{Z}_t \cong \mathbb{Z}_{st}$. Figure 1 illustrates the construction when $s = 9$ and $t = 5$ and is typical of the general case for $s \equiv 1 \pmod 4$; Figure 2 does the same for $s = 7$ and $t = 5$ and is typical of the general case for $s \equiv 3 \pmod 4$.

We describe the construction for $s \equiv 1 \pmod 4$ first. In the first coordinates of the first $2t$ elements we alternate between a_1 and a_2, using them each t times in total. Similarly, in the first coordinates of the next $2t$ elements we alternate between a_3 and a_4. Repeat until the end of the alternation between $a_{(s-3)/2}$ and $a_{(s-1)/2}$. The next t elements have first coordinate $a_{(s+1)/2}$. Now resume alternating, with the next $2t$ elements alternating between $a_{(s+3)/2}$ and $a_{(s+5)/2}$ and keep doing so until the last $2t$ elements have alternated between a_{s-1} and a_s.

In the second coordinates we put all of $\mathbf{c}$ in order, then all of $\mathbf{c}$ in reverse order and keep alternating between these.

Thinking of this as a path through $\mathbb{Z}_s \times \mathbb{Z}_t$ we get a picture much like that in Figure 1. The path visits each vertex once and so uses each element of $\mathbb{Z}_s \times \mathbb{Z}_t$ once.

Figure 1: The construction of Theorem 5.2 when $s = 9$ and $t = 5$.

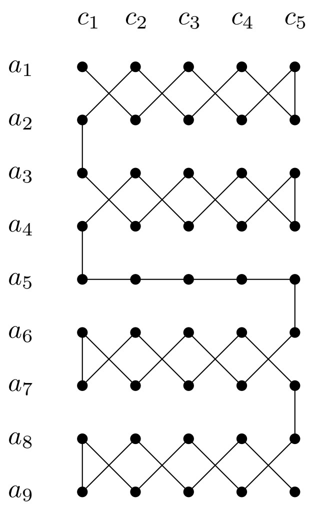
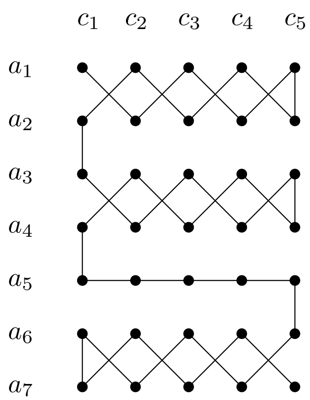


Figure 2: The construction of Theorem 5.2 when $s = 7$ and $t = 5$.



Now consider the sequence of differences. Differences of the form $(0, y)$ arise from horizontal lines. These appear the correct number of times as the central t elements are $(a_{(s+1)/2}, c_1), (a_{(s+1)/2}, c_2), \ldots, (a_{(s+1)/2}, c_t)$ and these are the only ones to give rise to a 0 in the first coordinate of the differences. This also guarantees the matryoshka property and, after a translation by $(-a_{(s+1)/2}, 0)$, that the inner terrace is a copy of $\mathbf{c}$.

Elements of the form $(x, 0)$ arise from vertical lines. These appear at and only at positions $t, 2t, \ldots, (s-1)t$ and these elements are

$$(b_1, 0), (b_2, 0), \ldots, (b_{s-1}, 0)$$

respectively. As $\mathbf{a}$ is a terrace we have the required number of elements of the form $(x, 0)$ among the differences.

Elements of the form (x, y) with $x \neq 0 \neq y$ arise from diagonal lines. These diagonal lines come in pairs in an "$\times$" shape in which the two lines are either both traverseed upwards or both traversed downwards.

Such an $\times$ in a cell with top-left corner (a_i, c_j) and bottom-left corner (a_{i+1}, c_{i+1}) will give rise to (b_i, d_j) and $(b_i, -d_j)$ (when traversed downwards) or $(-b_i, d_j)$ and $(-b_i, -d_j)$ (when traversed upwards) . When there is such an $\times$, we will have exactly one more with $(\pm b_i, d_{j'})$ and $(\pm b_i, -d_{j'})$ where $d_{j'} = \pm d_j$. Again the signs on the b_i depend on the the direction the lines are traversed, but again they are the same as each other. This means that we have at least enough occurrences of the type $(\pm b_i, \pm d_j)$ whenever there is an $\times$ corresponding to $\pm b_i$. Further, the jumpstarter property of $\mathbf{a}$ guarantees that we will not have any such $\times$ with the $b_{i'}$ such that $b_{i'} = \pm b_i$. Therefore we have each element of the form (x, y) with $x \neq 0 \neq y$ an appropriate number of times in the sequence of differences.

The construction and argument for $s \equiv 3 \pmod 4$ are very similar. The only difference in the construction is that initial alternations between a_1 and a_2, then a_3 and a_4 are repeated until the end of the alternation between $a_{(s-1)/2}$ and $a_{(s+1)/2}$. They restart with alternations between $a_{(s+5)/2}$ and $a_{(s+7)/2}$. Figure 2 illustrates this case of the construction.

The argument that this gives a matryoshka terrace is the same as before. However, in this case the matryoshka terrace is not perfect as the copy of $\mathbf{c}$ starts at position $t(s+1)/2 + 1$ of the terrace. $\square$

In the above proof the condition that $\gcd(s, t) = 1$ is used only to make the resultant group cyclic: $\mathbb{Z}_s \times \mathbb{Z}_t \cong \mathbb{Z}_{st}$. Without this condition the construction still gives matryoshka terraces (with the obvious definition) for non-cyclic groups.

Example 5.3 *Take the centred Owens jumpstarter terrace* $\mathbf{a}$ *for* $\mathbb{Z}_9$ *and let* $\mathbf{c}$ *be the following centred terrace for* $\mathbb{Z}_5$*:*

$$\begin{aligned}\mathbf{a} &= (4,5,2,7,0,1,8,3,6)\\ \mathbf{c} &= (1,2,0,3,4)\end{aligned}$$

(The terrace $\mathbf{c}$ *is an example of the first known infinite family of terraces called the Lucas-Walecki-Williams terraces; see [14], for example, for more details.) We construct a* $(45 \supset 5)$ *matryoshka terrace for* $\mathbb{Z}_{45}$ *from these using Theorem 5.2. (We leave* $\mathbb{Z}_{45}$ *written as* $\mathbb{Z}_9 \times \mathbb{Z}_5$ *to make it easier to track the method of construction.)*

$$(4,1),(5,2),(4,0),(5,3),(4,4),(5,4),(4,3),(5,0),(4,2),(5,1),$$
$$(2,1),(7,2),(2,0),(7,3),(2,4),(7,4),(2,3),(7,0),(2,2),(7,1),$$
$$(0,1),(0,2),(0,0),(0,3),(0,4),$$
$$(1,4),(8,3),(1,0),(8,2),(1,1),(8,1),(1,2),(8,0),(1,3),(8,4),$$
$$(3,4),(6,3),(3,0),(6,2),(3,1),(6,1),(3,2),(6,0),(3,3),(6,4).$$

We now remove the need for the coprimality condition:

Theorem 5.4 *Let* s *and* t *be odd. Given a jumpstarter terrace* $\mathbf{a}$ *for* $\mathbb{Z}_s$ *and a terrace* $\mathbf{c}$ *for* $\mathbb{Z}_t$ *we can construct a* $(st \supset t)$ *matryoshka terrace for* $\mathbb{Z}_{st}$ *with* $\mathbf{c}$ *as the inner terrace. If* $s \equiv 1 \pmod 4$ *then the constructed matryoshka terrace is perfect.*

Proof. The construction is very similar to that of Theorem 5.2. We embed $\mathbb{Z}_t$ in $\mathbb{Z}_{st}$ in the natural way, take $\mathbf{c}$ to be the terrace as before, and choose elements of the cosets in $\mathbb{Z}_{st}/\mathbb{Z}_t$ to be "compatible" with the terrace $\mathbf{a}$ to make the construction work. This generalises Theorem 5.2 analgously to the generalisation of Bailey's direct product construction for starter-translate terraces of Anderson and Ihrig [1, 10].

Let $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_s)$ be a sequence in $\mathbb{Z}_{st}$ with differences $(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{s-1})$ given by $\bar{b}_i = \bar{a}_{i+1} - \bar{a}_i$ for each i. Say that $\bar{\mathbf{a}}$ is *compatible* with the terrace $\mathbf{a} = (a_1, a_2, \ldots a_s)$ in $\mathbb{Z}_{st}/\mathbb{Z}_t \cong \mathbb{Z}_s$ if each $\bar{a}_i$ is in the coset a_i and whenever $b_i = b_j$ in the differences of $\mathbf{a}$ we have $\bar{b}_i = \bar{b}_j$ and whenever we have $b_i = -b_j$ in the differences of $\mathbf{a}$ we have $\bar{b}_i = -\bar{b}_j$. Given a terrace for $\mathbb{Z}_s$ we can always choose a compatible sequence for it in $\mathbb{Z}_{st}$ by selecting each element in turn, observing the conditions as we proceed [1].

Now, performing the construction of Theorem 5.2 on $\bar{\mathbf{a}}$ and $\mathbf{c}$ gives us exactly the terrace we need. The compatibility condition ensures that the $\times$ pieces give us the required differences for very similar reasons to the direct product case. $\square$

Example 5.5 *Take* $\mathbf{a}$ *to be the following compatible choice in* $\mathbb{Z}_{25}$ *of the Owens terrace for* $\mathbb{Z}_5$ *and* $\mathbf{c}$ *to be the centred LWW terrace as in the previous example, but now the appropriate subgroup for* $\mathbf{c}$ *is* $\{0, 5, 10, 15, 20, 25\}$*:*

$$\begin{aligned} \mathbf{a} &= (2, 23, 0, 4, 6) \\ \mathbf{c} &= (5, 10, 0, 15, 20) \end{aligned}$$

Then Theorem 5.4 gives the following perfect $(25 \supset 5)$ *matryoshka terrace for* $\mathbb{Z}_{25}$*:*

$$(7, 8, 2, 13, 22, 18, 17, 23, 12, 3; 5, 10, 0, 15, 20; 24, 21, 4, 16, 9, 11, 14, 6, 19, 1).$$

We defined jumpstarter terraces with the "jump" as close to the middle of the terrace as possible in order to place the internal terrace of the matryoshka terrace as centrally as possible. However, the jump may be placed anywhere and Theorems 5.2 and 5.4 still go through (with the exception that the terraces in the $1 \mod 4$ case will not be perfect). More precisely, the 2-sequencing $(b_1, b_2, \ldots, b_{s-1})$ of the terrace for $\mathbb{Z}_s$ (s odd) must contain exactly one occurrence from each set $\{x - x\}$ for $x \in \mathbb{Z}_s \setminus \{0\}$ among the elements

$$b_1, b_3, \ldots b_{y-2}, b_y; b_{y+3}, b_{y+5}, \ldots, b_{s-1}$$

for some odd y, where the semi-colon represents the jump. (We thank an anonymous referee for this observation.)

Theorem 5.4 allows us to construct matryoshka terraces with arbitrary nesting structures and perfect matryoshka terraces with arbitrary nesting structures provided that none but possibly the innermost terrace has order congruent to $3 \pmod 4$.

Corollary 5.6 *Let* $s_1, s_2, \ldots, s_k$ *be odd. There is a*

$$(s_1 s_2 \cdots s_k \supset s_2 \cdots s_k \supset \cdots \supset s_k)$$

matryoshka terrace for $\mathbb{Z}_{s_1 s_2 \cdots s_k}$*. Further, if* $s_1 \equiv s_2 \equiv \cdots \equiv s_{k-1} \equiv 1 \pmod 4$ *then there is a perfect matryoshka terrace for* $\mathbb{Z}_{s_1 s_2 \cdots s_k}$*.*

Proof. Repeatedly apply Theorem 5.4 using the Owens terrace as the jump-starter one. □

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