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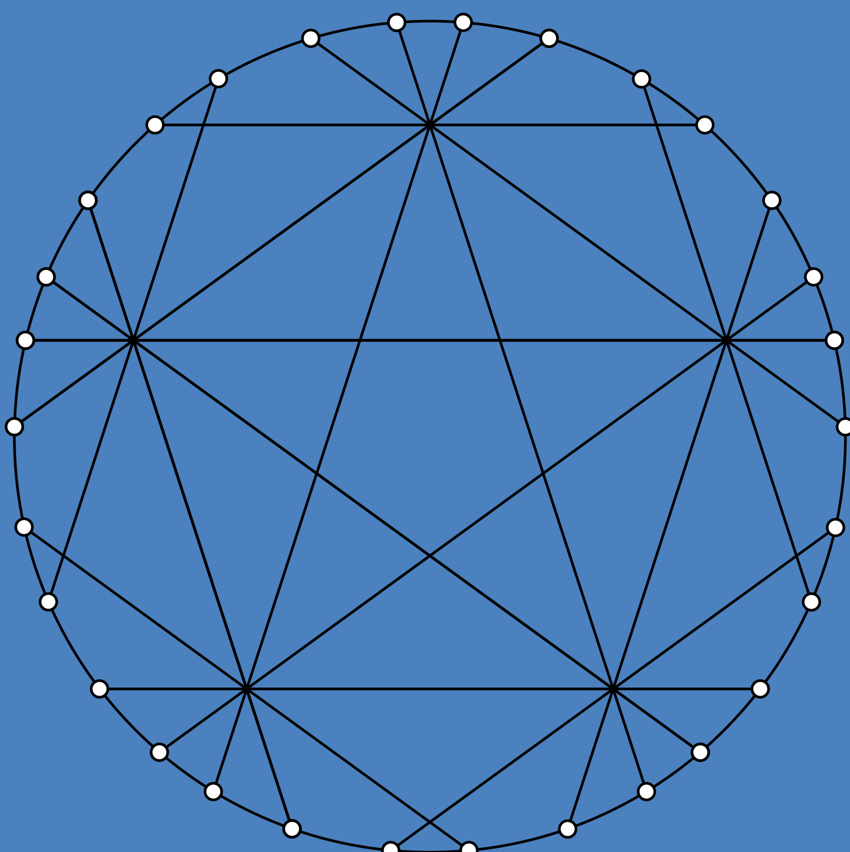

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Rainbow Cycle Designs

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Abstract: A k-edge-coloring of a graph G is a mapping from $E(G)$ into $\{1, 2, \ldots, k\}$. If, in addition, incident edges of G receive distinct colors, then the coloring is a proper edge-coloring. A subgraph H of an edge-colored graph G is a rainbow subgraph provided all the edges of H are of distinct colors.

In this paper, we study the existence of rainbow cycle designs of complete graphs in which the edge-coloring is either prescribed or arbitrarily given.

1 Introduction and preliminaries

In an edge-colored graph, a subgraph is monochromatic if all its edges receive the same color and the subgraph is a heterochromatic (or multicolored, rainbow) subgraph provided all its edges receive distinct colors. The study of finding either monochromatic or heterochromatic subgraphs in an edge-colored graph has a long history, see [10]. Most of the effort has been on finding monochromatic subgraphs in edge-colored complete graphs since the result is closely related to graph decompositions and the existence of Ramsey numbers.

As to finding rainbow subgraphs, we shall focus on the graphs which are properly edge-colored, i.e., incident edges receive distinct colors. Both of them are interesting topics in research. In this paper, we shall study the existence of rainbow cycles in properly edge-colored complete graphs. The following results are known.

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Theorem 1.1 ([5]). *For each $m \geq 3$, there exists a proper $(2m-1)$-edge-coloring such that K_{2m} can be decomposed into m isomorphic rainbow spanning trees.*

Theorem 1.2 ([6]). *For each $m \geq 2$, there exists a proper $(2m+1)$-edge-coloring of K_{2m+1} such that K_{2m+1} can be decomposed into m rainbow Hamilton cycles.*

An H-design of G is a decomposition of G such that all its members are isomorphic to H, denoted by $H \mid G$. Furthermore, if G is edge-colored and each member H is a rainbow subgraph, then we have a rainbow H-design of G, denoted by $H \mid_R G$. In case that $G \cong K_n$, we simply call it a rainbow H-design of order n.

Therefore, we obtain a rainbow Hamilton cycle design of order $2m+1$. Motivated by the above results, we are interested in the following problems.

Problem 1. *Can we find a $\chi'(G)$-edge-coloring and an H-decomposition of G such that each member of the decomposition is a rainbow subgraph?*

Problem 2. *Given a properly edge-coloring of G, can it be decomposed into subgraphs such that each member is isomorphic to H and also each member is a rainbow subgraph?*

Both of the above problems are easy to solve if the subgraphs do have certain structure, for example triangles and stars. But, it won't be that trivial if we have a larger subgraph to consider. Since our focus is on complete graph of order n, the H-decomposition obtained in Problem 1 will be referred to as a "*weak*" rainbow H-design of order n and the one obtained in Problem 2 is a "*strong*" rainbow H-design of order n.

In this study, we shall mainly focus on cycle decomposition of K_n. A graph G is called k-sufficient if the order of G, $|G| \geq k$, each vertex is of even degree and k divides the size of G. The following cycle-decomposition is well known.

Theorem 1.3 ([1]). *K_n (respectively $K_n - F$) can be decomposed into k-cycles if and only if K_n (respectively $K_n - F$) is k-sufficient. Here, F is a 1-factor of K_n provided n is even.*

2 The main results

Throughout of this section, for the prescribed edge-colorings we shall use the following edge-colorings. Since they are easy to see, we omit their proofs.

Lemma 2.1. *Let $V(K_{2m+1}) = \{v_i | i \in \mathbb{Z}_{2m+1}\}$. Let $\varphi : E(K_{2m+1}) \to \mathbb{Z}_{2m+1}$ be defined as follows: $\varphi(\{v_i, v_j\}) \equiv i + j \pmod{2m+1}$. Then, φ is a proper $(2m+1)$-edge-coloring of K_{2m+1}.*

Lemma 2.2. *Let $K_{m,m} = (A, B)$ where $A = \{a_i | i \in \mathbb{Z}_m\}$ and $B = \{b_i | i \in \mathbb{Z}_m\}$. Let π be an edge-coloring of $K_{m,m}$ such that $\pi(\{a_i, b_j\}) \equiv j - i \pmod{m}$. Then, π is a proper m-edge-coloring of $K_{m,m}$.*

For the decomposition of K_{2m+1} and $K_{m,m}$ we need the following labelings. For more information about this idea, the readers may refer to [11, 12].

Definition 2.3. Let f be a labeling of a graph H of size k defined on $\mathbb{Z}_{2k+1}$, i.e., $f : V(H) \to \mathbb{Z}_{2k+1}$. Let $e = \{x, y\}$ be an edge of H. Then, $\min\{(2k+1) - |f(x) - f(y)|, |f(x) - f(y)|\}$ is known as the (circular) difference of e. f is a ρ-labeling of H if all edges of H receive distinct differences. Furthermore, f is a bipartite p-labeling provided that f is a ρ-labeling and there exists a λ such that for each edge e, exactly one of the two vertices of e receives a label at most λ.

For example, if we use $(0, 4, 2, 3)$ to denote the labeling of (v_0, v_1, v_2, v_3), then we have a bipartite ρ-labeling defined on $\mathbb{Z}_9$.

It is well-known that if H is of size k and H has a ρ-labeling defined on $\mathbb{Z}_{2k+1}$, then $H \mid K_{2k+1}$. The construction is known as the cyclic decomposition. If, furthermore, H has a bipartite ρ-labeling defined on $\mathbb{Z}_{2k+1}$, then $H \mid K_{2tk+1}$ for all $t \in \mathbb{N}$. For convenience, we list the above results as lemmas.

Lemma 2.4 ([12]). *If $|E(H)| = k$ and H has a ρ-labeling defined on $\mathbb{Z}_{2k+1}$, then K_{2k+1} can be decomposed cyclically into copies of H. Moreover, if ρ is a bipartite labeling, then K_{2tk+1} can be decomposed cyclically into copies of H.*

In order to obtain rainbow designs of complete graphs, we need following.

Definition 2.5. Let f be a labeling of H defined on $\mathbb{Z}_{2k+1}$. Let $e = \{x, y\}$ be an edge of H. Then $f(x) + f(y) \pmod{2k+1}$ is called the weight of e. If the weights of all edges in H are distinct, then f is called a rainbow labeling of H.

The reason that this labeling is called "rainbow" is due to the edge-coloring we use for K_{2k+1} in Lemma 2.1.

Theorem 2.6. *For each $l = 4t$, $t \in \mathbb{N}$, C_l has a rainbow bipartite ρ-labeling defined on $\mathbb{Z}_{2l+1}$.*

Proof. The proof follows by giving the labelings.

Case 1. $t = 2k$, $k \in \mathbb{N}$.

For convenience, let $C_l = (a_0, b_0, a_1, b_1, \ldots, a_{2t-1}, b_{2t-1}) = (a_0, b_0, a_1, b_1, \ldots, a_{4k-1}, b_{4k-1})$. Let f be the labeling defined as follows:

$$f(a_i) = i, \quad 0 \le i \le 4k-1;$$

$$f(b_i) = \begin{cases} 8t - 3i - 3 & 0 \le i \le k-2; \\ 8t - 3i - 4 & k-1 \le i \le 3k-2; \\ 8t - 3i - 5 & 3k-1 \le i \le 4k-2; \\ 8t & i = 4k-1. \end{cases}$$

Now, it is a routine matter to check that f is indeed a rainbow bipartite ρ-labeling defined on $\mathbb{Z}_{2l+1}$.

Case 2. $t = 2k+1$, $k \in \mathbb{N} \cup \{0\}$.

Again we can find the labeling of $C_l = (a_0, b_0, a_1, b_1, \ldots, a_{4k+1}, b_{4k+1})$ directly. Let f be the labeling defined as follows:

$$f(a_i) = i, \quad 0 \le i \le 4k+1;$$

$$f(b_i) = \begin{cases} 8t - 3i - 3 & 0 \le i \le k-1; \\ 8t - 3i - 4 & k \le i \le 3k; \\ 8t - 3i - 5 & 3k+1 \le i \le 4k; \\ 8t & i = 4k+1. \end{cases}$$

□

For convenience of checking, we present two examples for reference.

Example 1. $t = 6$ $(k = 3)$

The labeling of C_{24} is $(0, 45, 1, 42, 2, 38, 3, 35, 4, 32, 5, 29, 6, 26, 7, 23, 8, 19, 9, 16, 10, 13, 11, 48)$. Therefore, the differences are: $\langle 4, 5, 8, 9, 13, 14, 17, 18, 21, 22, 24, 23, 20, 19, 16, 15, 11, 10, 7, 6, 3, 2, 12, 1\rangle$ and the weights (mod 49) are: $\langle 45, 46, 43, 44, 40, 41, 38, 39, 36, 37, 34, 35, 32, 33, 30, 31, 27, 28, 25, 26, 23, 24, 10, 48\rangle$.

Example 2. $t = 9$, $t = 2k + 1$. $(k = 4)$

The labeling of C_{36} is $(0, 69, 1, 66, 2, 63, 3, 60, 4, 56, 5, 53, 6, 50, 7, 47, 8, 44, 9, 41, 10, 38, 11, 35, 12, 32, 13, 28, 14, 25, 15, 22, 16, 19, 17, 72)$. Therefore, the differences are: $\langle 4, 5, 8, 9, 12, 13, 16, 17, 21, 22, 25, 26, 29, 30, 33, 34, 36, 35, 32, 31, 28, 27, 24, 23, 20, 19, 15, 14, 11, 10, 7, 6, 3, 2, 18, 1\rangle$ and the weights are: $\langle 69, 67, 68, 65, 66, 63, 64, 60, 61, 58, 59, 56, 57, 54, 55, 52, 53, 50, 51, 48, 49, 46, 47, 44, 45, 41, 42, 39, 40, 37, 38, 35, 36, 18, 72\rangle$.

Corollary 2.7. *For each $l = 4t$, $k \in \mathbb{N}$, there exists a weak rainbow l-cycle design of order $2lk + 1$.*

Proof. By adapting the coloring defined in Lemma 2.1 and the bipartite ρ-labeling of C_l $(l = 4t)$ defined on $\mathbb{Z}_{2l+1}$ (Theorem 2.6), we conclude the proof. $\square$

As a matter of fact, if $l = 2^s$, $s \geq 2$, then an l-cycle design of order n exists if and only if $n \equiv 1 \pmod{2l}$. Therefore, we also have the following

Corollary 2.8. *For $l = 2^s$, $s \geq 2$, a weak rainbow l-cycle design of order n exists if and only if $n \equiv 1 \pmod{2l}$.*

For the case that the cycle length is odd, we also obtain a rainbow ρ-labeling for $l \equiv 3 \pmod 4$.

Theorem 2.9. *There exists a rainbow ρ-labeling for $l \equiv 3 \pmod 4$ defined on $\mathbb{Z}_{2l+1}$.*

Proof. We present its labeling directly.

Let $l = 4t + 3$ and the labeling f be defined on $\mathbb{Z}_{2l+1}$ as follows,

$$f(a_i) = i, \quad 0 \le i \le 2t + 1;$$
$$f(b_i) = \begin{cases} 8t - 3i + 4, & 0 \le i \le \lceil \frac{t-3}{2} \rceil; \\ 8t - 3i + 3, & \lceil \frac{t-1}{2} \rceil \le i \le \lceil \frac{3t-1}{2} \rceil; \\ 8t - 3i + 2, & \lceil \frac{3t+1}{2} \rceil \le i \le 2t. \end{cases}$$

Then, it is a routine mater to check that f is a rainbow ρ-labeling defined on $\mathbb{Z}_{2l+1}$. □

Example 3. $l = 4t + 3$ and $t = 6$.

The labeling of C_{27} is $(0, 52, 1, 49, 2, 46, 3, 42, 4, 39, 5, 36, 6, 33, 7, 30, 8, 27, 9, 24, 10, 20, 11, 17, 12, 14, 13)$. Therefore, the differences are: $\langle 3, 4, 7, 8, 11, 12, 16, 17, 20, 21, 24, 25, 27, 26, 23, 22, 19, 18, 15, 14, 10, 9, 6, 5, 2, 1, 13\rangle$ and the weights (mod 55) are $\langle 52, 53, 50, 51, 48, 49, 45, 46, 43, 44, 41, 42, 39, 40, 37, 38, 35, 36, 33, 34, 30, 31, 28, 29, 26, 27, 13\rangle$.

Since a cyclic shift will not change the order of differences and also keep the weights distinct, we have the following corollary.

Corollary 2.10. *There exists a weak a rainbow l-cycle design of order $2l + 1$ for each $l \equiv 3 \pmod 4$.*

We remark here that for cycle length $l \equiv 1$ or $2 \pmod 4$, there are weak rainbow cycle designs for small orders. But, no systematic decomposition is obtained at this moment.

3 Decomposing $K_{2m,2m}$

If $l \equiv 0 \pmod 4$ and $K_{kl,kl}$ is edge-colored using the coloring given in Lemma 2.2, then $K_{kl,kl}$ can be decomposed cyclically into rainbow l-cycles.

Theorem 3.1. *If $l \equiv 0 \pmod 4$, then $C_l \mid_R K_{kl,kl}$ for each $k \ge 1$.*

Proof. It suffices to find a cycle C_l in $K_{l,l}$ such that all the l edges are of l distinct differences and thus distinct colors. For convenience, let $l = 4t$, $V(K_{l,l}) = A \cup B$ where $A = \{a_i | i \in \mathbb{Z}_l\}$ and $B = \{b_i | i \in \mathbb{Z}_l\}$. Therefore, the bipartite difference of a_i and b_j is $i - j \pmod l$. Since the pattern of the cycle can be seen easily, we list the cycle directly: $Z_1 : (a_0, b_l, a_1, b_{l-1}, \ldots, a_{t-1}, b_{3t}, a_{t+1}, b_{3t-1}, a_{t+1}, b_{3t-2}, \ldots, a_{2t}, b_{2t})$. So, the differences are: $\langle l, l-1, l-2, \ldots, 2t+1, 2t-1, 2t-2, \ldots, 1, 0, 2t\rangle$ as desired.

Now, for the C_l-decomposition of $K_{kl,kl}$, we define Z_j for $2 \leq j \leq k$. Since Z_1 is obtained in a bipartite graph, Z_j can be defined as follows: (1) Fixed a_j for $j \in \mathbb{Z}_l$ and (2) add $(j-1)l$ to the indices of b_i's. Hence, in Z_j, the differences are: $\langle jl, jl-1, \ldots, 1, 0, 2t+(j-1)l\rangle$. This implies that $K_{kl,kl}$ can be decomposed in l-cycles cyclically. Furthermore, each cycle is a rainbow cycle. $\square$

For the case $l \equiv 2 \pmod 4$, we have a weaker results.

Theorem 3.2. *If $l \equiv 2 \pmod 4$, then $C_l \mid_R K_{kl,kl}$ for each even integer k.*

Proof. Let $l = 4t+2$. It suffices to find two base cycles C and D for the decomposition of $K_{2l,2l}$ into l-cycles. Note that the differences we reserve for them are $\{0, 1, 2, \ldots, l-2, l\}$ for C and $\{l-1, l+1, l+2, \ldots, 2l-1\}$ for D. Now, by using the notation in Theorem 3.1, let

$$C = (a_0, a_l, a_2, b_{l-1}, a_3, b_{l-2}, \ldots, a_{\frac{l}{2}+1}, b_{\frac{l}{2}+1})$$

and

$$\begin{aligned} D &= (a_0, b_{2l-1}, a_1, b_{2t-2}, \ldots, a_t, b_{2l-t-1}, a_{t+2}, b_{2l-t-2}, \\ &\qquad \ldots, a_{\frac{l}{2}-1}, b_{2l-2t}, a_{2t+1}, b_{2l-2t-2}) \\ &= (a_0, b_{8t+3}, a_1, b_{8t+2}, \ldots, a_t, b_{7t+3}, a_{t+2}, b_{7t+2}, \ldots, a_{2t}, b_{6t+4}, a_{2t+1}, b_{6t+2}). \end{aligned}$$

Since all the differences in C and D are all distinct, the proof follows by a similar argument as that in Theorem 3.1. $\square$

By using Theorem 3.2, and the fact that

$$C_6 \mid_R K_{13} \quad \text{(base cycle: } (0, 5, 6, 8, 11, 4))$$

and

$$C_{10} \mid_R K_{21} \quad \text{(base cycle: } (0, 11, 20, 7, 6, 3, 1, 15, 9, 5)),$$

we are able to obtain the following results:

Proposition 3.3. *$C_6 \mid_R K_{12t+1}$ for each $t \in \mathbb{N}$.*

Proof. Let $V(K_{12t+1}) = \{\infty\} \cup \{(i,j) | i \in \mathbb{Z}_t \text{ and } j \in \mathbb{Z}_{12}\}$. Since both K_{13} and $K_{12,12}$ can be decomposed into 6-cycles, $C_6 \mid K_{12t+1}$. For convenience, we shall use the base cycles used in $C_6 \mid_R K_{13}$ and $C_6 \mid_R K_{12,12}$ respectively. Now, all we need is an edge-coloring of K_{12t+1} which uses $12t+1$ colors.

Case 1. t is even.

Color each complete graph induced by $\{\infty\} \cup \{(i,j) | j \in \mathbb{Z}_{12}\}$ for $i \in \mathbb{Z}_t$ with colors $\{0, 1, 2, \ldots, 12\}$ using Lemma 2.1. Let π be a $(t-1)$-edge-coloring of K_t where $V(K_t) = \mathbb{Z}_t$. Let the set of colors be $\{1, 2, \ldots, t-1\}$.

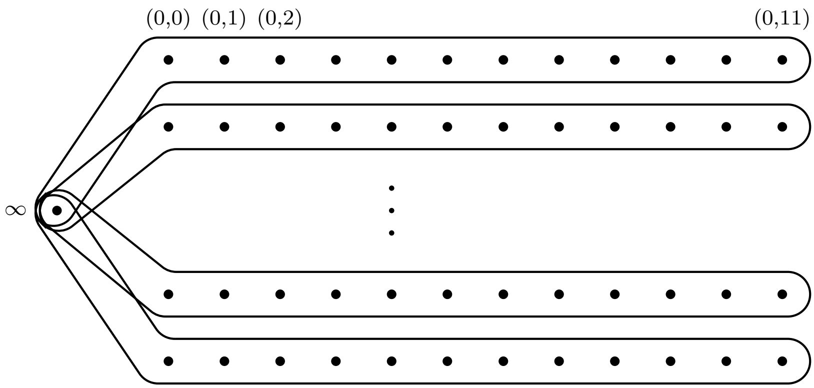


Figure 1: Basic construction $K_{12,12}$

Now, color the complete bipartite graph induced by $\{(i_1, j) | j \in \mathbb{Z}_{12}\} \cup \{(i_2, j) | j \in \mathbb{Z}_{12}\}$ with $\pi(\{i_1, i_2\}) \cdot 12 + j$, $j \in \mathbb{Z}_{12}$. (Using Lemma 2.2)

Since all the 6-cycles obtained in either K_{13} or $K_{12,12}$ are rainbow 6-cycles, the proof follows.

Case 2. t is odd.

We adjust the edge-coloring accordingly. First, we find a t-edge-coloring of K_t with $V(K_t) = \{v_i | i \in \mathbb{Z}_t\}$, called π. Let the set of colors used be $\{0, 1, \ldots, t-1\}$. Clearly, for each color $s \in \mathbb{Z}_t$, s is missing at exactly one vertex v_s. Now, we reserve the set colors $\{0\} \cup \{s \cdot 12 + j + 1 \mid j \in \mathbb{Z}_{12}\}$ for the complete graph K_{13} induced by $\{\infty\} \cup \{(s, j) | j \in \mathbb{Z}_{12}\}$ (using Lemma 2.1). Furthermore, we use $\{s \cdot 12 + j + 1 | j \in \mathbb{Z}_{12}\}$ to color the complete bipartite graph $K_{12,12}$ induced by $\{(i_1, j) | j \in \mathbb{Z}_{12}\} \cup \{(i_2, j) | j \in \mathbb{Z}_{12}\}$ (using Lemma 2.2) where $\pi(\{v_{i_1}, v_{i_2}\}) = s$. Based on this edge-coloring, all 6-cycles in the decomposition are rainbow 6-cycles, this concludes the proof of this case. □

Proposition 3.4. *$C_{10} \mid_R K_{20t+1}$ for each $t \in \mathbb{N}$.*

Proof. It follows by the same argument. □

In fact, this idea can be applied to find $C_l \mid_R K_{2lt+1}$ as long as we know that both $C_l \mid_R K_{2l+1}$ and $C_l \mid_R K_{2l,2l}$.

By way of the same technique, we are able to find some weak rainbow cycle designs of $K_{2m} - F$ where F is a 1-factor of K_{2m}. The following cycle decomposition is known.

Theorem 3.5 ([1])**.** *For all $k \geq 3$, $C_k \mid K_{2m} - F$ if and only if $K_{2m} - F$ is k-sufficient.*

Proposition 3.6. *A weak rainbow l-cycle design of $K_{2l+2} - F$ exists provided $l \equiv 0$ or $3 \pmod 4$.*

Proof. First, we give an edge-coloring of $K_{2l+2} - F$ which uses $2l$ colors. Let $V(K_{2l+2}) = \mathbb{Z}_{2l+2}$. By letting $\{2l + 1, i\}$ be colored i and retain the edge-coloring defined for K_{2l+1} in Lemma 2.1, we have $(2l+1)$-edge-coloring of K_{2l+2}. Subsequently, we can delete the edges colored $2l + 1$ to obtain a $2l$-edge-coloring of $K_{2l+2} - F$.

Now, in fact, we can use similar base cycles in Theorem 2.6 and 2.9 to construct weak rainbow l-cycle designs of $K_{2l+2} - F$ for respective cases $l \equiv 0$ or $3 \pmod 4$.

For $l \equiv 0 \pmod 4$, we use $l = 24$ as an example to explain the similarity between them. In $C_{24} \mid_R K_{49}$, the labeling of C_{24} is

$$(0, 46, 1, 43, 2, 39, 3, 36, 4, 33, 5, 29, 6, 26, 7, 23, 8, 19, 9, 16, 10, 13, 11, 49)$$

and the labeling of C_{24} in $C_{24} \mid_R (K_{50} - F)$ is

$$(0, 45, 1, 42, 2, 38, 3, 35, 4, 32, 5, 29, 6, 26, 7, 23, 8, 19, 9, 16, 10, 13, 11, 48).$$

The difference between them will be the labeling of b_i's. In order to have the same difference, for $0 \leq i \leq k$, and $i = 2k$, where $l = 4k+2$, the new label is $f(b_i)+1$. So, in fact the labeling of C_{24} in the decomposition of $K_{50} - F$ is also a bipartite ρ-labeling.

On the other hand, for $l \equiv 3 \pmod 4$, we can obtain a new labeling (ρ-labeling) for C_l in $K_{2l+2} - F$ similarly. This concludes the proof. $\square$

We remark here that if rainbow path-decompositions are being considered, then a rainbow bipartite ρ-labeling can be obtained by a similar manner. There are also special cases which can be done. For example, if n is an odd integer and $k+1$ is a factor of n, then a weak rainbow ρ_{k+1}-design exists. This is a direct consequence of the existence of a rainbow Hamilton cycle design of order n (Theorem 1.2).

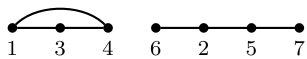
4 Strong rainbow cycle designs

In what follows, we consider the case when the edge-coloring is arbitrarily given. In an early work, Fu et al. [7] proved that if K_{2m} is $(2m-1)$-edge-colored, then we can find at least $\lfloor \frac{(2m+9)^{1/2}}{3} \rfloor$ rainbow spanning trees (not necessarily be isomorphic). For isomorphic rainbow spanning trees, we can find three so far, [8]. So, it is interesting to know how many rainbow k-cycles, $k \leq 2m+1$, can we find in a $(2m+1)$-edge-colored K_{2m+1} or a $2m$-edge-colored $K_{2m,2m}$ for each $m \in \mathbb{N}$.

Theorem 4.1. *For each $m \geq 3$, there exists a strong rainbow 4-cycle design of $K_{2m,2m}$.*

Proof. Let φ be an arbitrarily given edge-coloring of $K_{2m,2m} = (A, B)$ where $A = \{a_i | i \in \mathbb{Z}_{2m}\}$ and $B = \{b_i | i \in \mathbb{Z}_{2m}\}$. Now, we consider the subgraph G induced by $\{a_0, a_1\} \cup B$. Clearly, if we can decompose G into rainbow 4-cycles, then the proof follows by taking two other vertices of A in turn and apply the same decomposition.

For convenience, let $c_i = \varphi(a_0 b_i)$, and $d_i = \varphi(a_1 b_i)$, $i \in \mathbb{Z}_{2m}$. Since φ is a proper edge-coloring, for each $i \in \mathbb{Z}_{2m}$, $c_i \neq d_i$. Moreover, $c_i \neq c_j$ and $d_i \neq d_j$, $0 \leq i \neq j \leq 2m-1$. Now, we define a graph H where $V(H) = \{c_i | i \in \mathbb{Z}_{2m}\} \cup \{d_j | j \in \mathbb{Z}_{2m}\}$ and $E(H) = \{\{c_i, d_i\} | i \in \mathbb{Z}_{2m}\}$. Therefore H has exactly $2m$ edges. Moreover, $\Delta(H) \leq 2$ since both $\{c_i | i \in \mathbb{Z}_{2m}\}$ and $\{d_j | j \in$

$\mathbb{Z}_{2m}\}$ are $2m$-sets. $\deg_H(c_i) = 2$ (respectively $\deg_H(d_j) = 2$) if and only if $c_i = d_{j'}$ for some $j' \in \mathbb{Z}_{2m}$ (respectively $d_j = c_{i'}$ for some $i' \in \mathbb{Z}_{2m}$). For example, if $(c_0, c_1, c_2, c_3, c_4, c_5) = (1, 2, 3, 4, 5, 6)$ and $(d_0, d_1, d_2, d_3, d_4, d_5) = (3, 5, 4, 1, 7, 2)$ then $H \cong$ 

.

Claim. H can be decomposed into m matchings of size 2 if $m \geq 3$.

By Vizing's Theorem [13], $\chi'(H) \leq 3$ since $\Delta(H) \leq 2$. This implies that H has an equitable m-edge-coloring such that each color class is of size 2, [14]. Thus, we have the claim.

Finally, we observe that a matching of size two induces a rainbow 4-cycle. Let $\{i_1, j_1\}$ and $\{i_2, j_2\}$ be the edges in a 2-matching. Therefore, in π, $c_{i_1} \to d_{j_1}$ and $c_{i_2} \to d_{j_2}$. Since for some x and y, $\varphi(a_0 b_x) = c_{i_1}$, $\varphi(a_1 b_x) = d_{j_1}$, $\varphi(a_0 b_y) = c_{i_2}$ and $\varphi(a_1 b_y) = d_{j_2}$, these four distinct colors are used in the cycle (a_0, b_x, a_1, b_y) which is a rainbow 4-cycle. This completes the proof. $\square$

Corollary 4.2. *Let $K_{n_1, n_2, \ldots, n_k}$ be a complete t-partite graph with an arbitrary proper edge-coloring. If each n_i, $i = 1, 2, \ldots, t$, is even and $n_i \geq 6$, then there exists a strong rainbow 4-cycle design of $K_{n_1, n_2, \ldots, n_t}$.*

In order to determine whether an $(8t+1)$-edge-colored 4-sufficient complete graph K_{8t+1} can be decomposed into rainbow 4-cycles, we need to solve the following problem first.

Problem. *Let K_9 be properly edge-colored with at least 9 colors. Can this graph be decomposed into rainbow 4-cycles?*

Clearly, if this problem can be solved, then we can apply the well-known $(8t + 1)$-construction and Theorem 4.1 to prove that a strong rainbow 4-cycle design of K_n exists if and only if $n \equiv 1 \pmod 8$. So far, we are not able to settle this problem. But, we are able to prove this result except if $n = 9$ and 17.

Proposition 4.3. *For each $t \geq 3$, a strong 4-cycle design of K_{8t+1} exists.*

Proof. Suppose we are given an arbitrary proper edge-coloring of K_{8t+1}. First, note that $K_{2,2t} \mid K_{8t+1}$ since $K_{2,2t}$ has the following bipartite ρ-labeling φ defined on $\mathbb{Z}_{8t+1}$: let $K_{2,2t} = \langle \{a_0, a_1\}, \{b_0, b_1, \ldots, b_{2t-1}\} \rangle$, $\varphi(a_0) = 0$, $\varphi(a_1) = 2t$, and $\varphi(b_i) = i + 2t$, $i \in \mathbb{Z}_{2t}$. By adapting the colors on each $K_{2,2t}$, we have an edge-colored $K_{2,2t}$. Now, the proof follows by the argument used in Theorem 4.1. $\square$

Concluding remark.

It seems quite possible to obtain a weak rainbow l-cycle design of K_n or $K_n - F$ as long as they are k-sufficient. At least, we have shown some of the cases are possible:

(1) $l = 4t$ and $n = 2l_{k+1}$, $l \in \mathbb{N}$;

(2) $l = 4t + 3$ and $n = 2l + 1$;

(3) $l = 0$ or $3 \pmod 4$, $C_l \mid_R K_{2l+2} - F$.

As to finding a strong rainbow cycle designs, we believe that it is much harder when the cycle length is getting large.

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