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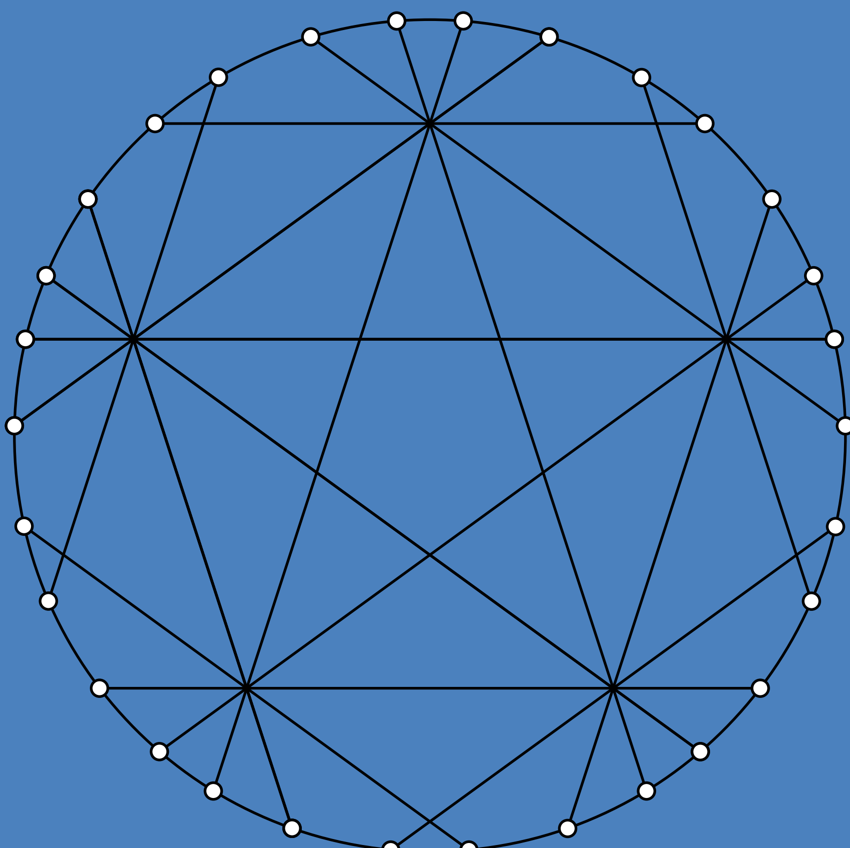

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The length distribution for burn-off chip-firing games on complete graphs

Dave Perkins[†]

Mathematics Department, Hamilton College, Clinton NY 13323, USA
dperkins@hamilton.edu

P. Mark Kayll[‡]

Department of Mathematical Sciences, University of Montana, Missoula MT 59812, USA
mark.kayll@umontana.edu

Abstract: We continue our investigations of burn-off chip-firing games from [*Discrete Math. Theor. Comput. Sci.* **15** (2013), 121–132], [*Australas. J. Combin.* **68** (2017), 330–345], and [On lengths of burn-off chip-firing games, 2016, submitted]. The middle article introduced randomness by choosing successive seeds uniformly from the vertex set of a simple graph. The length ℓ of a game is the number of vertices that fire (by sending a chip to each neighbour and annihilating one chip) as an excited chip configuration settles to a relaxed state. This article determines explicitly the game-length distribution $(p_\ell)_{\ell=0}^{n}$ in a long course of burn-off games on a complete graph K_n. Thus we recover the corresponding enumeration results obtained by Cori, Dartois, and Rossin [Avalanche polynomials of some families of graphs, *Mathematics and computer science III*, Trends Math., 81–94, Birkhäuser, 2004]. We give two proofs of our main theorem: one working from first principles; the other invoking a result from the third

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listed article. Additionally, we include a combinatorial verification that (p_ℓ) defines a probability distribution; this depends on a century-old identity known to Dziobek [Eine Formel der Substitutionstheorie, *Sitzungsber. Berl. Math. Ges.* **16** (1917), 64–67].

Keywords: chip-firing, burn-off game, complete graph, relaxed legal configuration, game-length probability, limiting distribution, Markov chain

1 Introduction

This article continues our investigations in [28, 29, 37] of a variant of chip-firing games—burn-off games—in which each iteration simulates the loss of energy from a complex system. Postponing a few definitions until §1.1, we begin by recalling the Markov chain $(X_k)_{k\geq 0}$ from [37]. The state space of (X_k) is the set $\mathcal{R}$ of relaxed legal configurations on a connected graph $G = (V, E)$. A transition from a given configuration in $\mathcal{R}$ is determined by randomly seeding a vertex and relaxing the resulting configuration. (See before Theorem 2.8 for a more precise definition of (X_k).) We proved in [37] that (X_k) is irreducible, has a doubly stochastic transition matrix, and therefore has a uniform stationary distribution. In [29], we exemplified this result by combining it with theorems counting the pairs $(\sigma, v) \in \mathcal{R} \times V$ corresponding to possible game lengths. This shed light on the game-length distribution in a long burn-off game sequence for a general graph.

In this paper, we narrow our scope to complete graphs K_n and consequently sharpen the results in [29]. Whereas the expression in Theorem 2.7 below (from [29]) for the number of pairs (σ, v) resulting in a game of length $\ell > 0$ is generally not closed, in the case of complete graphs, we find it to be tractable. For integers $m \geq 1$ and $\ell \in \{0, 1, \ldots, n\}$, let $\Lambda_m(\ell)$ denote the number of games of length ℓ occurring during the first m transition epochs of (X_k) played on K_n. Our main result determines the distribution of $\Lambda_m(\cdot)/m$ over a long course of burn-off games:

1.1 Theorem. *With probability one,*

$$\lim_{m\to\infty} \frac{\Lambda_m(\ell)}{m} = \begin{cases} \dbinom{n}{\ell} \dfrac{\ell^{\ell-1}(n-\ell+1)^{n-\ell-1}}{n(n+1)^{n-1}} & \text{for } 1 \leq \ell \leq n, \\ \dfrac{n-1}{n+1} & \text{for } \ell = 0. \end{cases} \tag{1}$$

Of course, we aren't the first authors to study chip-firing games on complete graphs; a recent reference is [44]. Nor are we alone in considering (X_k); see, e.g., [26] for the latest on this Markov chain. If we weren't the first to examine (X_k)—starting in the early 2000's for [36]—then we arrived at this process independently. And though others have investigated game length (e.g., [23, 33, 42]), Theorem 1.1 marks the first published version of the limiting distribution of $\Lambda_m(\cdot)/m$. This is not to say the required combinatorics is new. Prior to the completion of [36], the enumeration results needed for Theorem 1.1 also appeared in [14]—part of the abelian sandpile and parking function literature. Aside from two new proofs, an advantage of our approach is to focus the results through a probabilistic lens based on our analysis in [29, 37] of the Markov chain.

Brief background on chip firing

In solving some geometry/linear algebra problems, Spencer [41] introduced his so-called 'balancing game': start with a pile of N chips at the centre of a long path; in the first step, move $\lfloor N/2 \rfloor$ chips to the right and $\lfloor N/2 \rfloor$ chips to the left (leaving one in place if N is odd). In the second step, the game continues with the two new piles, and so on. Subsequently, Anderson et al. [1] considered a variant of the balancing game, after which Björner et al. studied a natural generalization to graphs [7] and then to digraphs [6]. For graphs, their 'chip-firing game' is almost the same as burn-off chip firing (cf. §1.1 below), except the threshold for a vertex v to fire is $\deg(v)$ (cf. $\deg(v) + 1$) and firing v eliminates no chips (cf. one chip).

In several guises, chip firing has been studied deeply, by various approaches, and touches on numerous concepts: e.g., legal game sequences [6, 7], the chromatic polynomial [5], the Tutte polynomial [4, 35], the critical group [3], and G-parking functions [2]. The chip-firing game is essentially a disguised 'abelian sandpile model', introduced earlier by Dhar [16]; see also [18]. For other results related to algebraic aspects and properties, see, e.g., [12, 34]. Chip firing on graphs evidently concerns—and indeed unites—many concepts in graph theory and algebra.

Motivation for burn-off chip firing

In a (standard) chip-firing game, the total number of chips residing on the host graph remains constant throughout the game's execution. So one might guess that a game initialized with sufficiently many chips will continue indefinitely, i.e., have infinite game length. This intuition is correct, but the question of when a game is finite has a more subtle answer, found by Björner et al. [7] (see §1.1 below for terminology). In the statement of their theorem, G is a connected graph of order n, size m $(= |E(G)|)$, and N denotes the total number of chips in a configuration on G.

1.2 Theorem ([7]).
(a) If $N > 2m - n$, then all chip-firing games on G have infinite length;
(b) if $m \leq N \leq 2m - n$, then there exists an initial configuration on G leading to a game of finite length and also one leading to a game of infinite length;
(c) if $N < m$, then all chip-firing games on G have finite length.

In a paper predating even [7], Tardos [42] showed that for games of finite length, the length is bounded by a polynomial in n (namely $O(n^4)$). See the paragraph preceding Lemma 2.2 below for more references to length investigations in the chip-firing literature.

One motivation for introducing burn-off chip-firing games in [36] was to force all games to have finite length; see [28] for other motivations and [3] for an earlier-defined game equivalent to burn-off games. Finiteness follows because in these games, the total number of chips decreases by one during each firing event. This opens the door for studying the probability distribution of game length during a long course of suitably randomized burn-off games. And indeed, such a study for complete graphs is the purpose of the present paper.

Outline

We hazard no further literature review and instead point the reader to the surveys [24, 34], to the book [22], to the thorough yet concise piece [31], and to our earlier papers [28, 29, 37]. The rest of this article is organized as follows. First (in §1.1), we review the basic chip-firing notions, including the undefined terms already encountered. Section 2 records several earlier supporting results. Section 3 presents two propositions counting germane subsets of 'relaxed legal configurations' on complete graphs. These are needed for our main proof of Theorem 1.1 in Section 4. We end that section with three short passages: one offering an alternate proof of the main case in (1); one specifying the connection between our results and those in [14]; and one confirming combinatorially that (1) defines a probability distribution.

We close with the brief Section 5, which contrasts our main result with extant conclusions within the self-organized criticality literature typically accompanying chip-firing studies.

Notation and terminology

All graphs in this paper are finite, simple, undirected, and usually complete. We use 'general graph' when we wish to emphasize that a graph may not be complete (or even connected, for that matter). The order of a graph $G = (V, E)$ is denoted by n ($:= |V|$) and the number of its spanning trees by $\tau = \tau(G)$. See [9] for omitted graph theory items and [21] for probability background. Finite Markov chains are also introduced in [40] and [43]. A reference addressing chip firing specifically is [22].

1.1 Burn-off chip firing

Beginning with a *(chip) configuration* on G—i.e., a function $\sigma\colon V \to \mathbb{N}$—a *burn-off (chip-firing) game* plays as follows. For a vertex v, if $\sigma(v)$ exceeds $\deg_G(v)$, then v can *fire*, meaning it sends one chip to each neighbour and eliminates one chip from further play. Formally, when v fires, σ morphs to a *successor* configuration σ' such that

$$\sigma'(u) = \begin{cases} \sigma(v) - \deg_G(v) - 1 & \text{if } u = v, \\ \sigma(u) + 1 & \text{if } uv \in E(G), \\ \sigma(u) & \text{otherwise.} \end{cases} \tag{2}$$

A *relaxed* configuration is one for which no vertex can fire. To start a burn-off game, we add a chip to a selected vertex v (called a *seed*) in a relaxed configuration σ; this is called *seeding* σ at v. Writing $\mathbf{1}_v$ for the configuration with a total of one chip, on v only, we capture 'seeding σ at v' algebraically by passing from σ to $\sigma + \mathbf{1}_v$. Just prior to seeding, if v happened to be *critical*, meaning $\sigma(v) = \deg_G(v)$, then from $\sigma + \mathbf{1}_v$, we fire v, which may trigger a neighbour u of v to become *supercritical*, meaning its new chip count exceeds $\deg_G(u)$. If so, we fire u, which may cause another vertex to become supercritical. The game continues this cascade until reaching a relaxed configuration, called a *relaxation* of $\sigma + \mathbf{1}_v$. The game *length* equals the number of vertex firings, possibly zero, in passing from the initial relaxed configuration to the final one.

'Legal' configurations are those typically encountered in a long game sequence; to define these formally, as we did in our earlier papers [28, 29, 37], we begin by calling a configuration *supercritical* if every vertex is supercritical. Now consider what happens when a burn-off game is played in reverse. Looking at (2), we see that to start in a configuration σ' and *reverse-fire* a vertex v (each of whose neighbours u necessarily satisfies $\sigma'(u) \geq 1$) means to modify σ' to a configuration σ such that

$$\sigma(u) = \begin{cases} \sigma'(v) + \deg_G(v) + 1 & \text{if } u = v, \\ \sigma'(u) - 1 & \text{if } uv \in E(G), \\ \sigma'(u) & \text{otherwise.} \end{cases}$$

A configuration σ is *legal* if there exists a reverse-firing sequence starting with σ and ending with a supercritical configuration. Throughout this paper, we use $\mathcal{R} = \mathcal{R}(G)$ to denote the set of relaxed legal configurations on G, which is usually a complete graph. We write $r(G) := |\mathcal{R}(G)|$ and, for a subgraph H of G, denote by $r(H)$ the number of relaxed legal configurations on H.

2 Precursive results

In the Introduction, we glossed over the well-definedness of the length of a burn-off game. The following early chip-firing result settles this question and shows that the relaxation of a configuration is uniquely determined.

2.1 Lemma ([16, 19])**.** *If $\sigma_0, \sigma_1, \ldots, \sigma_\ell$ is a sequence of configurations on a general graph, each a successor of the one before and $\sigma'_0, \sigma'_1, \ldots, \sigma'_k$ is another such sequence such that $\sigma'_0 = \sigma_0$ and both of σ_ℓ, σ'_k are relaxed, then $k = \ell$, each vertex fires the same number of times in both sequences, and $\sigma'_\ell = \sigma_\ell$.*

Thus, in a burn-off game, the vertices can be fired in any order without affecting the length or final configuration of the game. Lemma 2.1 has appeared elsewhere, including [7] and [24], the latter containing a succinct proof.

As noted in the Introduction (under Motivation), burn-off games always have finite length. Within the general chip-firing literature, finding nontrivial bounds for the game length has been addressed, e.g., in [23], [33], and [42]. We shall need the following elementary result from [36]; see [37] for a published proof.

2.2 Lemma ([36]). *During a burn-off game that starts with a relaxed legal configuration on a general graph, no vertex fires more than once.*

The next tool characterizes the relaxed legal configurations on complete graphs. It was proved originally in [36], followed by a published version in [28].

2.3 Lemma ([36]). *A relaxed configuration $\sigma\colon V \to \mathbb{N}$ on K_n is legal if and only if it is possible to relabel V as $w_1, \ldots, w_n$ so that $\sigma(w_i) \geq i-1$ for $1 \leq i \leq n$.*

The following four results from [28, 36] and [29, 36] enumerate configurations σ subject to various constraints. The first of these focusses on legal (but not necessarily relaxed) configurations on K_n. For $n \geq 0$ and $m \geq \max\{0, n-1\}$, let $L_{n,m}$ denote the number of such configurations satisfying $\sigma(v) \leq m$ for each $v \in V(K_n)$.

2.4 Proposition ([28, 36]). *If $n \geq 0$ and $m \geq \max\{0, n-1\}$, then $L_{n,m} = (m-n+2)(m+2)^{n-1}$.*

We stress that the main cases in Proposition 2.4 have $n \geq 1$ and $m \geq n-1$; the (boundary) cases with $n = 0$ and $m \geq 0$, which correspond to $L_{0,m} = 1$, don't contradict the result but satisfy it merely by convention.

Because $r(K_n) = L_{n,n-1}$, we obtain

2.5 Corollary ([28, 36]). *The number of relaxed legal configurations on K_n is $(n+1)^{n-1}$.*

The preceding observation originally led us to the connection between burn-off games on complete graphs and the enumeration of spanning trees therein; see [28].

Two of the main results from [29] apply to general graphs G, but here, we shall apply them only to complete graphs. Together they count the pairs $(\sigma, v) \in \mathcal{R}(G) \times V(G)$ such that seeding σ at v results in a game of length $\ell \geq 0$. In the first of these, the *cone* G^* means the graph obtained from G by adding a new vertex x adjacent to every vertex of G, and t_v denotes the number of spanning trees of $G^* - xv$.

2.6 Theorem ([29, 36])**.** *The number of pairs* (σ, v) *resulting in a game of length zero is* $\sum_{v \in V} t_v$*.*

In Section 4, when invoking Theorem 2.6, we again use 'x' for the universal vertex introduced in defining G^*.

When $\ell > 0$, our count involves the set $\mathcal{T}_{v,\ell}$ of subtrees of G of order ℓ and including v; it also employs the counting function r defined at the end of §1.1.

2.7 Theorem ([29, 36])**.** *The number of pairs* (σ, v) *resulting in a game of length* $\ell > 0$ *is*

$$\sum_{v \in V} \sum_{T \in \mathcal{T}_{v,\ell}} r(G - T).$$

Finally, we return to the Markov chain (X_k) introduced in Section 1. To be more precise about its transitions, given $X_k \in \mathcal{R}$, the next state is determined by choosing $v \in V$ uniformly at random and taking X_{k+1} to be the relaxation of $X_k + \mathbf{1}_v$. For integers $m \geq 1$ and states σ, we denote by $N_m(\sigma)$ the number of visits of (X_k) to σ during the first m transition epochs. The uniformity of (X_k)'s stationary distribution yields the following result:

2.8 Theorem ([37])**.** *For general graphs* G*,*

$$\Pr\left\{\lim_{m \to \infty} \frac{N_m(\sigma)}{m} = \frac{1}{|\mathcal{R}|}\right\} = 1 \text{ for all } \sigma \in \mathcal{R} \text{ (irrespective of the initial state).}$$

So with high probability, the long-term proportion of time that (X_k) spends in any given state is equally spread across the states. We shall need only the specialization of Theorem 2.8 to complete graphs:

2.9 Corollary. *For complete graphs* K_n*,*

$$\Pr\left\{\lim_{m \to \infty} \frac{N_m(\sigma)}{m} = \frac{1}{(n+1)^{n-1}}\right\} = 1 \text{ for all } \sigma \in \mathcal{R}(K_n).$$

3 Enumeration in $\mathcal{R}$

Our proof of Theorem 1.1 relies on a couple of results counting certain subsets of $\mathcal{R} = \mathcal{R}(K_n)$. The definitions in this paragraph concern complete graphs and apply to any fixed $\sigma \in \mathcal{R}$. Notice that a critical vertex will fire with the addition of a single chip. Call a non-critical vertex w *unstable* if, for some critical vertex v, when v is seeded, the resulting firing sequence includes the firing of w. Call a vertex a *dud* if it won't fire during any burn-off game initiated on σ. Either a vertex will fire when seeded (so is critical) or will not (so is unstable or a dud). But an unstable vertex will (eventually) fire when a critical one is seeded, whereas a dud will not. Since each vertex must be one of these three types, V is partitioned into the subsets of critical vertices, unstable vertices, and duds.

Given a configuration σ, let c denote the number of critical vertices and u the number of unstable vertices. Notice that there are only two possible game lengths that can result when a vertex is seeded: either the vertex will not fire, resulting in a game of length zero; or the vertex will fire, resulting in a game of length $c+u$ (by Lemmas 2.1 and 2.2). For this reason, we will profit from counting the configurations that have c critical vertices and u unstable ones, for fixed nonnegative integers c and u. We handle the case $c = 1$ separately after considering the case $c \geq 2$.

3.1 Proposition. *If c, u, and d are integers such that $2 \leq c \leq n$, $0 \leq u \leq n-c$, and $d = n-(c+u)$, then the number of relaxed legal configurations on K_n with c critical vertices, u unstable vertices, and d duds is*

$$\binom{n}{d}(d+1)^{d-1}\binom{n-d}{u}(c-1)(n-d-1)^{u-1}. \tag{3}$$

Proof. Let $\mathcal{Q}$ be the set of configurations satisfying the hypotheses. We shall determine the size of $\mathcal{Q}$ by first examining the chip placements on the duds and deriving the intermediate count

$$|\mathcal{Q}| = \binom{n}{d}L_{d,d-1}\binom{n-d}{u}L_{u,n-d-3}. \tag{4}$$

If a critical vertex of $\sigma \in \mathcal{Q}$ is seeded, then all $c+u$ critical and unstable vertices will fire, contributing $c+u$ chips to each dud. Therefore, each dud contains at most $n-(c+u)-1 = d-1$ chips, for otherwise the added chips will cause it to fire. Since there are $\binom{n}{d}$ ways to choose which of the n vertices will be duds and $L_{d,d-1}$ ways to place chips on these duds, the first two factors in (4) account for the chip placement on the duds.

Now we account for the number of ways to place chips on the u unstable vertices of σ. Since vertices containing $n-1$ chips are critical, unstable vertices contain at most $n-2$ chips. The greatest number of chips that can be added during a game to an unstable vertex is $c+(u-1)$, which implies that the smallest number of chips that an unstable vertex can contain at the outset is $n-(c+(u-1)) = d+1$. Thus, unstable vertices contain between $d+1$ and $n-2$ chips.

Within the set of u unstable vertices, there is (at least) one that contains the fewest chips, and it may contain anywhere from $d+1$ to $n-2$ chips. If two unstable vertices v and w both contain $d+1$ chips, then there are at most $c+(u-2)$ other vertices that can fire and add chips to v and w. These additional chips will increase the number of chips on v and w to at most

$$(d+1)+c+(u-2) = c+d+u-1 = n-1,$$

which is not enough for either of them to fire. Thus, both v and w are duds, a contradiction. Therefore, if the unstable vertex containing the fewest chips contains $d+1$ chips, then the unstable vertex containing the next-to-fewest chips must contain between $d+2$ and $n-2$ chips.

By a similar argument, we see that the vertex containing the k^{th} fewest chips (for $1 \leq k \leq u$) among the unstable vertices must contain at least $d+k$ and at most $n-2$ chips. (Note that $d+u \leq n-2$ because we are considering the case when $c \geq 2$.)

Consider the vector $(x_1, x_2, \ldots, x_u)$, where x_k is the number of chips on the unstable vertex containing the k^{th} fewest chips. We have just seen that $d+k \leq x_k \leq n-2$. Subtracting $d+1$ yields $k-1 \leq x_k-(d+1) \leq n-d-3$, for $1 \leq k \leq u$. Since $x_k \geq d+1$, we see that counting the number of ways to distribute chips onto the unstable vertices is equivalent to counting the number of relaxed legal configurations on K_u where each vertex can have at most $n-d-3$ chips (cf. Lemma 2.3). Thus, we have $L_{u,n-d-3}$ ways to distribute these chips. Since there are $\binom{n-d}{u}$ ways to choose which of the $n-d$ non-duds will be unstable and $L_{u,n-d-3}$ ways to place chips on these unstable vertices, the last two factors in (4) account for the chip placement on the unstable vertices.

Once we have placed chips on all the duds and unstable vertices, it remains only to place chips on the critical vertices. Since all of these must contain $n-1$ chips, and we have no remaining choice as to which vertices are critical (with the unstable vertices and duds already decided), the placement of these final chips is uniquely determined; thus, we've established (4). Now Proposition 2.4 reduces the right side of (4) to (3). $\square$

Because (3) vanishes when $c = 1$, it doesn't apply in this case, but we can handle it with a simpler argument.

3.2 Proposition. *The number of relaxed legal configurations on K_n with one critical vertex is n^{n-1}.*

Proof. A relaxed legal configuration with just one critical vertex cannot have any unstable vertices because, when the critical vertex fires, it gives just one chip to each other vertex, which is not sufficient to fire any of the non-critical ones. Thus, in the proof of Proposition 3.1, once we count the number of ways to distribute the chips on the duds, the remaining choice for the critical vertex is determined. We find that the number of relaxed legal configurations on K_n with $c = 1$ critical vertex and $d = n - 1$ duds is

$$\binom{n}{d} L_{d,d-1} = \binom{n}{n-1} L_{n-1,n-2} = n^{n-1}.$$

□

4 Proof of Theorem 1.1

To understand the limiting behaviour of $\Lambda_m(\ell)/m$, we consider the random variable Λ, defined as the length of a burn-off game on K_n when the seed v is chosen uniformly from $V = V(K_n)$ and, independently, the starting configuration σ is chosen uniformly from $\mathcal{R} = \mathcal{R}(K_n)$.

We distinguish three cases: $\ell = 0$; $\ell = 1$; and $2 \leq \ell \leq n$.

Case $\ell = 0$

First we apply Theorem 2.6 to find the number Z of pairs (σ, v) leading to $\Lambda = 0$; for each $v \in V$, we need to evaluate $t_v = \tau(K_n^* - xv)$. As $K_n^* = K_{n+1}$, a reduced Laplacian of $K_n^* - xv$ (eliminating the row/column of v) is the $n \times n$ matrix

$$A = \begin{bmatrix} n & -1 & -1 & & -1 & -1 \\ -1 & n & -1 & \cdots & -1 & -1 \\ -1 & -1 & n & & -1 & -1 \\ & \vdots & & \ddots & & \vdots \\ -1 & -1 & -1 & & n & -1 \\ -1 & -1 & -1 & \cdots & -1 & n-1 \end{bmatrix}.$$

The Matrix-Tree Theorem—which specializes to give $t_v = \det(A)$ (see, e.g., [9])—and a bit of algebra show that

$$t_v = (n-1)(n+1)^{n-2},$$

and Theorem 2.6 yields

$$Z = \sum_{v \in V} t_v = n(n-1)(n+1)^{n-2}. \tag{5}$$

Conditioning on $(\sigma, v) \in \mathcal{R} \times V$, we find that

$$\Pr\{\Lambda = 0\} = \sum_{(\sigma,v) \in \mathcal{R} \times V} \Pr\{\Lambda = 0 \mid (\sigma, v)\} \Pr\{(\sigma, v)\}. \tag{6}$$

In (5), we have the number of factors $\Pr\{\Lambda = 0 \mid (\sigma, v)\}$ in (6) taking the value 1 (with the rest being 0). From the definition of Λ, the factors $\Pr\{(\sigma, v)\}$ obey the uniform distribution on $\mathcal{R} \times V$. Thus, (6) simplifies to

$$\Pr\{\Lambda = 0\} = \frac{n(n-1)(n+1)^{n-2}}{n(n+1)^{n-1}} = \frac{n-1}{n+1}. \tag{7}$$

Corollary 2.9 shows that (X_k) with high probability in the limit visits the states $\sigma \in \mathcal{R}$ uniformly. In the event of this uniform visitation, the relation (7) specifies the probability of a length-zero game. Therefore, when $\ell = 0$, Theorem 1.1 follows.

Case $\ell = 1$

Here we condition on $\sigma \in \mathcal{R}$ to find that

$$\Pr\{\Lambda = 1\} = \sum_{\sigma \in \mathcal{R}} \Pr\{\Lambda = 1 \mid \sigma\} \Pr\{\sigma\}. \tag{8}$$

Proposition 3.2 shows that the number of nonzero factors $\Pr\{\Lambda = 1 \mid \sigma\}$ in (8) is n^{n-1}; with our seed choices being uniform, each of these factors is $1/n$ because $\Lambda = 1$ exactly when the seed coincides with the critical vertex. Again, the definition of Λ gives the uniform distribution on $\mathcal{R}$ for the factors $\Pr\{\sigma\}$, which leads to

$$\Pr\{\Lambda = 1\} = \frac{n^{n-2}}{(n+1)^{n-1}}. \tag{9}$$

Similarly to the first case, Corollary 2.9 together with (9) leads to Theorem 1.1 when $\ell = 1$. Notice that (9) agrees with (1) in this case.

Case $2 \leq \ell \leq n$

Here things are more complicated. Start at (8), with 'ℓ' in place of '1':

$$\Pr\{\Lambda = \ell\} = \sum_{\sigma \in \mathcal{R}} \Pr\{\Lambda = \ell \mid \sigma\} \Pr\{\sigma\}. \tag{10}$$

As in the case $\ell = 1$, the factors $\Pr\{\sigma\}$ are all $1/(n+1)^{n-1}$, so it remains to determine the sum

$$\mathcal{S} := \sum_{\sigma \in \mathcal{R}} \Pr\{\Lambda = \ell \mid \sigma\}.$$

With reference to Proposition 3.1, let c, d, and u denote, resp., the critical, dud, and unstable vertex counts of a given configuration $\sigma \in \mathcal{R}$. Those σ contributing nonzero terms to $\mathcal{S}$ are the ones with c between 2 and ℓ because we're considering the case $\ell \geq 2$, while $c > \ell$ leads to games of length either zero or exceeding ℓ. When indeed $2 \leq c \leq \ell$, the corresponding probability in $\mathcal{S}$ is c/n (as one of the c critical vertices must be chosen in order to have a nontrivial game) for a game of length

$$\ell = c + u \tag{11}$$

(cf. Lemmas 2.1 and 2.2). These remarks and Proposition 3.1 show that

$$\mathcal{S} = \sum_{c=2}^{\ell} \binom{n}{d} (d+1)^{d-1} \binom{n-d}{u} (c-1)(n-d-1)^{u-1} \left(\frac{c}{n}\right). \tag{12}$$

The constraints on c, d, u, and ℓ in Proposition 3.1 and (11) allow the elimination of d and u from (12), after which we simplify further using the

Binomial Theorem:

$$\begin{aligned}
\mathcal{S} &= \sum_{c=2}^{\ell} \binom{n}{n-\ell} (n-\ell+1)^{n-\ell-1} \binom{\ell}{\ell-c} (c-1)(\ell-1)^{\ell-c-1} \left(\frac{c}{n}\right) \\
&= \binom{n}{\ell} \frac{(n-\ell+1)^{n-\ell-1}}{n} \sum_{c=2}^{\ell} \binom{\ell}{c} c(c-1)(\ell-1)^{\ell-c-1} \\
&= \binom{n}{\ell} \frac{(n-\ell+1)^{n-\ell-1}}{n} \ell(1+(\ell-1))^{\ell-2} \\
&= \binom{n}{\ell} \frac{\ell^{\ell-1}(n-\ell+1)^{n-\ell-1}}{n}.
\end{aligned}$$

This expression for $\mathcal{S}$, our earlier remarks, and (10) together show that

$$\Pr\{\Lambda = \ell\} = \binom{n}{\ell} \frac{\ell^{\ell-1}(n-\ell+1)^{n-\ell-1}}{n(n+1)^{n-1}}. \tag{13}$$

Now Corollary 2.9 and (13) yield Theorem 1.1 when $2 \leq \ell \leq n$, and this final case of the proof is complete. □

We should point out that—though the second and third cases in the preceding proof eventually coalesce in (1)—we needed to address them separately because (3) vanishes when $c = 1$, so we couldn't roll the $\ell = 1$ case into the more delicate final one.

4.1 Alternate proof when $1 \leq \ell \leq n$

In [29], we included an example employing Theorem 2.7. As a further showcase for this result, we apply it in a second approach to the main cases within the proof of Theorem 1.1.

Fix a positive integer ℓ at most n. For each vertex v of K_n, we enumerate the subtrees of order ℓ including v. There are $\binom{n-1}{\ell-1}$ ways to choose the $\ell-1$ vertices $u \neq v$ for these subtrees; each choice results in a clique K_ℓ of K_n, for which there are $\ell^{\ell-2}$ spanning trees (by Cayley's Formula [11]). Now we delete each such subtree T in turn, counting the number $r(K_n - T)$ of relaxed legal configurations on the resulting graph. When T is deleted from K_n, the graph $K_{n-\ell}$ results. By Corollary 2.5, $r(K_n - T) = r(K_{n-\ell}) = (n-\ell+1)^{n-\ell-1}$. Thus, Theorem 2.7 yields the expression

$$n\binom{n-1}{\ell-1} \ell^{\ell-2}(n-\ell+1)^{n-\ell-1} \tag{14}$$

for the number of pairs $(\sigma, v) \in \mathcal{R} \times V$ resulting in a game of length $\ell \geq 1$ on K_n. Resurrecting Λ from the first proof (so using the uniform distribution on $\mathcal{R} \times V$), we find that

$$\Pr\{\Lambda = \ell\} = \frac{n\binom{n-1}{\ell-1}\ell^{\ell-2}(n-\ell+1)^{n-\ell-1}}{n(n+1)^{n-1}} = \binom{n}{\ell}\frac{\ell^{\ell-1}(n-\ell+1)^{n-\ell-1}}{n(n+1)^{n-1}},$$

which is (13), now for $\ell \neq 0$. Invoking Corollary 2.9 completes the proof as before. $\square$

4.2 Connection with sandpile avalanches

As noted in the Introduction, the companion enumeration results to Theorem 1.1 appeared in the abelian sandpile literature contemporaneously with our work leading to [36]. We confess to being unaware of them until submitting this paper's manuscript for publication, when a referee set us straight. In this short section, we cement the connection.

First, our determination of Z in (5) is the chip-firing version of [14, Proposition 4.3]. Second, our expression (14) for the number of pairs $(\sigma, v) \in \mathcal{R} \times V(K_n)$ producing nonzero game lengths is exactly [14, Proposition 4.4] in chip-firing terms. And we mean "exactly": the algebraic expressions in (5) and (14) are virtually identical to their versions in [14], an uncanny coincidence considering the two sets of authors were working without knowledge of each other.

4.3 Confirmation that (1) defines a probability distribution

In [27], we gave a combinatorial proof that the values in (1) sum to one. For completeness, we include that proof here.

4.1 Proposition. *If n is a positive integer, then*

$$\frac{n-1}{n+1} + \sum_{\ell=1}^{n}\binom{n}{\ell}\frac{\ell^{\ell-1}(n-\ell+1)^{n-\ell-1}}{n(n+1)^{n-1}} = 1. \tag{15}$$

Proof. It's convenient to rewrite (15) in the equivalent form

$$\sum_{\ell=1}^{n} \binom{n+1}{\ell} \ell^{\ell-2}(n+1-\ell)^{n-1-\ell}\ell(n+1-\ell) = 2n(n+1)^{n-1}. \tag{16}$$

To see that (16) holds, first observe that the right side enumerates the pairs $(T, \vec{e})$, where T is a spanning tree of K_{n+1} for which one edge e (among its n) has been distinguished and oriented (in one of two possible directions). The left side also enumerates these pairs. Given $(T, \vec{e})$, notice that deleting the oriented edge $\vec{e}$ from T leaves behind a spanning forest of K_{n+1} with two components L, U (that we may consider ordered from lower to upper). If $|V(L)| = \ell$, for an integer ℓ with $1 \leq \ell \leq n$, then $|V(U)| = n + 1 - \ell$. Conversely, given such a spanning forest, we can recover $(T, \vec{e})$ by selecting a vertex x of L and a vertex y of U and letting $\vec{e} = (x, y)$. On the left side of (16), the factor $\binom{n+1}{\ell}$ accounts for the selection of $V(L)$ (hence for the selection of $V(U)$). Since L, U are, resp., spanning trees of the induced (complete) subgraphs $K_{n+1}[V(L)]$, $K_{n+1}[V(U)]$, the factors $\ell^{\ell-2}$ and $(n+1-\ell)^{n-1-\ell}$ are delivered by Cayley's Formula. Finally, the factors ℓ, $(n+1-\ell)$ count the number of ways to select the vertices $x \in V(L)$ and $y \in V(U)$ determining $\vec{e}$. $\square$

Remarkably, (16) dates (at least) to 1917 and has appeared in numerous works, including [8, 10, 13, 15, 20, 32, 38, 39], none of which used our proof above; see [27] for a survey.

5 Concluding remarks

Absence of a power law

Since their creation (by Dhar [16]), abelian sandpile models (chip-firing games) have been associated with a physical phenomenon called 'self-organized criticality'; see, e.g., [25]. A hallmark of such systems is a power law relating size and frequency of system events. In our context, this would translate to a power law relating the game-length ℓ and the relative frequency $\Lambda_m(\ell)/m$ of games with this length (for large m):

$$\frac{\Lambda_m(\ell)}{m} \overset{?}{\propto} \ell^{-\gamma} \tag{17}$$

for a constant $\gamma > 0$.

Our main result shows that the question in (17) has a negative answer for complete graphs. Indeed, for $1 \leq \ell \leq n$, the expression in (1) is invariant under the transformation $\ell \mapsto (n+1)-\ell$. So for such ℓ, the random variable Λ defined at the start of Section 4 satisfies

$$\Pr\{\Lambda = \ell\} = \Pr\{\Lambda = (n+1) - \ell\}. \tag{18}$$

The symmetry evinced in (18) precludes any power law like (17). This doesn't suggest any contradiction between existing results tying sandpile models with power laws (cf. [25, 30]). Rather, it shows that burn-off chip-firing games on *complete* graphs do not fall within the realm of self-organized critical systems. This should be contrasted with such games (e.g., Biggs' dollar game [3], equivalent to our games) on grid- or lattice-graphs (cf. [17]).

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References

[1] R. Anderson, L. Lovász, P. Shor, J. Spencer, É. Tardos, and S. Winograd. Disks, balls, and walls: analysis of a combinatorial game. *Amer. Math. Monthly* **96** (1989), 481–493.

[2] B. Benson, D. Chakrabarty, and P. Tetali. G-parking functions, acyclic orientations and spanning trees. *Discrete Math.* **310** (2010), 1340–1353.

[3] N. L. Biggs. Chip-firing and the critical group of a graph. *J. Algebraic Combin.* **9** (1999), 25–45.

[4] N. Biggs. The Tutte polynomial as a growth function. *J. Algebraic Combin.* **10** (1999), 115–133.

[5] N. Biggs and P. Winkler. Chip-firing and the chromatic polynomial. Research Report LSE-CDAM-97-03, Centre for Discrete and Applicable Mathematics, London School of Economics, London, U.K., 1997, 10pp.

[6] A. Björner and L. Lovász. Chip-firing games on directed graphs. *J. Algebraic Combin.* **1** (1992), 305–328.

[7] A. Björner, L. Lovász, and P. W. Shor. Chip-firing games on graphs. *European J. Combin.* **12** (1991), 283–291.

[8] G. Bol. Über eine kombinatorische Frage. *Abh. Math. Sem. Univ. Hamburg* **12** (1937), 242–245.

[9] J. A. Bondy and U. S. R. Murty. *Graph theory*, vol. 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[10] R. G. Busacker and T. L. Saaty. *Finite graphs and networks: An introduction with applications*. McGraw-Hill Book Co., New York-Toronto-London-Sydney, 1965.

[11] A. Cayley. A theorem on trees. In *The Collected Mathematical Papers of Arthur Cayley*, vol. 13, pp. 26–28, Cambridge University Press, Cambridge, 1897, (Quart. J. Math. **23** (1889), 376–378).

[12] F. Chung and R. B. Ellis. A chip-firing game and Dirichlet eigenvalues. *Discrete Math.* **257** (2002), 341–355, Kleitman and combinatorics: a celebration (Cambridge, MA, 1999).

[13] R. B. Cooper. Derivation from queueing theory of an identity related to Abel's generalization of the binomial theorem, which is useful in graph theory. *Management Sci.* **19** (1972/73), 582–584.

[14] R. Cori, A. Dartois, and D. Rossin. Avalanche polynomials of some families of graphs. In *Mathematics and computer science. III*, Trends Math., pp. 81–94, Birkhäuser, Basel, 2004.

[15] J. Dénes. The representation of a permutation as the product of a minimal number of transpositions, and its connection with the theory of graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **4** (1959), 63–71.

[16] D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.* **64** (1990), 1613–1616.

[17] D. Dhar. The abelian sandpile and related models. *Physica A: Statistical Mechanics and its Applications* **263** (1999), 4–25, Proceedings of the 20th {IUPAP} International Conference on Statistical Physics.

[18] D. Dhar, P. Ruelle, S. Sen, and D.-N. Verma. Algebraic aspects of abelian sandpile models. *J. Phys. A* **28** (1995), 805–831.

[19] P. Diaconis and W. Fulton. A growth model, a game, an algebra, Lagrange inversion, and characteristic classes. *Rend. Sem. Mat. Univ. Politec. Torino* **49** (1991), 95–119 (1993), Commutative algebra and algebraic geometry, II (Italian) (Turin, 1990).

[20] O. Dziobek. Eine Formel der Substitutionstheorie. *Sitzungsber. Berl. Math. Ges.* **16** (1917), 64–67.

[21] W. Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition, John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[22] C. Godsil and G. Royle. *Algebraic graph theory*, vol. 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[23] J. van den Heuvel. Algorithmic aspects of a chip-firing game. *Combin. Probab. Comput.* **10** (2001), 505–529.

[24] A. E. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, and D. B. Wilson. Chip-firing and rotor-routing on directed graphs. In *In and out of equilibrium. 2*, vol. 60 of *Progr. Probab.*, pp. 331–364, Birkhäuser, Basel, 2008. Corrected in `arXiv:0801.3306v4 [math.CO]`, 20 Jun 2013.

[25] E. Ivashkevich and V. Priezzhev. Introduction to the sandpile model. *Physica A: Statistical Mechanics and its Applications* **254** (1998), 97–116.

[26] D. C. Jerison, L. Levine, and J. Pike. Mixing time and eigenvalues of the abelian sandpile Markov chain. Preprint on `arXiv:1511.00666 [math.PR]`, 2 Nov 2015, 42pp.

[27] P. M. Kayll and D. Perkins. Combinatorial proof of an Abel-type identity. *J. Combin. Math. Combin. Comput.* **70** (2009), 33–40.

[28] P. M. Kayll and D. Perkins. A chip-firing variation and a new proof of Cayley's Formula. *Discrete Math. Theor. Comput. Sci.* **15** (2013), 121–132.

[29] P. M. Kayll and D. Perkins. On lengths of burn-off chip-firing games. Submitted, 2016, 20pp.

[30] L. Laurson, M. J. Alava, and S. Zapperi. Power spectra of self-organized critical sandpiles. *J. Stat. Mech. Theory Exp.* (2005), L11001 (electronic), 8 pp.

[31] L. Levine and J. Propp. What is ... a sandpile? *Notices Amer. Math. Soc.* **57** (2010), 976–979.

[32] L. Lovász. *Combinatorial problems and exercises.* Second edition, North-Holland Publishing Co., Amsterdam, 1993.

[33] L. Lovász and P. Winkler. Mixing of random walks and other diffusions on a graph. In *Surveys in combinatorics, 1995 (Stirling)*, vol. 218 of *London Math. Soc. Lecture Note Ser.*, pp. 119–154, Cambridge Univ. Press, Cambridge, 1995.

[34] C. Merino. The chip-firing game. *Discrete Math.* **302** (2005), 188–210.

[35] C. Merino López. Chip firing and the Tutte polynomial. *Ann. Comb.* **1** (1997), 253–259.

[36] D. Perkins. *Investigations of a chip-firing game.* ProQuest LLC, Ann Arbor, MI, 2005, Thesis (Ph.D.)–University of Montana.

[37] D. Perkins and P. M. Kayll. A chip-firing variation and a Markov chain with uniform stationary distribution. *Australas. J. Combin.* **68** (2017), 330–345.

[38] A. Rényi. Some remarks on the theory of trees. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **4** (1959), 73–85.

[39] J. Riordan. *Combinatorial identities.* John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[40] S. M. Ross. *Stochastic processes.* Wiley Series in Probability and Statistics: Probability and Statistics, Second edition, John Wiley & Sons, Inc., New York, 1996.

[41] J. Spencer. Balancing vectors in the max norm. *Combinatorica* **6** (1986), 55–65.

[42] G. Tardos. Polynomial bound for a chip firing game on graphs. *SIAM J. Discrete Math.* **1** (1988), 397–398.

[43] H. C. Tijms. *A first course in stochastic models.* John Wiley & Sons, Ltd., Chichester, 2003.

[44] W. Zhuang, W. Yang, L. Zhang, and X. Guo. Properties of chip-firing games on complete graphs. *Bull. Malays. Math. Sci. Soc.* **38** (2015), 1463–1469.