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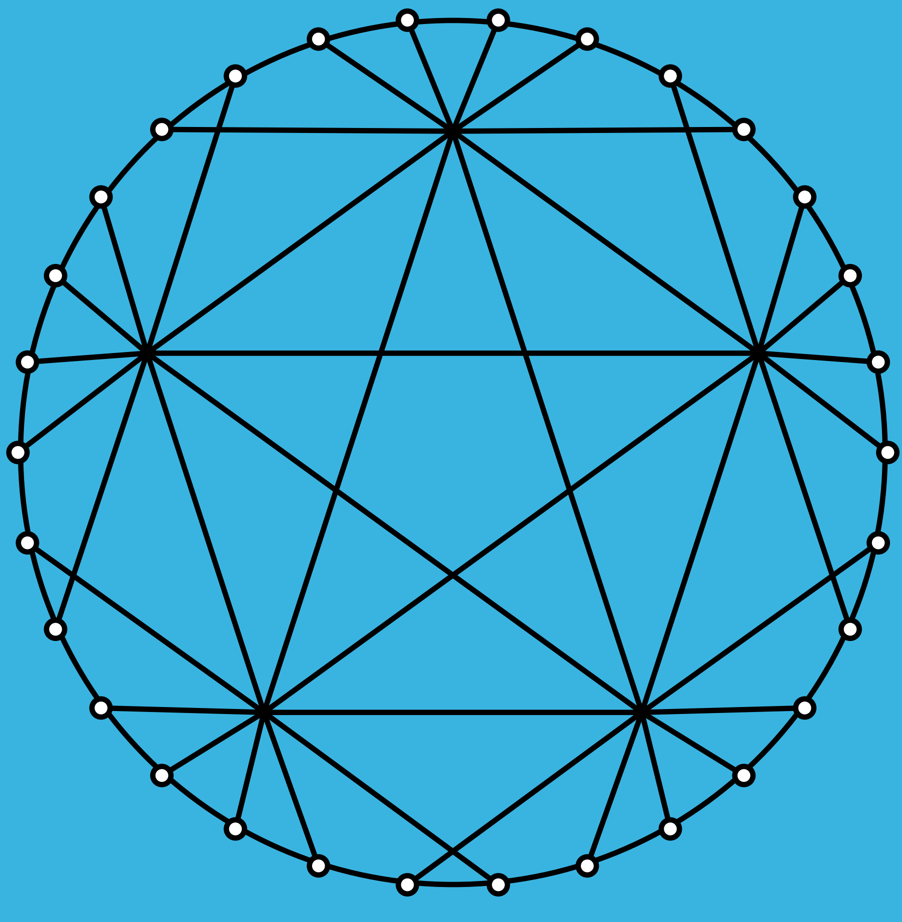

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Almost every non-negative integer contains . . .

Izak Broere*

Department of Mathematics and Applied Mathematics, University of Pretoria
izak.broere@up.ac.za

Abstract: If the number of non-negative integers with n digits with a prescribed property P is $f(n)$ and the sequence of fractions $(\frac{f(n)}{10^n})$ has limit 1 as n tends to infinity, we shall say that *almost every non-negative integer has property* P.

In this note we first show that almost every non-negative integer, when written in its (standard) decimal expansion, contains every digit. Then we prove a much stronger result: for every fixed finite sequence $\mathbf{s}$ of digits, almost every non-negative integer contains $\mathbf{s}$ as a subsequence of its decimal expansion.

These results can both be deduced from a suitable Borel-Cantelli Lemma, a result from probability theory. In spite of that we offer elementary (and quite different) proofs.

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1 Introduction

A well-known statement from number theory is that the prime numbers have zero density in the set of positive integers; a fact which was perhaps known to Gauss already around 1800 and which follows from the *Prime Number Theorem* – see for example [4]. In this theorem the distribution of primes is considered by estimating $\pi(N)$, the number of primes which are less than or equal to a given positive integer N; the result is that it is asymptotically equal to $\frac{N}{log(N)}$, where $log(N)$ is the natural logarithm of N. This means that the fraction of positive integers which are less than or equal to N and prime is about $\frac{1}{log(N)}$; a value which decreases as N increases.

In this note we first consider the property that a non-negative integer, when written in its (standard) decimal expansion, contains every digit and show that in some sense almost every non-negative integer has this property. Then we also show that, for every fixed finite sequence $\mathbf{s}$ of digits, almost every non-negative integer contains $\mathbf{s}$ as a subsequence of its decimal expansion too; a result which is clearly quite stronger.

2 Preliminaries

We consider only non-negative integers and represent each in terms of its decimal expansion. To be precise: by writing $k = a_n a_{n-1} \ldots a_1$ for a non-negative integer k, we mean that the value of this non-negative integer is

$$k \quad = a_n 10^{n-1} + a_{n-1} 10^{n-2} + \cdots + a_2 10 + a_1. \tag{1}$$

In this, we tacitly assume that any a_i may be 0 so that the description of k uses n digits even though this description may contain leading zeros: this choice will facilitate the counting arguments needed for the proofs of the results below.

In this note we prove the two results mentioned in Section 1. The only prerequisites for this exercise are to

• know some counting techniques, which can be obtained from any one of a number of introductory texts on Combinatorics – for example from [2]; and

• know some elementary Analysis, which can be found in a number of textbooks on the subject – for example in [3].

Clearly we have to choose some reasonable way to express this notion of “almost every non-negative integer has this or that property P” in terms of some limit process which takes cognisance of the fact that we are dealing with an infinite set of numbers. The way we have chosen to do this is to count the number of non-negative integers $f(n)$ with n digits with property P. Then we consider the fraction $\frac{f(n)}{10^n}$ since 10^n is the number of non-negative integers with n digits and finally we calculate the limit of this fraction as n tends to infinity. If the value of this limit is 1, we shall say that *almost every non-negative integer has property P.*

These results can both be deduced from a suitable Borel-Cantelli Lemma (see for example [1]), a result involving probability spaces. In spite of that we offer elementary (and quite different) proofs.

3 Two properties

3.1 P is the property to contain all digits

In the next lemma, we count those integers in which all ten digits do not appear. The counting technique used in the proof of the lemma is the Principle of Inclusion and Exclusion.

Lemma 1. *The number of non-negative integers with n digits which do not contain all digits is* $h(n) = \binom{10}{1} \times 9^n - \binom{10}{2} \times 8^n + \binom{10}{3} \times 7^n - \cdots + \binom{10}{9} \times 1^n$.

Proof. As prescribed by the Principle of Inclusion and Exclusion, we first count the number of non-negative integers with n digits in which a given, fixed digit does not appear, then we subtract the number of non-negative integers with n digits in which two given, fixed digits do not appear, then we add the number of non-negative integers with n digits in which three given, fixed digits do not appear, and so on till nine of the ten digits do not appear. This counting process delivers
$h(n) = (9^n + 9^n + \cdots + 9^n) - (8^n + 8^n + \cdots + 8^n) + (7^n + 7^n + \cdots + 7^n) - \cdots + (1^n + 1^n + \cdots + 1^n)$. Note that in the above expression for $h(n)$ there are $\binom{10}{1} = 10$ terms of the form 9^n, $\binom{10}{2} = 45$ terms of the

from 8^n, $\binom{10}{3} = 120$ terms of the from 7^n, . . . , and, finally, $\binom{10}{9} = 10$ terms of the from 1^n. Hence the above formula can be simplified to $h(n) = 10 \times 9^n - 45 \times 8^n + 120 \times 7^n - \cdots + 10 \times 1^n$. □

Note that the counting argument in the proof of Lemma 1 can be facilitated using Stirling numbers of the second kind by counting nonsurjective mappings from an n-set to a k-set; see Exercise 13 on p. 319 of [2] for example.

Corollary 1. *The number of non-negative integers $f(n)$ with n digits in which all digits appear is $f(n) = 10^n - h(n) = 10^n - 10 \times 9^n + 45 \times 8^n - 120 \times 7^n + \cdots - 10 \times 1^n$.*

Finally we have

Theorem 1. *Almost every non-negative integer contains every digit.*

Proof. By the above results we have that $lim_{n\to\infty} \frac{f(n)}{10^n} = 1$ since the terms of $h(n)$ contain only powers of $k \in \{9, 8, \ldots, 1\}$, and for each of these terms we have $lim_{n\to\infty} \frac{k^n}{10^n} = 0$. □

3.2 P is the property to contain a prescribed sequence

It is easy to see that 0 (1, 20) of the non-negative integers between 0 and 9 (99, 999 respectively) contain the sequence of digits 12 as a sub"sequence" of their decimal expansions. A computer search[1] on this kind of question has revealed that, for the given sequence $\mathbf{s} = 12$, we have the following values for the fraction of integers between 0 and $10^n - 1$ containing $\mathbf{s}$ as a subsequence.

n	2	3	4	5	6	7	8	9	10	11	12
Fraction	.01	.02	.0299	.0397	.0494	.0590	.0685	.0779	.0872	.0965	.1056

More such searches give rise to the following values: For $\mathbf{s} = 123$

[1] The author is indebted to Derek Broere who kindly wrote a programme to facilitate this search

n	3	4	5	6	7	8	9	10	11	12
Fraction	.001	.002	.003	.0039	.0049	.00599	.00699	.007985	.008979	.009972

for $\mathbf{s} = 1234$,

n	4	5	6	7	8	9	10	11	12
Fraction	.0001	.0002	.0003	.0004	.00049	.00059	.00069	.00079	.00089

for $\mathbf{s} = 12345$,

n	5	6	7	8	9	10	11	12
Fraction	.00001	.00002	.00003	.00004	.00005	.00006	.00007	.00008

and, finally, for $\mathbf{s} = 123456$,

n	6	7	8	9	10	11	12
Fraction	.000001	.000002	.000003	.000004	.000005	.000006	.0000069

It is clear that the values of n used in these searches are still too small to support the conjecture that almost every non-negative integer contains a given sequence as a subsequence. However, we are able to show that almost every non-negative integer contains *any* fixed finite sequence of digits as a subsequence. Although this shows that our Theorem 1 is not nearly the strongest result of its kind, we present both results since they have different proof techniques.

In order to make this claim precise, let $\mathbf{s} = s_1 s_2 \ldots s_r$ be any sequence of digits (which we write without commas between the s_i's and will consider as fixed in the sequel). Also, for a given such sequence $\mathbf{s}$ and a positive integer n we define a partition of the set $N_n = \{x \mid x \text{ is an integer}, 0 \leq x \leq 10^n - 1\}$ of integers with n digits as follows:
$A_n = \{x \mid x \in N_n, \mathbf{s} \text{ is a subsequence in the decimal expansion of } x\}$, and $B_n = N_n - A_n$.

We are interested in the sequence (p_n) of fractions, where $p_n = \frac{|A_n|}{10^n}$, since by showing that $\lim_{n\to\infty} p_n = 1$ we will establish the fact that almost every non-negative integer contains the sequence $\mathbf{s}$ in its decimal expansion. We first formulate and prove a useful lemma.

Lemma 2. *$p_{n+1} \geq p_n$ for every $n \geq 1$, i.e., the sequence (p_n) of fractions is an increasing sequence.*

Proof. Let $w_1, w_2, \ldots, w_t$, with $t = p_n \times 10^n$, be a complete list of all the non-negative integers between 0 and $10^n - 1$ which contain the given sequence $\mathbf{s}$ of digits.

Interpreting each w_i as a sequence of digits then allows us to create the following list of $10 \times t$ non-negative integers between 0 and $10^{n+1} - 1$, each of which contains this given sequence of digits:

$$0w_1, 0w_2, \ldots, 0w_t, 1w_1, 1w_2, \ldots, 1w_t, \ldots, 9w_1, 9w_2, \ldots, 9w_t.$$

The $10t$ entries in this list are all different, since if iw_p and jw_q are any two of them, we either have $i \neq j$ or $i = j$ and $w_p \neq w_q$; and in both cases $iw_p \neq jw_q$.

Hence the fraction p_{n+1} of non-negative integers between 0 and $10^{n+1} - 1$ which contain the sequence $\mathbf{s}$ of digits is at least $\frac{10 \times t}{10^{n+1}} = \frac{10 \times p_n \times 10^n}{10^{n+1}} = p_n$, i.e., $p_{n+1} \geq p_n$. □

Finally we have

Theorem 2. *Let any fixed finite sequence of digits be given. Then almost every non-negative integer contains this given sequence of digits.*

Proof. We have established in the above lemma that the sequence (p_n) of fractions is increasing; it is also bounded since $0 \leq p_n \leq 1$ for every n. Hence it is convergent (by the *Principle of Monotone Sequences*); see 3.4.1 of [3]. Suppose $lim_{n \to \infty} p_n = p$. By the boundedness of (p_n) we have that $p \leq 1$ and it is thus sufficient to prove that $p \geq 1$, which we now do.

In order to do it, we first estimate the value of p_{n+r} by constructing elements of A_{n+r} from elements of A_n and from B_n. Note first that $|A_n| = p_n \times 10^n$ and $|B_n| = (1 - p_n) \times 10^n$.

- For every integer $a_1 a_2 \ldots a_n \in A_n$ we construct 10^r integers in A_{n+r} by taking the integers $a_1 a_2 \ldots a_n x_1 x_2 \ldots x_r$ with $x_1, x_2, \ldots, x_r$ arbitrary digits.

• For every integer $b_1b_2\dots b_n \in B_n$ we construct one integer in B_{n+r} by using the digits in $\mathbf{s}$ to create the integer $b_1b_2\dots b_ns_1s_2\dots s_r$.
We remark that all the integers constructed above are different; hence $|A_{n+r}| \geq |A_n|\times 10^r + |B_n|$, i.e., $p_{n+r}\times 10^{n+r} \geq p_n\times 10^n\times 10^r + (1-p_n)\times 10^n$ which yields $p_n + \frac{1-p_n}{10^r} \leq p_{n+r}$.

Combining the last inequality with the fact that (p_n) is increasing and taking the limit we obtain $p_n + \frac{1-p_n}{10^r} \leq p_{n+r} \leq p_{n+r+1} \leq \cdots \leq p$, i.e., $p_n \leq \frac{10^r p-1}{10^r-1}$ for every n. Taking the limit again, we obtain $p \leq \frac{10^r p-1}{10^r-1}$, i.e., $p \geq 1$. □

4 Conclusion

In the context of our results one can clearly also consider, for a property P which an integer may or may not have, the sequence of fractions (q_n) where q_n is the fraction of integers from $\{1, 2, \dots, n\}$ having property P. The relationship of such an approach to the approach followed in the sections above is that we have studied the subsequence (q_{10^n}) of such a sequence for the two properties we defined. For these two properties, the full sequence (q_n) is not increasing.

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