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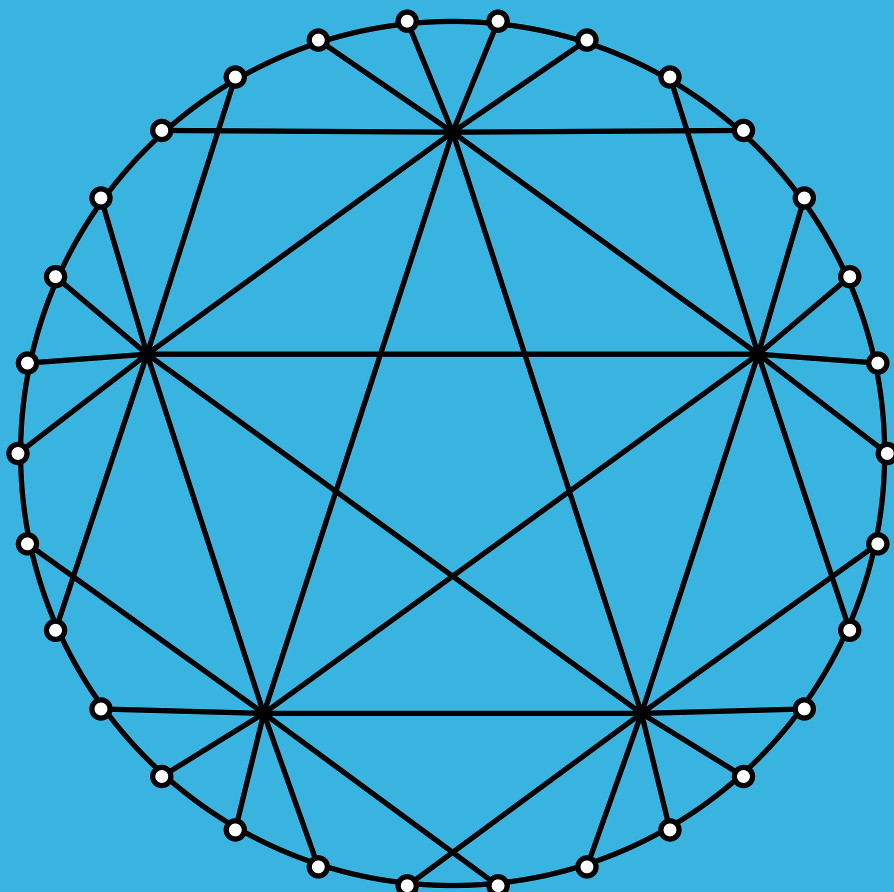

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$K_{1,3}$-subdivision representations with tolerance 1 and 2

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Abstract: Consider a simple undirected graph $G = (V, E)$. A family of subtrees, $\{S_v\}_{v \in V}$, of a tree H is called an (H, t)-representation of G provided $uv \in E$ if and only if $|V(S_u) \cap V(S_v)| \geq t$. Let H_m denote the $K_{1,3}$-subdivision with center node x and three leaves, each of distance m from x and let $\mathcal{H}(t)$ denote the set of (H_m, t)-representable graphs for some positive integer m. In this paper we show that any graph G in $\mathcal{H}(t)$ is also in $\mathcal{H}(t + 1)$ for all t and use this result to proof $\mathcal{H}(1) = \mathcal{H}(2)$. We also characterize the set of all trees in $\mathcal{H}(1)$ and hence in $\mathcal{H}(2)$

Keywords: chordal, host tree, subdivision, tolerance representation

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1 Introduction

In this paper we consider tree tolerance representations of graphs.

Definition 1. *Let $G = (V, E)$ be a simple graph, H a tree and $t > 0$. Then G is called* ***(H,t)-representable*** *if there exists a family of subtrees of H, $\{S_v\}_{v \in V}$, such that*

$$uv \in E \leftrightarrow |V(S_u) \cap V(S_v)| \geq t.$$

In this case we call $\{S_v\}_{v \in V}$ an H $t$olerance representation of G with **tolerance** t or an *(H,t)-representation* of G. Also note that the tree H is referred to as the **host tree** of the representation.

Definition 2. *We will denote by H_m the $K_{1,3}$-subdivision with x the center node of degree three and three leaves, each of distance m from x. See Figure 1.*

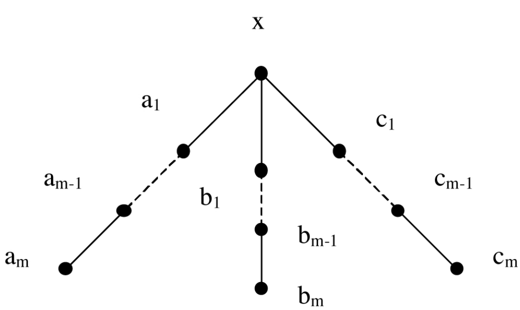

Figure 1: H_m

We denote the three subpaths of H_m, beginning at $x = a_0 = b_0 = c_0$, as follows:

$$P_a = a_0, a_1, a_2, \ldots, a_m;$$
$$P_b = b_0, b_1, b_2, \ldots, b_m;$$

and

$$P_c = c_0, c_1, c_2, \ldots, c_m.$$

Figure 2 illustrates an H_2 tolerance representation of C_4 with tolerance 3.

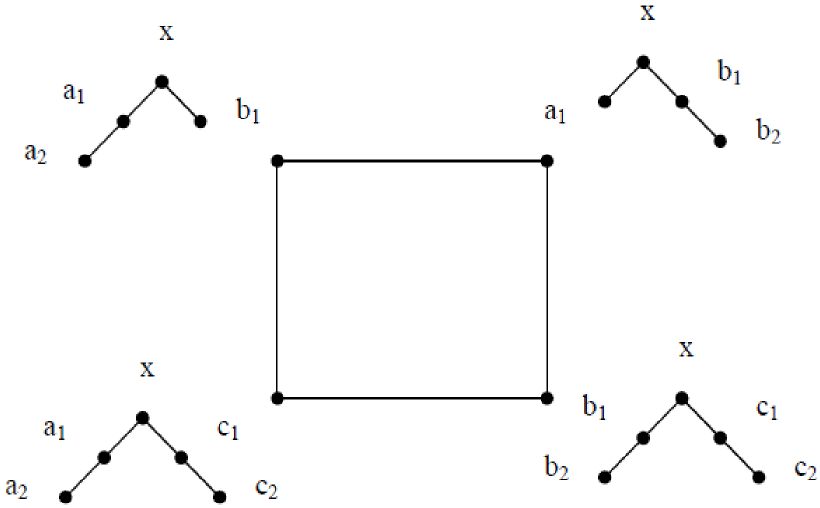

Figure 2: H_2 tolerance representation of C_4

Definition 3. *The class of all graphs that are* (H_m, t)*-representable for some* m *is denoted* $\mathcal{H}(t)$*.*

A well established theorem due to Buneman, Gavril and Walter shows that when the tolerance is limited to one, the tree representable graphs are the chordal graphs. See [1] [2] and [4]. Here we restrict the host tree to H_m and look at implications of increasing tolerance.

Theorem 1. *Let* G *be a graph in* $\mathcal{H}(t)$*. Then* G *is in* $\mathcal{H}(t+1)$*.*

Proof. Let $G = (V, E)$ be a graph in $\mathcal{H}(t)$ for some $t \geq 1$ with (H_m, t)-representation $\mathcal{F} = \{S_v\}_{v \in V}$. We use this representation to create an

$(H_{m+1}, t+1)$-representation, $\mathcal{F}''$, for G through the following steps.

step 1: Define the following subgraph of G:

$$G_a =< \{v \in V | a_i \in V(S_v) \text{ for some } i \in \{0, 1, 2, ..., m\}\} > .$$

Consider $v \in V(G_a)$ and let i be the maximum value such that $a_i \in V(S_v)$. Now define $S'_v = S_v \cup \{a_{i+1}\}$. Repeat this process for each $v \in V(G_a)$. We claim that $\{S'_v\}_{v \in V(G_a)}$ is an $(H_{m+1}, t+1)$-representation for the graph G_a. Indeed, take i and j to be the maximum values for which $a_i \in V(S_u)$ and $a_j \in V(S_v)$. Without loss of generality, assume $i \geq j$. If $a_j \in V(S_u)$, then $a_{j+1} \in V(S'_u)$ and $|V(S'_u) \cap V(S'_v)| = |V(S_u) \cap V(S_v)| + 1$. If $a_j \notin V(S_u)$, then $|V(S'_u) \cap V(S'_v)| \leq 1$ since $|V(S_u) \cap V(S_v)| = 0$. Hence, if $uv \in E(G_a)$, then $a_j \in V(S_u)$ and $|V(S'_u) \cap V(S'_v)| \geq t+1$. Furthermore, if $uv \notin E(G_a)$, then $|V(S_u) \cap V(S_v)| \leq t-1$ and $|V(S'_u) \cap V(S'_v)| \leq t$.

step 2: Define the following subgraphs of G:

$$G_b =< \{v \in V | S_v \subseteq P_b - \{x\}\} >$$
$$G_c =< \{v \in V | S_v \subseteq P_c - \{x\}\} >$$

Observe that $V(G) = V(G_a) \cup V(G_b) \cup V(G_c)$. Repeat the process done in step 1 with G_a for each G_b and G_c to obtain an $(H_{m+1}, t+1)$-representation, $\{S'_v\}_{v \in V(G_b)}$ for G_b and an $(H_{m+1}, t+1)$-representation, $\{S'_v\}_{v \in V(G_c)}$ for G_c. Denote the resulting set of subtrees $\mathcal{F}' = \{S'_v\}_{v \in V}$.

We know that $\mathcal{F}'$ satisfies the edge and non-edge conditions necessary for an $(H_{m+1}, t+1)$-representation for G within each of the subgraphs G_a, G_b, and G_c. Now we address these conditions for edge and non-edge pairs of vertices of G that are not contained in the same subgraph. For any $u \in V(G_b)$ and $v \in V(G_c)$ we have $V(S'_u) \cap V(S'_v) = V(S_u) \cap V(S_v) = \emptyset$. Therefore, the non-edge condition $|V(S'_u) \cap V(S'_v)| \leq t$ is satisfied for $u \in V(G_b)$ and $v \in V(G_c)$. It remains to look at the edges and non-edges between G_a and G_b or G_a and G_c. Take $u \in V(G_a)$ and $v \in V(G_b)$ such that $uv \notin E$. We know $|V(S_u) \cap V(S_v)| \leq t - 1$. Adding a pendant node from P_a to S_u and a pendant node from P_b to S_v can only increase the intersection by at most one, since $S_v \subseteq P_b - \{x\}$. Therefore, $|V(S'_u) \cap V(S'_v)| \leq t$. Hence, the non-edge condition between G_a and G_b is satisfied. We can use a similar argument to show that the non-edge condition between G_a and G_c is satisfied as well. Finally, we address the edges between G_a and G_b or G_a and G_c, which will require modification of $\mathcal{F}'$.

step 3: In this last step we look at the edges between G_a and G_b, and observe that an analogous process may be used for the edges between G_a and G_c. Take $u \in V(G_a)$ and $v \in V(G_b)$ such that $uv \in E$. We know $|V(S_u) \cap V(S_v)| \geq t$. This implies that S_u contains at least t nodes from P_b and that $x = b_0 \in V(S_u)$. Let i and j be the maximum values such that $b_i \in V(S'_u)$ and $b_j \in V(S'_v)$. If $i \geq j$, then $|V(S'_u) \cap V(S'_v)| \geq t + 1$ since $b_j \in V(S'_u)$ and $S'_v = S_v \cup \{b_j\}$. Now suppose $i < j$. In this case, $b_{i+1} \in V(S'_v)/V(S'_u)$. Replace S'_u with $S''_u = S'_u \cup \{b_{i+1}\}$ and we have $|V(S''_u) \cap V(S'_v)| \geq t + 1$.

Now we confirm that we did not disrupt any existing non-edge conditions. Take $w \in V$ such that w is not adjacent to u. It suffices to show that $|V(S''_u) \cap V(S'_w)| \leq t$.

Case 1. $w \in G_c$

In this case $|V(S''_u) \cap V(S'_w)| = |V(S'_u) \cap V(S'_w)| \leq t$.

Case 2. $w \in G_b$

In this case $V(S'_u) \cap V(S'_w) = V(S_u) \cap V(S_w)$. Hence, adding b_{i+1} to S'_u can only increase $|V(S'_u) \cap V(S'_w)|$ by at most one. Thus, we have $|V(S''_u) \cap V(S'_w)| \leq |V(S'_u) \cap V(S'_w)| + 1 = |V(S_u) \cap V(S_w)| + 1 \leq t - 1 + 1 = t$.

Case 3. $w \in G_a$

Since $uv \in E$, $u \in V(G_a)$ and $v \in V(G_b)$, we know that S_u and S_v share at least t nodes from $P_b - \{x\}$. Hence, $i \geq t-1$. If $b_i \in S_w$, then we would have $|V(S_u) \cap V(S_w)| \geq t$, which is impossible as $uw \notin E$. Hence, $b_i \notin S_w$, which implies that $b_{i+1} \notin S_w$ and $|V(S''_u) \cap V(S'_w)| = |V(S'_u) \cap V(S'_w)| \leq t$.

We use an analogous process to accommodate the edges between G_a and G_c, adding nodes from P_c where necessary. Now let $S''_v = S'_v$ for those S'_v from $\mathcal{F}'$ that were not modified in step 3 and let $\mathcal{F}'' = \{S''_v\}_{v \in V}$.

In order to show $\mathcal{F}''$ is an $(H_{m+1}, t+1)$-representation for G, it remains to verify that $uv \notin E(G_a)$ implies that $|V(S''_u) \cap V(S''_v)| < t+1$. First, observe that $uv \notin E(G_a)$ implies that $uv \notin E$. If S''_u was obtained from S'_u by adding a vertex from P_b and S''_v was obtained from S'_v by adding a vertex from P_c, then $|V(S''_u) \cap V(S''_v)| = |V(S'_u) \cap V(S'_v)| < t+1$. Consider the situation where both S''_u and S''_v were obtained by adding a vertex from P_b to S'_u and S'_v, respectively. In this situation, S_u and S_v must both contain x and at least t nodes from P_b. However, this implies $|V(S_u) \cap V(S_v)| \geq t$, which is impossible, since $uv \notin E$. We can make an analogous argument for the situation where both S''_u and S''_v were obtained by adding a vertex from P_c.

The arguments above, show that $\mathcal{F}''$ is an $(H_{m+1}, t+1)$-representation for G. Therefore, $G \in \mathcal{H}(t+1)$ as was to be shown. □

We use Theorem 1 to show a graph G is in $\mathcal{H}(1)$ if and only if G is in $\mathcal{H}(2)$.

Theorem 2. $\mathcal{H}(1) = \mathcal{H}(2)$

Proof. $\mathcal{H}(1) \subseteq \mathcal{H}(2)$ follows from Theorem 1. It remains to show $\mathcal{H}(2) \subseteq \mathcal{H}(1)$. Consider a graph $G = (V, E)$ in $\mathcal{H}(2)$ with (H_m, t)-representation $\{S_v\}_{v \in V}$. Recall $x = a_0 = b_0 = c_0$. If $deg_{S_v}(x) = 1$ or $x \notin S_v$ then proceed as follows: For $S_v \in P_a$ let $S'_v = S_v - a_i$ where i is the smallest number such that $a_i \in S_v$. For $S_v \in P_b$ let $S'_v = S_v - b_j$ where j is the smallest number such that $b_j \in S_v$. For $S_v \in P_c$ let $S'_v = S_v - c_k$ where k is the smallest number such that $c_k \in S_v$. If $deg_{S_v}(x) > 1$ let $S'_v = S_v$. Now we show $\{S'_v\}_v$ is an $(H_m, 1)$-representation for G. Let $uv \in E$. Then $|V(S_u) \cap V(S_v)| \geq 2$ which implies that $|V(S'_u) \cap V(S'_v)| \geq 1$. Consider $uv \notin E$. Then $|V(S_u) \cap V(S_v)| < 2$. Also S_u or S_v must contain x with degree 1 or not contain x at all. Otherwise, $deg_{S_u}(x) \geq 2$ and $deg_{S_v}(x) \geq 2$ which implies $|V(S_u) \cap V(S_v)| \geq 2$. Without loss of generality assume that $deg_{S_u}(x) = 1$ or $x \notin S_u$. If $deg_{S_u}(x) = 1$ then x is the one vertex that S_u and S_v share and $|V(S'_u) \cap V(S'_v)| = 0$ since $x \notin S'_u$. If $x \notin S_u$ then

S_u and S_v must share a node from $P_a - x$, $P_b - x$ or $P_c - x$. Without loss of generality assume a_j is the one vertex in $S_u \cap S_v$. We can also assume that j is the smallest number such that $a_j \in S_u$ and hence a_j is the largest number such that $a_j \in S_v$. So $j+1$ is the smallest number such that $a_{j+1} \in S'_u$ and j is the largest number such that $a_j \in S'_v$. Therefore $|V(S'_u) \cap V(S'_v)| = 0$. □

Now we characterize trees in $\mathcal{H}(1)$ and $\mathcal{H}(2)$ beginning with some preliminaries.

Definition 4. *An* ***asteroidal triple*** *in a graph G is a set of 3 distinct vertices $\{x_1, x_2, x_3\}$ of G such that for each choice of distinct $i, j, k \in \{1, 2, 3\}$, there is an $x_i x_j$-path that x_k is not on or adjacent to.*

Definition 5. *G is called* ***n-asteroidal*** *if n is the largest integer for which there exists a set S of n points of G with the property that any three members of S form an asteroidal triple.*

Definition 6. *A graph G has* ***property W*** *if for any pair of asteroidal triples $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ of G, every path from u_i to u_j, $1 \leq i < j \leq 3$, is adjacent to every path from v_k to v_m, $1 \leq k < m \leq 3$.*

The following result by Walter was stated in [5] and shown in [4].

Theorem 3. *A connected chordal graph G is representable on $K_{1,3}$ if and only if G is at most 3-asteroidal and G satisfies property W.*

Note that representability on $K_{1,3}$ is equivalent to being in $\mathcal{H}(1)$.

Lemma 1. *Let T be a tree and $\{u_1, u_2, u_3\}$ be an asteroidal triple of T. Then the u_1u_2-path, u_2u_3-path, and u_1u_3-path all share exactly one common vertex.*

Proof: Consider an asteroidal triple, $\{u_1, u_2, u_3\}$ in T. Observe that the path from u_3 to the u_1u_2-path does not contain u_1 or u_2. Indeed, as if so then either the u_2u_3-path would contain u_1 or the u_1u_3-path would contain u_2, which is impossible as $\{u_1, u_2, u_3\}$ is an asteroidal triple. Denote the path from u_3 to the u_1u_2-path as P and the vertex where P and the u_1u_2-path intersect as w. Observe that w is common to the u_1u_2-path, the u_2u_3-path, and the u_1u_3-path as desired. We know these paths cannot share more than one vertex as then T would contain a cycle. □

Definition 7. *A tree T has the* ***aster overlap property*** *if for every pair of asteroidal triples $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ in T if the following conditions hold.*
(i) The u_iu_j-paths and v_iv_j-paths for $i, j \in \{1, 2, 3\}$ all share exactly one common vertex, say w.
(ii) If u_i is distinct from v_j for $j = 1, 2,$ or 3, then either u_i is on the wv_k-path or v_k is on the wu_i-path for some $k \in \{1, 2, 3\}$.

Theorem 4. *A tree T has the aster overlap property iff T is at most 3-asteroidal and T satisfies property W.*

Proof: Let T be a tree with the aster overlap property. Then it is easy to see that T has property W. Now suppose for the sake of contradiction that T is N-asteroidal for $N \geq 4$. Then T has a 4-asteroidal set $\{u_1, u_2, u_3, u_4\}$. Hence $\{u_1, u_2, u_3\}$ and $\{u_2, u_3, u_4\}$ are two asteroidal triples in T. By the aster overlap property we know that the following paths all share exactly one common vertex: u_1u_2-path, u_2u_3-path, u_1u_3-path, u_3u_4-path, u_2u_4-path. Let us denote this common vertex as w. Furthermore, without loss of generality we can assume that u_4 is on the wu_1-path, since u_4 is distinct from u_1, u_2 and u_3. Now $\{u_1, u_2, u_4\}$ is also an asteroidal triple in T. However, there is no path from u_1 to u_2 that does not contain u_4. Contradiction.

Let T be a tree that is at most 3-asteroidal and has property W. Consider a pair of asteroidal triples $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ in T. From Lemma 1 we know that the u_1u_2-path, u_2u_3-path, and u_1u_3-path all share exactly one common vertex, say u. We also know that the v_1v_2-path, v_2v_3-path, and v_1v_3-path all share exactly one common vertex, say v. To show the first part of the aster overlap property, we will show that $u = v$. For the sake of contradiction suppose that $u \neq v$.

Claim 1 The only path from any u_iu_j-path, for $1 \leq i < j \leq 3$, to any v_kv_m-path, for $1 \leq k < m \leq 3$ is the uv-path.

Proof of claim 1: Let us suppose the contrary. Without loss of generality suppose there is a path from the u_1u_2-path to the v_1v_2-path that is distinct from the uv-path. Let us denote this path the xy-path. Then u, x, y, v, u is a cycle in T, which is impossible. □

Now we know that v_1, v_2, or v_3 must be distinct from u_1, u_2 and u_3. Without loss of generality suppose v_1 is distinct from u_i for $i \in \{1, 2, 3\}$.

Claim 2 $\{u_1, u_2, u_3, v_1\}$ forms a 4-asteroidal set in T.

Proof of claim 2: We already know that $\{u_1, u_2, u_3\}$ forms an asteroidal triple in T. So it remains to show that $\{u_i, u_j, v_1\}$ forms an asteroidal triple for any $i, j \in \{1, 2, 3\}$ and $i \neq j$. From Claim 1 we know v_1 is not adjacent to the u_1u_2-path, as this path contains u. Also, the u_iu-path, uv-path and vv_1-path combined, form a path from u_i to v_1 that is not adjacent to u_j. We can similarly justify that u_i is not adjacent to the u_jv_1-path. Hence our claim is shown. □

Claim 2 contradicts our assumption that T is at most 3-asteroidal. Hence $u = v$. Let us denote this vertex w. It remains to show that, if u_i is distinct from v_j for $j \in \{1, 2, 3\}$, then either u_i is on the wv_k-path or v_k is on the wu_i-path for some $k \in \{1, 2, 3\}$. We do so by supposing the contrary. Without loss of generality suppose that u_1 is distinct from v_j for $j \in \{1, 2, 3\}$, u_1 is not on the wv_k-path and v_k is not on the wu_1-path for $k \in \{1, 2, 3\}$. We know $d(w, u_1) \geq 2$, as if not then u_1 would be adjacent to the u_2u_3-path. Hence $\{v_1, v_2, v_3, u_1\}$ forms a 4-asteroidal set. Contradiction. □

Theorem 5. *A tree T is in $\mathcal{H}(1)$ iff T has the aster overlap property.*

Theorem 5 follows directly from Theorem 3 and Theorem 4.

Theorem 6. *A tree T is in $\mathcal{H}(2)$ iff T has the aster overlap property.*

Theorem 6 follows directly from Theorem 5 and Theorem 2.

References

[1] P. Buneman, A characterization of rigid circuit graphs, *Discrete Math.*, **9** (1974) 205-212.

[2] F. Gavril, The intersection graphs of subtrees in a tree are exactly the chordal graphs, *J. Combin. Theory Ser. B*, **16** (1974) 47-56.

[3] M.A. Saadi, *Some results on tree tolerance representations*, Ph.D. Thesis, Department of Mathematics, University of Rhode Island, 2001.

[4] J. Walter, *Representations of rigid cycle graphs*, Dissertation, Wayne State University (1972).

[5] J. Walter, Representations of chordal graphs as subtrees of a tree, *J. of Graph Theory*, **2** (1978), 265-267.