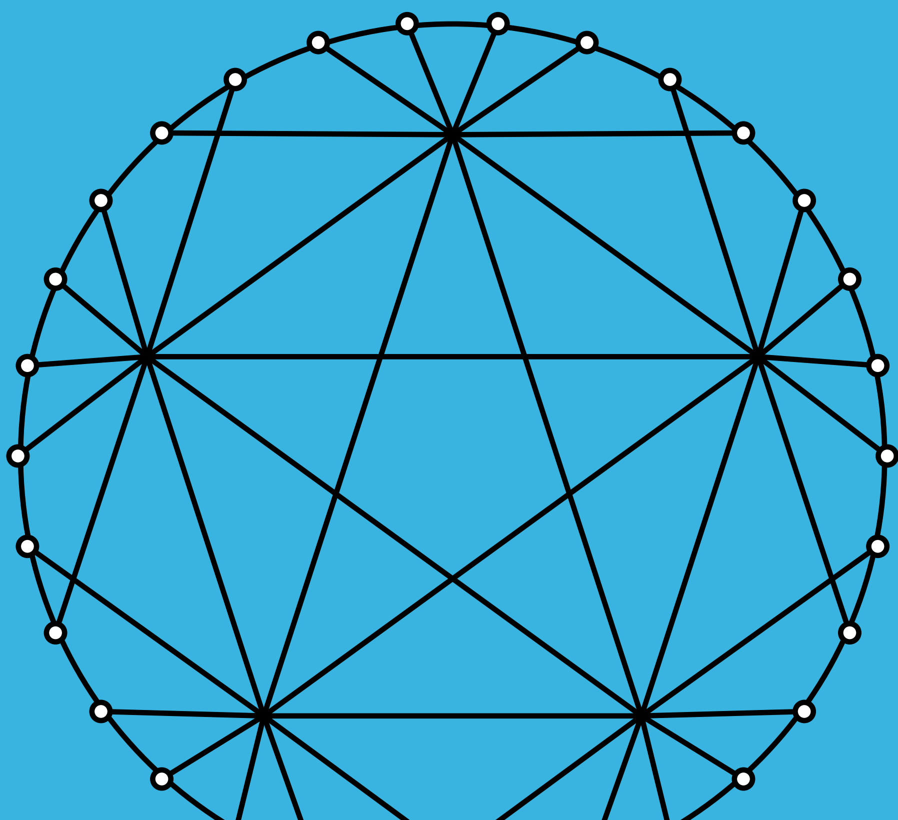
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 82
February 2018



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **82** (2018), Pages 80–95

The harmonious number of graphs and related parameters

R. Ichishima

College of Humanities and Sciences, Nihon University, 3-25-40 Sakurajousui Setagaya-ku, Tokyo 156-8550, Japan
ichishima@msh.biglobe.ne.jp

F.A. Muntaner-Batle and A. Oshima

Graph Theory and Applications Research GroupA, School of Electrical Engineering and Computer Science, Faculty of Engineering and Built Environment, The University of Newcastle, NSW 2308 Australia
famb1es@yahoo.es and akitoism@yahoo.co.jp

Abstract: In this paper, we introduce the notion of (strong) harmonious number of a graph, and provide a necessary condition for a graph to have finite harmonious number. We also provide sufficient conditions for a graph to have infinite (strong) harmonious number. In addition, we examine the relations between (strong) harmonious numbers and other parameters that have previously been studied in the area of graph labelings. As applications of these, we determine the formulas for the (strong) harmonious numbers of some 2-regular graphs and all complete bipartite graphs, which lead us to formulas for other parameters of the same classes of graphs.

1 Introduction

In this paper, we will deal only with finite graphs without loops or multiple edges. Notation will be standard, and we will follow the notation in [2].

Received: 5 June 2017
Accepted: 13 September 2017

The *vertex set* and *edge set* of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Given any two graphs G and H, their *union*, denoted by $G \cup H$, is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The union of any finite number of graphs is defined similarly. For integers a and b with $a \leq b$, the set $\{x \in \mathbb{Z} | a \leq x \leq b\}$ is denoted by $[a, b]$, where $\mathbb{Z}$ denotes the set of integers.

An extensive survey of graph labelings as well as their applications has been written by Gallian [9]. As he pointed out in his survey, most graph labeling methods trace their origin to one introduced by Rosa [22] in 1967 or one given by Graham and Sloane [12] in 1980. Rosa [22] called a function f a *β-valuation* of a graph G with q edges if $f : V(G) \to [0, q]$ is an injective function such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. A few years later, Golomb [11] called such labelings *graceful* and this is the term that has been most commonly used since then. A *graceful graph* is a graph that has a graceful labeling. Rosa [22] also introduced the concept of α-valuations (a particular type of graceful labelings) as a tool for decomposing the complete graph into isomorphic subgraphs. A graceful labeling f of a graph G is called an *α-valuation* if there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. It follows that a graph that admits an α-valuation is necessarily bipartite.

The *gamma-number*, $\gamma(G)$, of a graph G is the smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. This definition first appeared in a paper by Golomb [11] who used the term 'gracefulness' and showed that $\gamma(G) < +\infty$ for every graph G. If G is a graph of size q with $\gamma(G) = q$, then G is graceful. Thus, the gamma-number of a graph G is a measure of how close G is to being graceful.

A restriction of the gamma-number was recently introduced in [15]. The *strong gamma-number*, $\gamma_s(G)$, of a graph G is the smallest positive integer n such that $\gamma(G) = n$ with the additional property that there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. The strong gamma-number is defined to be $+\infty$, otherwise. It is clear that if G is a graph such that $\gamma_s(G) < +\infty$, then G is necessarily bipartite. The converse is also shown to be true in [15]. If G is a graph of size q with $\gamma_s(G) = q$, then G has an α-valuation. Thus, the strong gamma-number of a graph G can be regarded as a measure of how close G is to having an α-valuation.

Harmonious graphs were first studied by Graham and Sloane [12] in connection with error-correcting codes and channel assignment problems. They defined a graph G with q edges to be *harmonious* if there exists an injective function $f : V(G) \to \mathbb{Z}_q$ such that each $uv \in E(G)$ is labeled $f(u) + f(v) \pmod{q}$ and the resulting edge labels are distinct. Such a function is called a *harmonious labeling.* If G is a tree (so that $|E(G)| = |V(G)| - 1)$ exactly two vertices are labeled the same; otherwise, the definition is the same.

We are now in a position to provide the definitions for the key concepts that are discussed in this paper. The *harmonious number*, $\eta(G)$, of a graph G with q edges is defined to be either the smallest positive integer n for which there exists an injective function $f : V(G) \to \mathbb{Z}_{n+1}$ such that each $uv \in E(G)$ is labeled $f(u) + f(v) \pmod{q}$ and the resulting edge labels are distinct or $+\infty$ if there exists no such integer n. If such functions exist, then we call them *harmonious numberings.* The *strong harmonious number*, $\eta_s(G)$, of a graph G is defined to be either the smallest positive integer n such that $\eta(G) = n$ with the additional property that there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$ or $+\infty$ if there exists no such integer n. A harmonious numbering f of a graph G is called a *strong harmonious numbering* if there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. As in the case of strong gamma-number, if G is a graph such that $\eta_s(G) < +\infty$, then G is necessarily bipartite.

By means of the above definitions, the parameters $\eta(G)$ and $\eta_s(G)$ can be regarded as measures of how close a graph G is to being harmonious, and we have the following relations among the two parameters.

Lemma 1. *If G is a graph of order p other than a tree, then*

$$p - 1 \leq \eta(G) \leq \eta_s(G).$$

The notion of edge-magic labelings was introduced in 1970 by Kotzig and Rosa [16]. These labelings were originally called magic valuations by them. These were rediscovered in 1996 by Ringel and Lladó [21] who coined one of the now popular terms for them: edge-magic labelings. For a graph G of order p and size q, a bijective function $f : V(G) \cup E(G) \to [1, p + q]$ is called an *edge-magic labeling* if $f(u) + f(v) + f(uv)$ is a constant for each $uv \in E(G)$. If such a labeling exists, then G is called an *edge-magic graph.* In 1998, Enomoto et al. [3] defined a slightly restricted version of an edge-magic labeling f of a graph G by requiring that $f(V(G)) = [1, |V(G)|]$. Such a labeling was called by them *super edge-magic* (see [6] and [17] for the significance of super edge-magic labelings). Thus, a *super edge-magic* *graph* is a graph that admits a super edge-magic labeling.

Due to the following lemma found in [6], it is sufficient to exhibit the vertex labeling of a super edge-magic graph.

Lemma 2. *A graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to [1, |V(G)|]$ such that the set*

$$\{f(u) + f(v)\,|uv \in E(G)\}$$

consists of $|E(G)|$ consecutive integers.

In 2001, Muntaner-Batle [18] defined the concept of special super edge-magic labelings of bipartite graphs. Let G be a bipartite graph with partite sets X and Y. If G has a super edge-magic labeling f with the property that $f(X) = [1, |X|]$ and $f(Y) = [|X| + 1, |V(G)|]$, then f is called a *special super edge-magic labeling*. Oshima [19] subsequently called such labelings *consecutively super edge-magic*. In this paper, we prefer to use the latter terminology to emphasize the property that a consecutively super edge-magic labeling uses consecutive integers in each partite set. Also, we refer a bipartite graph with a consecutively super edge-magic labeling as a *consecutively super edge-magic graph.*

The following lemma found in [18] provides us with a necessary and sufficient condition for a bipartite graph to be consecutively super edge-magic.

Lemma 3. *Let G be a bipartite graph with partite sets X and Y. Then G is consecutively super edge-magic if and only if there exists a bijective function $f : V(G) \to [1, |V(G)|]$ such that $f(X) = [1, |X|]$, $f(Y) = [|X| + 1, |V(G)|]$ and the set*

$$\{f(u) + f(v)|uv \in E(G)\}$$

consists of $|E(G)|$ consecutive integers.

For every graph G, Kotzig and Rosa [16] proved that there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This motivated them to define the edge-magic deficiency. The *edge-magic deficiency*, $\mu(G)$, of a graph G is the smallest nonnegative integer n for which $G \cup nK_1$ is edge-magic.

We next provide the definitions for two types of parameters that play an important role in the study of the (strong) harmonious number.

Inspired by Kotzig-Rosa notion, the super edge-magic deficiency was analogously defined by Figueroa-Centeno et al. [8]. The *super edge-magic deficiency*, $\mu_s(G)$, of a graph G is either the smallest nonnegative integer n with the property that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. If G is a graph with $\mu_s(G) = 0$, then G is super edge-magic. Thus, the super edge-magic deficiency of a graph G is a measure of how close G is to being super edge-magic.

The concept of the consecutively super edge-magic deficiency naturally arose in the study of consecutively super edge-magic properties of graphs (see [20]). The *consecutively super edge-magic deficiency*, $\mu_c(G)$, of a graph G is either the smallest nonnegative integer n with the property that $G \cup nK_1$ is consecutively super edge-magic or $+\infty$ if there exists no such integer n. If G is a graph with $\mu_c(G) = 0$, then G is consecutively super edge-magic. Thus, the consecutively super edge-magic deficiency of a graph G is a measure of how close G is to being consecutively super edge-magic.

2 Basic results

In this section, we provide a necessary condition for a graph to have finite harmonious number as well as sufficient conditions for a graph to have infinite (strong) harmonious number.

An *Eulerian circuit* of a graph G is a circuit containing all of the edges and vertices of G. A graph having an Eulerian circuit is called an *Eulerian graph.* A necessary condition for an Eulerian graph to have finite harmonious number is presented next.

Lemma 4. *If G is an Eulerian graph of even size q such that $\eta(G) < +\infty$, then $q \equiv 0 \pmod 4$.*

Proof. Let $C : x_0, x_1, \ldots, x_{q-1}, x_q = x_0$ be an Eulerian circuit of G, where q is even, and let $\eta(G) = n$ for some positive integer n. Then there exists a harmonious numbering $f : V(G) \to \mathbb{Z}_{n+1}$ such that $f(x_i) = a_i$ for each $i \in [0, q]$, where $a_i = a_j$ if $x_i = x_j$. Thus, the label of the edge $x_{i-1}x_i$ is $a_{i-1} + a_i \pmod q$, which implies that

$$\{a_{i-1} + a_i \pmod q | i \in [1, q]\} = [0, q-1].$$

Hence, the sum of the labels of the edges of G is

$$\sum_{i=1}^{q} (a_{i-1} + a_i) = 2 \sum_{i=0}^{q-1} f(x_i) \equiv 0 \pmod 2,$$

that is, the sum of the edge labels of G is even. However, the sum of the edge labels is

$$\sum_{i=1}^{q} (a_{i-1} + a_i) \equiv \sum_{i=0}^{q-1} i \equiv q(q-1)/2 \pmod q;$$

so $q(q-1)/2$ is even. Consequently, $4|q(q-1)$, which implies that $4|q$ or $4|q-1$ so that $q \equiv 0 \pmod 4$, since q is even. □

The following result is easily obtained by taking the contraposive of Lemma 4 and then applying Lemma 1.

Corollary 1. *If G is an Eulerian graph such that $|E(G)| \equiv 2 \pmod 4$, then*

$$\eta(G) = \eta_s(G) = +\infty.$$

A graph G is defined to be an *even graph* if all of its vertices have even degree. With this definition in hand, it is now possible to extend the preceding corollary to the sufficient condition for an even graph to have infinite (strong) harmonious number.

Theorem 1. *If G is an even graph such that $|E(G)| \equiv 2 \pmod 4$, then*

$$\eta(G) = \eta_s(G) = +\infty.$$

Proof. Since G is an even graph, it follows that every component of G is Eulerian. Let $G_1, G_2, \ldots, G_k$ $(k \geq 1)$ be the components of G. For each $i \in [1,k]$, let $C_i : x_0^i, x_1^i, \ldots, x_{q_i-1}^i, x_{q_i}^i = x_0^i$ be an Eulerian circuit of G_i, where $q_i = |E(G_i)|$. In light of Lemma 1, it suffices to show that $\eta(G) = +\infty$ when G is an even graph such that $|E(G)| \equiv 2 \pmod 4$. For this purpose, let $q = |E(G)|$ and suppose, to the contrary, that $\eta(G) = n$ for some positive integer n. Then there exists a harmonious numbering $f : V(G) \to \mathbb{Z}_{n+1}$ such that $f\left(x_j^i\right) = a_j^i$ for each $i \in [1,k]$ and $j \in [0, q_i]$, where $a_s^i = a_t^i$ if $x_s^i = x_t^i$. Thus, the label of the edge $x_{j-1}^i x_j^i$ is $a_{j-1}^i + a_j^i \pmod q$, which implies that

$$\left\{a_{j-1}^i + a_j^i \pmod q | i \in [1,k] \text{ and } j \in [1, q_i]\right\} = [0, q-1].$$

Notice that

$$\sum_{j=1}^{q_i} \left(a_{j-1}^i + a_j^i\right) = 2 \sum_{j=0}^{q_i - 1} f\left(x_j^i\right) \equiv 0 \pmod 2$$

for all $i \in [1, k]$. Hence, the sum of the labels of edges of G is

$$\sum_{i=1}^{k} \sum_{j=1}^{q_i} \left(a_{j-1}^i + a_j^i\right) \equiv 0 \pmod 2,$$

that is, the sum of the edge labels of G is even. However, the sum of edge labels is

$$\sum_{i=1}^{k} \sum_{j=1}^{q_i} \left(a_{j-1}^i + a_j^i\right) \equiv \sum_{i=0}^{q-1} i \equiv q\left(q-1\right)/2 \pmod q.$$

Consequently, $4|q\left(q-1\right)$, which implies that $4|q$ or $4|q-1$ so that $q \equiv 0 \pmod 4$ or $q \equiv 1 \pmod 4$. This contradicts the hypothesis that $q \equiv 2 \pmod 4$. □

3 Relations among parameters

In this section, we exhibit the relations between (strong) harmonious numbers and parameters that have been discussed in the introduction of this paper. We also establish lower and upper bounds for the strong harmonious number, which leads us to a sufficient condition for a graph G to have $\eta\left(G\right) = \eta_s\left(G\right)$. Moreover, we provide a sufficient condition of graphs G for which $\mu_s\left(G\right) = \mu_c\left(G\right)$.

We begin with the following result that gives us an upper bound for the harmonious number of a graph in terms of its super edge-magic deficiency and order.

Theorem 2. *For every graph G of order p,*

$$\eta\left(G\right) \leq \mu_s\left(G\right) + p - 1.$$

Proof. Let G be a graph of order p and size q. Notice that the result is trivial when $\mu_s\left(G\right) = +\infty$. Thus, assume that $\mu_s\left(G\right) < +\infty$, and let $H \cong G \cup nK_1$, where $n = \mu_s\left(G\right)$ for some nonnegative integer n. It follows from Lemma 2 that there exists a super edge-magic labeling $f : V(H) \to [1, n+p]$ such that

$$\{f(u) + f(v) \,|uv \in E(H)\} = [s, s+q-1],$$

where $s = \min\{f(u) + f(v) \,|uv \in E(H)\}$. At this point, define the bijective function $g : V(H) \to [0, n+p-1]$ such that $g(v) = f(v) - 1$ for all $v \in V(H)$. If we consider the restriction of g to $V(G)$, then we obtain

$$\{g(u) + g(v) \,|uv \in E(G)\} = [s-2, s+q-3],$$

which is a set of q consecutive integers, and

$$\max\{g(v) \,|v \in V(G)\} \leq |V(H)| - 1 = n + p - 1.$$

It is now immediate that the edge labels induced by $g(u) + g(v) \pmod q$ for each $uv \in E(H)$ are distinct. Therefore, $\eta(G) \leq n + p - 1$, implying that $\eta(G) \leq \mu_s(G) + p - 1$. □

It is important to notice that the bound given in Theorem 2 can be viewed as a lower bound for the super edge-magic deficiency of a graph, since previously nontrivial lower bound was not available for this parameter.

The following result provides us with an upper bound for the harmonious number of a graph in terms of its consecutively super edge-magic deficiency and order.

Theorem 3. *For every graph G of order p,*

$$\eta_s(G) \leq \mu_c(G) + p - 1.$$

Proof. Without loss of generality, assume that G is a bipartite graph of order p and size q such that $\mu_c(G) < +\infty$; otherwise, the result is trivial. Let $H \cong G \cup nK_1$, where $n = \mu_c(G)$ for some nonnegative integer n, and let X and Y be the partite sets of H, where $|X| = x$ and $|Y| = y$. By means of Lemma 3, there exists a consecutively super edge-magic labeling $f : V(H) \to [1, n+p]$ such that $f(X) = [1, x]$, $f(Y) = [x+1, n+p]$ and

$$\{f(u) + f(v) \,|uv \in E(G)\} = [s, s+q-1],$$

where $s = \min\{f(u) + f(v) \,|uv \in E(H)\}$. Then the bijective function $g : V(H) \to [0, n+p-1]$ defined by $g(v) = f(v) - 1$ for all $v \in V(H)$ provides that

$$g(X) = [0, x-1] \text{ and } g(Y) = [x, n+p-1],$$

and

$$g(u) + g(v) = f(u) + f(v) - 2$$

for each $uv \in E(H)$, where $u \in X$ and $v \in Y$. If we consider the restriction of g to $V(G)$, then we obtain

$$\{g(u) + g(v) \,|uv \in E(G)\} = [s-2, s+q-3],$$

which is a set of q consecutive integers, and

$$\max\{g(v)\,|v \in V(G)\} \leq |V(H)| - 1 = n + p - 1.$$

It is now immediate that the edge labels induced by $g(u) + g(v) \pmod q$ for each $uv \in E(H)$ are distinct. Therefore, $\eta_s(G) \leq n + p - 1$, implying that $\eta_s(G) \leq \mu_c(G) + p - 1$. □

It is interesting to notice that the bound presented in Theorem 3 can be viewed as a lower bound for the consecutively super edge-magic deficiency of a graph, since previously nontrivial lower bound was not known for this parameter.

Another parameter that plays an important role in the study of the strong harmonious number is the alpha-number defined and studied in [20]. The *alpha-number*, $\alpha(G)$, of a graph G is the smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c + |E(G)| - 1]$ for some positive integer c. Furthermore, the additional property that there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$ is required. The alpha-number of G is defined to be $+\infty$, otherwise. If G is a graph of size q with $\alpha(G) = q$, then G has an α-valuation. Thus, the alpha-number of a graph G can be regarded as a measure of how close G is to having an α-valuation.

The following result provides us with lower and upper bounds for the strong harmonious number of a graph in terms of the strong gamma-number and alpha-number.

Theorem 4. *For every graph G,*

$$\gamma_s(G) \leq \eta_s(G) \leq \alpha(G).$$

Proof. The upper bound for $\eta_s(G)$ is a direct consequence of Theorem 3 and the fact obtained in [14] that $\alpha(G) = \mu_c(G) + p - 1$ for every graph G of order p.

To verify the lower bound, assume that G is a bipartite graph and $\eta_s(G) < +\infty$; otherwise, the lower bound is immediate. Then there exists some positive integer n such that $n = \eta_s(G)$. It follows that there exists a strong harmonious numbering $f : V(G) \to \mathbb{Z}_{n+1}$, and hence there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. Without loss of generality, let G have partite sets

$$U = \{v \in V(G) \,|\, f(v) \leq \lambda\} \text{ and } V = \{v \in V(G) \,|\, f(v) > \lambda\}.$$

Now, consider the function $g : V(G) \to \mathbb{Z}_{n+1}$ such that

$$g(v) = \begin{cases} \lambda - f(v) & \text{if } v \in U, \\ f(v) & \text{if } v \in V. \end{cases}$$

Notice then that $\min\{g(v) \,|\, v \in V\} > \max\{g(v) \,|\, v \in U\}$. This implies that

$$|g(u) - g(v)| = g(v) - g(u) = f(u) + f(v) - \lambda$$

for each $uv \in E(G)$, where $u \in U$ and $v \in V$. Notice also that

$$f(u) + f(v) \geq \lambda + 1$$

for any $uv \in E(G)$. This implies that

$$|g(u) - g(v)| \geq 1$$

for any $uv \in E(G)$. Moreover, since the edge labels induced by $f(u)+f(v)$ for each $uv \in E(G)$ are distinct modulo $|E(G)|$, it follows that the edge labels induced by $|g(u) - g(v)|$ for each $uv \in E(G)$ are distinct. It only remains to notice that g is an injective function with the property that

$$\min\{g(u), g(v)\} \leq \lambda < \max\{g(u), g(v)\}$$

for each $uv \in E(G)$, and

$$\max\{g(v) \,|\, v \in V(G)\} = \max\{g(v) \,|\, v \in V\} = n.$$

Therefore, $\gamma_s(G) \leq n$, implying that $\gamma_s(G) \leq \eta_s(G)$. □

The bounds given in Theorem 4 are sharp. To see this, notice that every star $K_{1,n}$ has an α-valuation by labeling the central vertex with 0 and the remaining vertices with 1 through n. This implies that $\alpha(K_{1,n}) \leq n$ for every positive integer n. It is observed in [15] that $\gamma_s(G) \geq \max\{p-1, q\}$ for every graph G of order p and size q. This provides that $\gamma_s(K_{1,n}) \geq n$ for every positive integer n. By Theorem 4,

$$\gamma_s(K_{1,n}) = \eta_s(K_{1,n}) = \alpha(K_{1,n}) = n$$

for every positive integer n.

The upper bound presented in Theorem 4 together with Lemma 1 implies the following result, which provides us with a sufficient condition for a graph G to have $\eta(G) = \eta_s(G)$.

Corollary 2. *If G is a graph with an α-valuation that is not harmonious, then*

$$\eta(G) = \eta_s(G) = |E(G)|.$$

Later on, the above result will prove to be useful for computing the (strong) harmonious numbers of some graphs.

To present our next result, some definitions are required. The notion of sequential labelings was introduced in 1983 by Grace [10]. For a graph G of size q, an injective function $f : V(G) \to [0, q-1]$ (with the label q allowed if G is a tree) is called a *sequential labeling* if each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting set of edge labels is $[c, c+q-1]$ for some positive integer c. If such a labeling exists, then G is called a *sequential graph*.

We now consider the parameter that can be regarded as a measure of how close a graph is to being sequential. The concept of sequential number was first introduced in [5] for graphs without isolated vertices and was recently extended in [14] for any graph. The *sequential number*, $\sigma(G)$, of a graph G with q edges is either the smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting set of edge labels is $[c, c+q-1]$ for some positive integer c or $+\infty$ if there exists no such integer n.

It is known from [14] that if a graph G has an α-valuation and is not sequential, then $\sigma(G) = |E(G)|$. This implies that if G has an α-valuation and is not harmonious, then $\sigma(G) = |E(G)|$. It is also known from [14] that $\mu_s(G) = \sigma(G) + |V(G)| - 1$, implying that if G is a graph with an α-valuation that is not harmonious, then $\mu_s(G) = |E(G)| - |V(G)| + 1$. On the other hand, it was observed in [13] that if G is a graph with an α-valuation, then $\mu_c(G) = |E(G)| - |V(G)| + 1$. Summarizing these facts, we have the following relation, which provides us with a sufficient condition for a graph G to have $\mu_s(G) = \mu_c(G)$.

Corollary 3. *If G is a graph with an α-valuation that is not harmonious, then*

$$\mu_s(G) = \mu_c(G) = |E(G)| - |V(G)| + 1.$$

From Corollaries 2 and 3, we arrive at the following two relations, which will later serve as the tools for computing the (consecutively) super edge-magic deficiencies of some graphs.

Corollary 4. *If G is a graph with an α-valuation that is not harmonious, then*

$$\eta(G) = \mu_s(G) + |V(G)| - 1.$$

Corollary 5. *If G is a graph with an α-valuation that is not harmonious, then*

$$\eta_s(G) = \mu_c(G) + |V(G)| - 1.$$

We end this section with remarks on the relations obtained in the preceding two corollaries.

Notice that the relation of Corollary 4 shows the sharpness of the bound presented in Theorem 2, and that the problems of determining the harmonious number and the super edge-magic deficiency of certain graphs are equivalent. Notice also that the relation of Corollary 5 shows the sharpness of the bound given in Theorem 3, and that the problems of determining the strong harmonious number and the consecutively super edge-magic deficiency of certain graphs are equivalent.

4 Applications

As applications of Theorem 1 and Corollary 2, we determine, in this section, the (strong) harmonious numbers for some 2-regular graphs and all complete bipartite graphs. These together with Corollary 5 lead us to formulas for the consecutively super edge-magic deficiencies of the same classes of graphs.

To proceed, we mention the previously known results on 2-regular graphs. Youssef [24] obtained a harmonious labeling of the 2-regular graph mC_n if both m and n are odd, while Seoud et al. [23] observed that the 2-regular graph mC_n is not harmonious if either m or n is even. Rosa [22] verified that the cycle C_n has an α-valuation if and only if $n \equiv 0 \pmod 4$. Abrham and Kotzig [1] proved that the 2-regular graph $C_m \cup C_n$ has an α-valuation if and only if both m and n are even, and $m + n \equiv 0 \pmod 4$. This implies that the 2-regular graph $2C_n$ has an α-valuation if and only if n is even. Eshghi [4] established that every 2-regular bipartite graph with three components has an α-valuation if and only if the number of edges is a multiple of four except for $3C_4$.

Note that $\eta\,(3C_4) \leq 12$ is obtained by labeling the cycles in $3C_4$ with

$$0-7-2-8-0,\ 1-5-6-11-1,\ 3-10-4-12-3,$$

whereas $\eta_s\,(3C_4) \leq 13$ is obtained by labeling the cycles in $3C_4$ with

$$0-6-1-8-0,\ 3-7-5-11-3,\ 2-9-4-13-2.$$

Applying Theorem 1 and Corollary 2 with the aforementioned results on 2-regular graphs, we obtain the following formulas.

- $\eta\,(C_n) = \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \equiv 0 \pmod 4, \\ +\infty & \text{if } n \equiv 2 \pmod 4. \end{cases}$
- $\eta_s\,(C_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ +\infty & \text{if } n \text{ is odd or } n \equiv 2 \pmod 4. \end{cases}$
- $\eta\,(2C_n) = \eta_s\,(2C_n) = \begin{cases} 2n & \text{if } n \text{ is even,} \\ +\infty & \text{if } n \text{ is odd.} \end{cases}$
- $\eta\,(3C_n) = \begin{cases} 3n-1 & \text{if } n \text{ is odd,} \\ 3n & \text{if } n \equiv 0 \pmod 4, \\ +\infty & \text{if } n \equiv 2 \pmod 4. \end{cases}$
- $\eta_s\,(3C_n) = \begin{cases} 3n+1 & \text{if } n = 4, \\ 3n & \text{if } n \geq 8 \text{ and } n \equiv 0 \pmod 4, \\ +\infty & \text{if } n \text{ is odd or } n \equiv 2 \pmod 4. \end{cases}$

As we have seen in the previous section, $\eta_s\,(K_{1,n}) = n$ for every positive integer n. We now turn our attention briefly to the harmonious number of stars. For every positive integer n, the star $K_{1,n}$ is harmonious by labeling the central vertex with 0 and the remaining vertices with 0 through $n-1$, which implies that $\eta\,(K_{1,n}) \leq n-1$. On the other hand, we have $\eta\,(K_{1,n}) \geq n-1$ for every positive integer n, since it is true that $\eta\,(G) \geq \min\{p-1, q-1\}$ for every graph G of order p and size q. This establishes that $\eta\,(K_{1,n}) = n-1$ for every positive integer n.

We next consider a formula for the (strong) harmonious number of the complete bipartite graph that is not a star. Graham and Sloane [12] proved that the complete bipartite graph $K_{m,n}$ is harmonious if and only if either $m = 1$ or $n = 1$. On the other hand, Rosa [22] showed that every complete bipartite graph $K_{m,n}$ has an α-valuation. It follows by Corollary 2 that $\eta(K_{m,n}) = \eta_s(K_{m,n}) = mn$ for every two integers m and n with $m \geq 2$ and $n \geq 2$.

We end this section with applications of Corollaries 2 and 4.

Notice that the results on harmonious numbers found in this section and Corollary 4 give us the formulas for $\mu_s(C_n)$, $\mu_s(2C_n)$ and $\mu_s(3C_n)$, which were obtained in [7, 8], with relative ease.

Notice also that $\mu_s(K_{1,n}) = 0$ for every positive integer n (by labeling the central vertex with 1 and the remaining vertices with 2 through $n+1$). For every two integers m and n with $m \geq 2$ and $n \geq 2$, the formula $\mu_s(K_{m,n}) = (m-1)(n-1)$ is obtained by applying the above result on $\eta(K_{m,n})$ and Corollary 4. Thus, we have the following formula.

- $\mu_s(K_{m,n}) = (m-1)(n-1)$ for all positive integers m and n.

In the above, we resolve a conjecture of Figueroa-Centeno et al. [8] (see also [5] for an alternative proof).

Finally, notice that the results on strong harmonious numbers found in this section and Corollary 5 provide us with the following formulas.

- $\mu_c(C_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4, \\ +\infty & \text{if } n \text{ is odd or } n \equiv 2 \pmod 4. \end{cases}$

- $\mu_c(2C_n) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ +\infty & \text{if } n \text{ is odd}. \end{cases}$

- $\mu_c(3C_n) = \begin{cases} 2 & \text{if } n = 4, \\ 1 & \text{if } n \geq 8 \text{ and } n \equiv 0 \pmod 4, \\ +\infty & \text{if } n \text{ is odd or } n \equiv 2 \pmod 4. \end{cases}$

- $\mu_c(K_{m,n}) = (m-1)(n-1)$ for all positive integers m and n.

References

[1] J. Abrham and A. Kotzig, Graceful valuations of 2-regular graphs with two components, *Discrete Math.*, **150** (1996) 3–15.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Wadsworth & Brook/Cole Advanced Books and Software, Monterey, Calif. 1986.

[3] H. Enomoto, A. Lladó, T. Nakamigawa, and G. Ringel, Super edge-magic graphs, *SUT J. Math.*, **34** (1998) 105–109.

[4] K. Eshghi, The existence and construction of α-valuations of 2-regular graphs with 3 components, Ph. D. Thesis, Industrial Engineering Dept., University of Toronto, 1997.

[5] R.M. Figueroa-Centeno and R. Ichishima, On the sequential number and super edge-magic deficiency of graphs, *Ars Combin.*, **129** (2016) 157–163.

[6] R.M. Figueroa-Centeno, R. Ichishima, and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001) 153–168.

[7] R.M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, *J. Combin. Math. Combin. Comput.*, **55** (2005) 17–31.

[8] R.M. Figueroa-Centeno, R. Ichishima, and F.A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Ars Combin.*, **78** (2006) 33–45.

[9] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, (2016) #DS6.

[10] T. Grace, On sequential labelings of graphs, *J. Graph Theory* **7** (1983) 195–201.

[11] S.W. Golomb, How to number a graph, in *Graph Theory and Computing*, (ed. R.C. Read), Academic Press, New York (1972), 23–37.

[12] R.L. Graham and N.J. Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Math.*, **1** (1980) 382–404.

[13] R. Ichishima, F.A. Muntaner-Batle and A. Oshima, The consecutively super edge-magic deficiency of graphs and related concepts, *preprint.*

[14] R. Ichishima, F.A. Muntaner-Batle, and A. Oshima, On the strong sequential number of graphs, *preprint.*

[15] R. Ichishima and A. Oshima, Bounds for the gamma-number of graphs, *Util Math.*, *to appear.*

[16] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970) 451–461.

[17] S.C. López, F.A. Muntaner-Batle, and M. Rius-Font, Labeling constructions using digraph products, *Discrete Appl. Math.*, **161** (2013) 3005–3016.

[18] F.A. Muntaner-Batle, Special super edge-magic labelings of bipartite graphs, *J. Combin. Math. Combin. Comput.*, **39** (2001) 107–120.

[19] A. Oshima, Consecutively super edge-magic tree with diameter 4, *Rev. Bull. Calcutta Math. Soc.*, **15** (2007) 87–90.

[20] A. Oshima, R. Ichishima and F.A. Muntaner-Batle, New parameters for studying graceful properties of graphs, *Electron. Notes Discrete Math.*, **60** (2017) 3–10.

[21] G. Ringel and A. Lladó, Another tree conjecture, *Bull. Inst. Combin. Appl.*, **18** (1996) 83–85.

[22] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs* (Internat. Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris (1967), 349–355.

[23] M.A. Seoud, A. E.I. Abdel Maqsoud, and J. Sheehan, Harmonious graphs, *Util. Math.*, **47** (1995) 225–233.

[24] M.Z. Youssef, Two general results on harmonious labelings, *Ars Combin.*, **68** (2003) 225–230.