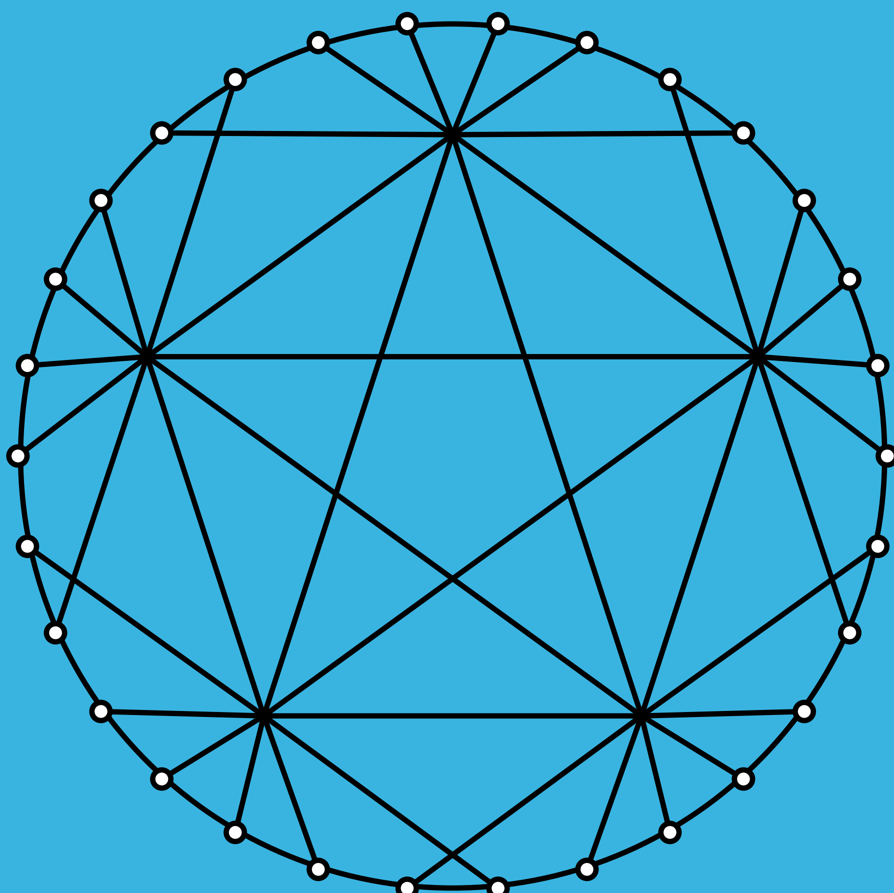
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 82
February 2018



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **82** (2018), Pages 42–50

Shifting property and other binomial identities for the bivariate Fibonacci polynomials

Emanuele Munarini

Dipartimento di Matematica, Politecnico di Milano, P.zza Leonardo da Vinci 32, 20133 Milano, Italy
emanuele.munarini@polimi.it

Abstract: We obtain the *shifting property* and some other combinatorial identities for the bivariate Fibonacci polynomials (which generalize the Fibonacci, Pell, Jacobsthal, Chebyschev, Fermat, Morgav-Voyce polynomials and the Horadam numbers). In particular, we specialize all these identities to the Chebyshev polynomials of the second kind.

Keywords: combinatorial sums, shifting property, Chebyshev polynomials.

AMS Classification: Primary 05A19; Secondary 11B65.

1 Introduction

The *bivariate Fibonacci polynomials* $F_n(x,y)$ are defined by the recurrence

$$F_{n+2}(x,y) = xF_{n+1}(x,y) + yF_n(x,y) \tag{1}$$

with the initial values $F_0(x,y) = 1$ and $F_1(x,y) = x$. Several classical numerical and polynomial sequences can be viewed as a specialization of this sequence. For instance, we have:

Received: 5 Nov 2017
Accepted: 10 Jan 2018

1. The *Fibonacci polynomials* $F_n(x) = F_n(x,1)$ defined by the recurrence $F_{n+2}(x) = xF_{n+1}(x)+F_n(x)$ with the initial values $F_0(x) = 1$ and $F_1(x) = x$. The *Fibonacci numbers* $f_n = F_n(1,1) = F_n(1)$, [10, A000045].

2. The *Pell polynomials* $P_n(x) = F_n(2x,1) = F_n(2x)$, [5], defined by the recurrence $P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x)$ with the initial values $P_0(x) = 1$ and $P_1(x) = 2x$. The *Pell numbers* $p_n = F_n(2,1) = P_n(1)$, [10, A000129].

3. The *Jacobsthal polynomials* $J_n(x) = F_n(1,2x)$, [6], defined by the recurrence $J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x)$ with the initial values $J_0(x) = J_1(x) = 1$. The *Jacobsthal numbers* $j_n = F_n(1,2) = J_n(1) = (2^{n+1} + (-1)^n)/3$, [10, A001045].

4. The *Chebyshev polynomials of the second kind* $U_n(x) = F_n(x,-1)$, [1, 4, 9], defined by the recurrence $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$ with the initial values $U_0(x) = 1$ and $U_1(x) = 2x$.

5. The *Fermat polynomials* $\varphi_n(x) = F_n(x,-2)$, [4], defined by the recurrence $\varphi_{n+2}(x) = x\varphi_{n+1}(x) - 2\varphi_n(x)$ with the initial values $\varphi_0(x) = 1$ and $\varphi_1(x) = x$. The *Lehmer numbers* $F_n(1,-2)$, [10, A107920].

6. The *Morgan-Voyce polynomials* $B_n(x) = F_n(x+2,-1) = U_n(x/2+2)$, [7, 11, 12], defined by the recurrence $B_{n+2}(x) = (x+2)B_{n+1}(x) - B_n(x)$ with the initial values $B_0(x) = 1$ and $B_1(x) = x+2$.

7. The *Horadam numbers* $W_n = F_n(p,-q)$, [2, 3], defined by the recurrence $W_{n+2} = pW_{n+1} - qW_n$ with the initial values $W_0 = 1$ and $W_1 = p$.

The bivariate Fibonacci polynomials have generating series

$$\sum_{n\geq 0} F_n(x,y)\, t^n = \frac{1}{1 - xt - yt^2} \tag{2}$$

and can be expressed as

$$F_n(x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k} y^k$$

$$F_n(x,y) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} (x^2+4y)^k x^{n-2k}$$

$$F_n(x,y)=\sum_{k=0}^{n}\binom{n+k+1}{2k+1}(x-2\sqrt{-y})^k(\sqrt{-y})^{n-k}\,.$$

Moreover, we have the Binet formula

$$F_n(x,y)=\frac{\alpha(x,y)^{n+1}-\beta(x,y)^{n+1}}{\sqrt{x^2+4y}} \tag{3}$$

where $\alpha(x,y)=\frac{x+\sqrt{x^2+4y}}{2}$ and $\beta(x,y)=\frac{x-\sqrt{x^2+4y}}{2}$ are the solutions of the equation $t^2-xt-y=0$. Using this formula, we can prove the identities

$$F_{2n+2}(x,y)=F_{n+1}(x,y)^2+yF_n(x,y)^2 \tag{4}$$

$$xF_{2n+3}(x,y)=F_{n+2}(x,y)^2-y^2F_n(x,y)^2\,. \tag{5}$$

Starting from recurrence (1), we obtain the *shifting property* for the bivariate Fibonacci polynomials, extending the shifting property

$$\sum_{k=0}^{n}\binom{n}{k}f_{k+1}=\sum_{k=0}^{n}\binom{n+1}{k+1}f_k \tag{6}$$

for the ordinary Fibonacci numbers obtained in [8]. Moreover, using identities (4) and (5), we prove, in a similar way, two other binomial identities resembling the shifting property. Clearly, all these identities can be specialized to the polynomials recalled at be beginning. In the final section, as an example, we specialize them for the Chebyshev polynomials.

2 Shifting property and main identities

We start by generalizing identity (6) to the polynomials $F_n(x,y)$.

Theorem 1. *For the bivariate Fibonacci polynomials, we have the shifting property*

$$\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}F_{k+1}(x,y)=\sum_{k=0}^{n}\binom{n+1}{k+1}x^{k+1}y^{n-k}F_k(x,y) \tag{7}$$

Proof. From recurrence (1), we have

$$\sum_{k=0}^{n-1} \binom{n}{k+1} \frac{x^{k+1}}{y^{k+1}} F_{k+2}(x,y) =$$
$$= \sum_{k=0}^{n-1} \binom{n}{k+1} \frac{x^{k+2}}{y^{k+1}} F_{k+1}(x,y) + \sum_{k=0}^{n-1} \binom{n}{k+1} \frac{x^{k+1}}{y^{k}} F_{k}(x,y)$$

that is

$$\sum_{k=1}^{n} \binom{n}{k} \frac{x^{k}}{y^{k}} F_{k+1}(x,y) =$$
$$= \sum_{k=1}^{n} \binom{n}{k} \frac{x^{k+1}}{y^{k}} F_{k}(x,y) + \sum_{k=0}^{n-1} \binom{n}{k+1} \frac{x^{k+1}}{y^{k}} F_{k}(x,y)$$

that is

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x^{k}}{y^{k}} F_{k+1}(x,y) - F_1(x,y) =$$
$$= \sum_{k=0}^{n} \left[\binom{n}{k} + \binom{n}{k+1} \right] \frac{x^{k+1}}{y^{k}} F_{k}(x,y) - xF_0(x,y)\,.$$

By the recurrence of the binomial coefficients and by the initial values $F_0(x,y) = 1$ and $F_1(x,y) = x$, we obtain the identity

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x^{k}}{y^{k}} F_{k+1}(x,y) = \sum_{k=0}^{n} \binom{n+1}{k+1} \frac{x^{k+1}}{y^{k}} F_{k}(x,y)$$

which is equivalent to identity (7). □

Notice that identity (7) can be obtained also by employing the general techniques related to Riordan matrices developed in [8]. However, the elementary approach used to prove Theorem 1 can also be used to obtain other identities similar to (7), such as the ones stated in next two theorems.

Theorem 2. *We have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} y^{n-k} F_{2k}(x,y) = \sum_{k=0}^{n} \binom{n+1}{k+1} y^{n-k} F_{k}(x,y)^2\,. \tag{8}$$

Proof. From identity (4), we have

$$\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{F_{2k+2}(x,y)}{y^{k+1}}=\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{F_{k+1}(x,y)^2}{y^{k+1}}+\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{F_k(x,y)^2}{y^k}$$

that is

$$\sum_{k=1}^{n}\binom{n}{k}\frac{F_{2k}(x,y)}{y^k}=\sum_{k=1}^{n}\binom{n}{k}\frac{F_k(x,y)^2}{y^k}+\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{F_k(x,y)^2}{y^k}\,.$$

that is

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_{2k}(x,y)}{y^k}-F_0(x,y)=\sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k+1}\right]\frac{F_k(x,y)^2}{y^k}-F_0(x,y)^2\,.$$

By the recurrence of the binomial coefficients and by the initial value $F_0(x,y)=1$, we obtain the identity

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_{2k}(x,y)}{y^k}=\sum_{k=1}^{n}\binom{n+1}{k+1}\frac{F_k(x,y)^2}{y^k}$$

which is equivalent to identity (8). □

Theorem 3. *We have the identity*

$$\begin{aligned}x\sum_{k=0}^{n}\binom{n}{k}(-1)^k y^{2n-2k}F_k(x,y)^2F_{2k+1}(x,y)&=\\=\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k y^{2n-2k}F_k(x,y)^2F_{k+1}(x,y)^2\,.\end{aligned}\tag{9}$$

Proof. From identity (5), we have

$$xF_{k+1}(x,y)^2F_{2k+3}(x,y)=F_{k+1}(x,y)^2F_{k+2}(x,y)^2-y^2F_k(x,y)^2F_{k+1}(x,y)^2\,.$$

Then, from this equation, we have

$$\begin{aligned}x\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{F_{k+1}(x,y)^2F_{2k+3}(x,y)}{y^{2k+2}}&=\\=\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{F_{k+1}(x,y)^2F_{k+2}(x,y)^2}{y^{2k+2}}&+\\-\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{F_k(x,y)^2F_{k+1}(x,y)^2}{y^{2k}}&\end{aligned}$$

that is

$$x \sum_{k=1}^{n} \binom{n}{k} \frac{F_k(x,y)^2 F_{2k+1}(x,y)}{y^{2k}} =$$

$$= \sum_{k=1}^{n} \binom{n}{k} \frac{F_k(x,y)^2 F_{k+1}(x,y)^2}{y^{2k}} - \sum_{k=0}^{n-1} \binom{n}{k+1} \frac{F_k(x,y)^2 F_{k+1}(x,y)^2}{y^{2k}}$$

that is

$$x \sum_{k=0}^{n} \binom{n}{k} \frac{F_k(x,y)^2 F_{2k+1}(x,y)}{y^{2k}} - xF_0(x,y)^2 F_1(x,y) =$$

$$= \sum_{k=0}^{n} \left[\binom{n}{k} + \binom{n}{k+1} \right] \frac{F_k(x,y)^2 F_{k+1}(x,y)^2}{y^{2k}} - F_0(x,y)^2 F_1(x,y)^2 .$$

By the recurrence of the binomial coefficients and by the initial values $F_0(x,y) = 1$ and $F_1(x,y) = x$, we obtain the identity

$$x \sum_{k=0}^{n} \binom{n}{k} \frac{1}{y^{2k}} \, F_k(x,y)^2 F_{2k+1}(x,y) = \sum_{k=0}^{n} \binom{n+1}{k+1} \frac{F_k(x,y)^2 F_{k+1}(x,y)^2}{y^{2k}}$$

which is equivalent to identity (9). □

3 Some generalizations

The *bivariate Lucas polynomials* are defined by the Binet formula

$$L_n(x,y) = \alpha(x,y)^n + \beta(x,y)^n . \tag{10}$$

They have generating series

$$\sum_{n \geq 0} L_n(x,y) \, t^n = \frac{2 - xt}{1 - xt - yt^2}$$

and they can be expressed in terms of the bivariate Fibonacci polynomials as $L_n(x,y) = F_n(x,y) + yF_{n-2}(x,y)$. In particular, we have the *Lucas numbers* $L_n = L_n(1,1)$, [10, A000032].

We have the following result.

Lemma 4. *For every* $m \geq 1$ *, we have the generating series*

$$\sum_{n\geq 0} F_{m(n+1)-1}(x,y)\, t^n = \frac{F_{m-1}(x,y)}{1 - L_m(x,y)t + (-y)^m t^2}\,. \tag{11}$$

Proof. By Binet formula (3), we have

$$\begin{aligned}
\sum_{n\geq 0} F_{mn-1}(x,y)\, t^n &= \sum_{n\geq 0} \frac{\alpha(x,y)^{mn} - \beta(x,y)^{mn}}{\sqrt{x^2+4y}}\, t^n \\
&= \frac{1}{\sqrt{x^2+4y}} \left(\frac{1}{1-\alpha(x,y)^m t} - \frac{1}{1-\beta(x,y)^m t} \right) \\
&= \frac{1}{\sqrt{x^2+4y}}\, \frac{(\alpha(x,y)^m - \beta(x,y)^m)t}{1 - (\alpha(x,y)^m + \beta(x,y)^m)t + (\alpha(x,y)\beta(x,y))^m t^2}\,.
\end{aligned}$$

By Binet formulas (3) and (10), and by dividing by t , we get series (11). □

Now, by Lemma 4, we can obtain next

Theorem 5. *For* $m \geq 1$ *, we have the identities*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} (-1)^{(m+1)(n-k)} y^{m(n-k)} L_m(x,y)^k F_{m(k+2)-1}(x,y) = \\
&\quad = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^{(m+1)(n-k)} y^{m(n-k)} L_m(x,y)^{k+1} F_{m(k+1)-1}(x,y)
\end{aligned}$$

$$\begin{aligned}
&F_{m-1}(x,y) \sum_{k=0}^{n} \binom{n}{k} (-1)^{(m+1)(n-k)} y^{m(n-k)} F_{m(2k+1)-1}(x,y) = \\
&\quad = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^{(m+1)(n-k)} y^{m(n-k)} F_{m(k+1)-1}(x,y)^2
\end{aligned}$$

and

$$\begin{aligned}
&A_m(x,y) \sum_{k=0}^{n} \binom{n}{k} (-1)^k y^{2m(n-k)} F_{m(k+1)-1}(x,y)^2 F_{2m(k+1)-1}(x,y) = \\
&\quad = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k y^{2m(n-k)} F_{m(k+1)-1}(x,y)^2 F_{m(k+2)-1}(x,y)^2\,.
\end{aligned}$$

where $A_m(x,y) = F_{m-1}(x,y) L_m(x,y)$.

Proof. By series (2) and (11), we have at once the identity

$$\frac{F_{m(n+1)-1}(x,y)}{F_{m-1}(x,y)} = F_n(L_m(x,y), -(-y)^m)\,.$$

So, by replacing x and y by $L_m(x,y)$ and $-(-y)^m$, respectively, in identities (7), (8) and (9), we obtain the three claimed identities. $\square$

4 Chebyshev polynomials

In this final section, we specialize the identities obtained above to the Chebyshev polynomials of the second kind $U_n(x) = F_n(2x,-1)$. Identities (7), (8) and (9) become

$$\sum_{k=0}^{n}\binom{n}{k}(2x)^k(-1)^{n-k}U_{k+1}(x) = \sum_{k=0}^{n}\binom{n+1}{k+1}(2x)^{k+1}(-1)^{n-k}U_k(x)$$

$$2x\sum_{k=0}^{n}\binom{n}{k}(-1)^kU_k(x)^2U_{2k+1}(x) = \sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^kU_k(x)^2U_{k+1}(x)^2$$

$$2x\sum_{k=0}^{n}\binom{n}{k}(-1)^kU_k(x)^2U_{2k+1}(x) = \sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^kU_k(x,y)^2U_{k+1}(x)^2\,.$$

Moreover, since $L_n(2x,-1) = 2T_n(x)$, the identities stated in Theorem 5 becomes

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}2^kT_m(y)^kU_{m(k+2)-1}(x) =$$
$$= \sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^{n-k}2^{k+1}T_m(x)^{k+1}U_{m(k+1)-1}(x)$$

$$U_{m-1}(x)\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}U_{m(2k+1)-1}(x) =$$
$$= \sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^{n-k}U_{m(k+1)-1}(x)^2$$

and

$$2U_{m-1}(x)T_m(x)\sum_{k=0}^{n}\binom{n}{k}(-1)^k U_{m(k+1)-1}(x)^2 U_{2m(k+1)-1}(x) =$$

$$= \sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k U_{m(k+1)-1}(x)^2 U_{m(k+2)-1}(x)^2 \,.$$

References

[1] M. Abramowitz, I. A. Stegun, (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York 1972.

[2] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, *Fibonacci Quart.*, **3** (1965), 161–176.

[3] A. F. Horadam, Generating functions for powers of a certain generalized sequence of numbers, *Duke Math. J.*, **32** (1965), 437–446.

[4] A. F. Horadam, Chebyshev and Fermat polynomials for diagonal functions, *Fibonacci Quart.*, **17** (1979), 328–333.

[5] A. F. Horadam, J. M. Mahon, Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, **23** (1985), 7–20.

[6] A. F. Horadam, Jacobsthal representation polynomials, *Fibonacci Quart.*, **35** (1997), 137–148.

[7] A. M. Morgan-Voyce, Ladder network analysis using Fibonacci numbers, *I. R. E. Trans. Circuit Theory*, **6.3** (1959), 321–322.

[8] E. Munarini, Shifting property for Riordan, Sheffer and connection constants matrices, *J. Integer Seq.*, **20** (2017), Article 17.8.2 (31 pages).

[9] T. J. Rivlin, *Chebyshev Polynomials*, Wiley, New York 1990.

[10] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, electronically published at `http://oeis.org/`.

[11] M. N. S. Swamy, Properties of the polynomials defined by Morgan-Voyce, *Fibonacci Quart.*, **4** (1966), 73–81.

[12] M. N. S. Swamy, Further properties of Morgan-Voyce polynomials, *Fibonacci Quart.*, **6** (1968), 167–175.