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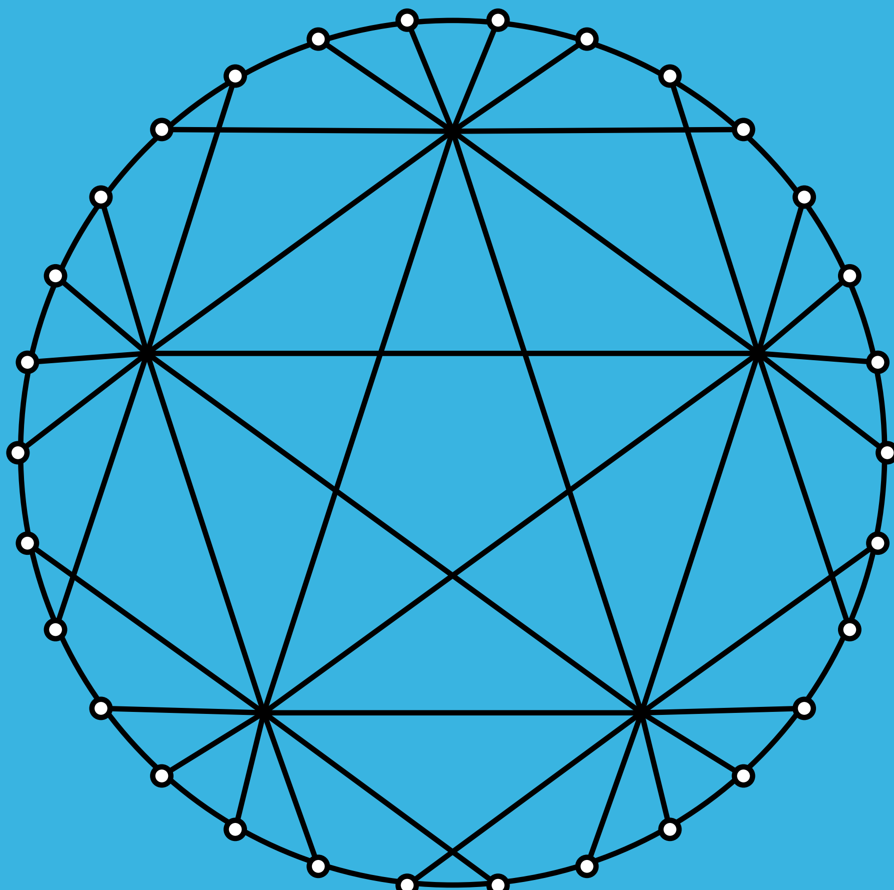

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New bounds on the biplanar crossing number of low-dimensional hypercubes: How low can you go?

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Abstract: In this note we provide an improved upper bound on the biplanar crossing number of the 8-dimensional hypercube. The k-planar crossing number of a graph $cr_k(G)$ is the number of crossings required when every edge of G must be drawn in one of k distinct planes. It was shown in [1] that $cr_2(Q_8) \leq 256$ which we improve to $cr_2(Q_8) \leq 128$. Our approach highlights the relationship between symmetric drawings and the study of k-planar crossing numbers. We conclude with several open questions concerning this relationship.

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1 Introduction

The traditional *crossing number* of a graph $G = (V, E)$, denoted by $cr(G)$, is the minimum number of edge crossings required to draw G in the 2-dimensional Euclidean plane. To study printed circuit boards, Owens [3] generalized the question: what is the minimum number of edge crossings required by a drawing that is allowed to carefully divide the edges of G among two different 2-dimensional Euclidean planes? Since then the definition has been extended to $k \geq 2$ planes [1].

Suppose that E is partitioned into k disjoint subsets, $E_1, E_2, ..., E_k$, and let $G_i = (V, E_i)$. Each G_i has some crossing number $cr(G_i)$. Suppose further that G_i will be drawn in the ith plane from a set of k distinct planes. The $k - planar\ crossing\ number\ of\ G$, denoted $cr_k(G)$ is then the minimum of

$$cr(G_1) + cr(G_2) + ... + cr(G_k)$$

over all partitions of the edge set E.

Trivially, letting $E_1 = E$ shows that $cr_k(G) \leq cr(G)$. The question remains: given the freedom to consider any partition of G's edges among k disjoint planes, how low can we drive the number of required crossings?

A significant challenge in designing a crossing-minimizing k-planar drawing of G is that, even for quite simple G_i, $cr(G_i)$ could be unknown. In this paper we consider the n-dimensional hypercube: the graph whose vertices are binary strings of length n and two vertices are adjacent if they differ in exactly one bit. For example: for Q_4, the 4-dimensional hypercube, it is known that $cr(Q_4) = 8$; however, the exact value of $cr(Q_d)$ is unknown for $d > 4$ [2].

The previous upper bound $cr_2(Q_8) \leq 256$ was given by a construction of Czabarka, Sýkora, Székely, and Vrťo in [1]. Czabarka et al. give a general construction for an upper bound on $cr_2(Q_d)$ that achieves 256 crossings when $d = 8$. Their approach specifies a bi-planar partition of the edges of Q_8 based on a set of lower-dimensional hypercube subgraphs. Their upper bound is minimized when these hypercube subgraphs are as-uniform-as-possible in size. In particular, for Q_8 their construction specifies sixteen disjoint Q_4 subgraphs in Plane 1 and a further sixteen disjoint Q_4 subgraphs in Plane 2. Recall that $cr(Q_4) = 8$, so drawing each disjoint copy of Q_4 optimally yields

$$cr_2(Q_8) \leq 16 \times 2 \times 8 = 256.$$

We now present our main result which improves on the the best known upper bound of $cr(Q_8)$ by a factor of 2.

Theorem 1 *There exists a 2-planar drawing of the 8-dimensional hypercube with 128 crossings so that* $cr_2(Q_8) \leq 128$.

2 A biplanar drawing of Q_8 with 128 crossings

To prove Theorem 1, we provide a biplanar drawing of Q_8 with 128 crossings. We improve the previous construction by plane-swapping edges to give a net reduction in total edge crossings. Our drawing consists of graphs G_1 and G_2 in Plane 1 and 2 respectively such that $G_1 \cong G_2$ where $cr(G_i) \leq 64$. We found several distinct bi-planar drawings of Q_8 with exactly 128 crossings which satisfy these conditions. For ease of exposition, we present a highly symmetric drawing.

We define a *depleted n-dimensional hypercube* to be a graph whose vertex set is $V(Q_n)$ and will refer to such graphs as *depleted n-cubes.* We will make use of depleted 5-cubes. To this end we introduce the following partition $V(Q_4) := C_1 \cup C_2$ where

$$C_1 := \{0000, 1000, 0010, 1010, 0011, 1011, 0001, 1001\}$$
$$C_2 := \{0111, 1111, 0101, 1101, 0100, 1100, 0110, 1110\}.$$

Note that C_1 and C_2 are disjoint.

For ease of notation, we denote $\hat{c} \in C_1$ and $\check{c} \in C_2$. Moreover, we let $b \in \{0, 1\}$ represent the usual binary-bit. Maintaining the notation of [1] we refer to each node of Q_8 by a length-8 binary string from $\{0, 1\}^8$. Given two binary strings s_1 and s_2 we write $s_1 s_2$, or $s_1 - s_2$ for readability, to be the usual string concatenation.

In our construction, each plane contains 512 edges, and furthermore, G_1 and G_2 are isomorphic. For exposition, suppose that we initially have a Plane 0 which contains all the edges and vertices of Q_8. Further suppose that there exist Planes 1 and 2 which each initially contain the vertices of Q_8 and no edges. We move every edge from Plane 0 to either Plane 1 or Plane 2 to create our biplanar partition. In Table 1, we describe explicitly the 512 edges we add to Plane 1.

Consider the set of pairs

$$P_1 := \{(0000, 1000), (0010, 1010), (0011, 1011), (0001, 1001)\} \subset \binom{C_1}{2}.$$

For $(\hat{c}_1, \hat{c}_2) \in P_1$ define the *depleted 5-cube of Type 1*, denoted $D_1(\hat{c}_1, \hat{c}_2)$, according to the Table 1.

$E(D_1(\hat{c}_1, \hat{c}_2))$ for $(\hat{c}_1, \hat{c}_2) \in P_1$ with $\hat{c} \in \{\hat{c}_1, \hat{c}_2\}$ and $b \in \{0, 1\}$.		
$(\hat{c} - b000, \hat{c} - b001)$	$(\hat{c} - b000, \hat{c} - b100)$	$(\hat{c} - b100, \hat{c} - b101)$
$(\hat{c} - b010, \hat{c} - b011)$	$(\hat{c} - b010, \hat{c} - b110)$	$(\hat{c} - b110, \hat{c} - b111)$
$(\hat{c} - b000, \hat{c} - b010)$	$(\hat{c} - b001, \hat{c} - b011)$	$(\hat{c} - b100, \hat{c} - b110)$
$(\hat{c} - b001, \hat{c} - b101)$	$(\hat{c} - b011, \hat{c} - b111)$	$(\hat{c} - b101, \hat{c} - b111)$
$(\hat{c} - 0101, \hat{c} - 1101)$	$(\hat{c} - 0111, \hat{c} - 1111)$	$(\hat{c} - 0110, \hat{c} - 1110)$
$(\hat{c} - 0100, \hat{c} - 1100)$		
$(\hat{c}_1 - 0000, \hat{c}_2 - 0000)$	$(\hat{c}_1 - 0100, \hat{c}_2 - 0100)$	$(\hat{c}_1 - 1100, \hat{c}_2 - 1100)$
$(\hat{c}_1 - 1001, \hat{c}_2 - 1001)$	$(\hat{c}_1 - 1101, \hat{c}_2 - 1101)$	$(\hat{c}_1 - 0101, \hat{c}_2 - 0101)$
$(\hat{c}_1 - 1000, \hat{c}_2 - 1000)$	$(\hat{c}_1 - 0001, \hat{c}_2 - 0001)$	

Table 1: Table of the 64 edges of *depleted 5-cubes of Type 1.*

The four *depleted 5-cubes of Type 1* are vertex disjoint (from the form of pairs in P_1). We present an eight-crossing drawing of a *depleted 5-cube of Type 1* in Figure 1, which proves the following claim.

Claim 1 *$cr(D_1(\hat{c}_1, \hat{c}_2)) \leq 8$.*

We similarly define $D_2(\check{c}_1, \check{c}_2)$, the *depleted 5-cube of Type 2*, according to Table 2 given

$$P_2 := \{(0111, 1111), (0101, 1101), (0100, 1100), (0110, 1110)\} \subset \binom{C_2}{2}.$$

Again, the four *depleted 5-cubes of Type 2* are vertex disjoint. An eight-crossing drawing of a *depleted 5-cube of Type 2* is given in Figure 2, which proves the following claim.

Claim 2 *$cr(D_2(\check{c}_1, \check{c}_2)) \leq 8$.*

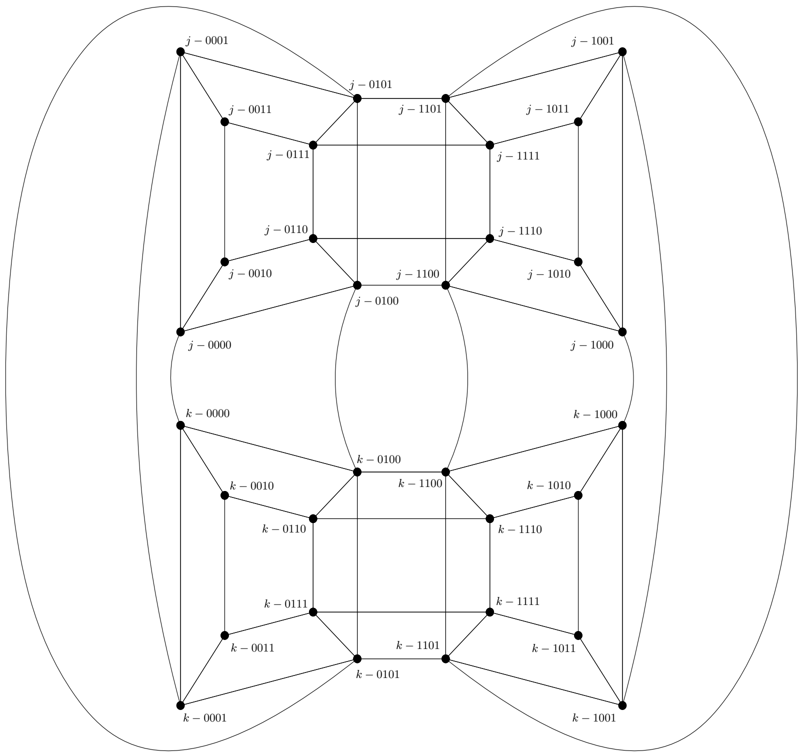

Figure 1: A drawing of $D_1(j,k)$ for $(j,k) \in P_1$ with eight crossings.

Each *depleted 5-cube* has 64 edges, so Plane 1 contains 512 edges. Further, no *depleted 5-cube of Type 1* shares a vertex with a *depleted 5-cube of Type 2*. This follows from the form of the pairs in P_1 and P_2 and the form of the edge sets described in Tables 1 and 2. Thus, these 512 edges can be drawn in Plane 1 with at most 64 crossings.

Remark 1 *Plane* 2 *contains all the edges of* Q_8 *which are not in Plane* 1*. Moreover,* $G_1 \cong G_2$*.*

We now provide a more illuminating description of the edges of Plane 2. The edges in Plane 2 have a symmetric representation in terms of the edges in Plane 1. Let $\rho : E(Q_8) \to E(Q_8)$ such that

$$\rho((v_p v_s, u_p u_s)) = (v_s v_p, u_s u_p)$$

$E(D_2(\check{c}_1, \check{c}_2))$ for $(\check{c}_1, \check{c}_2) \in P_2$ with $\check{c} \in \{\check{c}_1, \check{c}_2\}$ and $b \in \{0, 1\}$.		
$(\check{c} - b000, \check{c} - b001)$	$(\check{c} - b000, \check{c} - b100)$	$(\check{c} - b100, \check{c} - b101)$
$(\check{c} - b010, \check{c} - b011)$	$(\check{c} - b010, \check{c} - b110)$	$(\check{c} - b110, \check{c} - b111)$
$(\check{c} - b000, \check{c} - b010)$	$(\check{c} - b001, \check{c} - b011)$	$(\check{c} - b100, \check{c} - b110)$
$(\check{c} - b001, \check{c} - b101)$	$(\check{c} - b011, \check{c} - b111)$	$(\check{c} - b101, \check{c} - b111)$
$(\check{c} - 0011, \check{c} - 1011)$	$(\check{c} - 0001, \check{c} - 1001)$	$(\check{c} - 0000, \check{c} - 1000)$
$(\check{c} - 0010, \check{c} - 1010)$		
$(\check{c}_1 - 0110, \check{c}_2 - 0110)$	$(\check{c}_1 - 0111, \check{c}_2 - 0111)$	$(\check{c}_1 - 0011, \check{c}_2 - 0011)$
$(\check{c}_1 - 1111, \check{c}_2 - 1111)$	$(\check{c}_1 - 1110, \check{c}_2 - 1110)$	$(\check{c}_1 - 1010, \check{c}_2 - 1010)$
$(\check{c}_1 - 1011, \check{c}_2 - 1011)$	$(\check{c}_1 - 0010, \check{c}_2 - 0010)$	

Table 2: Table of 64 edges of *depleted 5-cubes of Type 2.*

where v_p is a prefix string of length four, $v_1v_2v_3v_4$, and v_s is a suffix string of length four, $v_5v_6v_7v_8$ that together define vertex $v = v_1v_2 \dots v_8$. Indeed ρ captures the symmetric relationship between edges in Plane 1 and the edges in Plane 2. Assuming an ordering on the vertices of Q_8 one can check that ρ is indeed a bijection. As an example, in Table 1 we assign edge (ĉb-000, ĉb-001) to Plane 1. So we send

$$\rho((\hat{c}b - 000, \hat{c}b - 001)) = (b000 - \hat{c}, b001 - \hat{c})$$

to Plane 2. If we let $\mathcal{P}_i$ be the set of edges partitioned into Plane i then $\mathcal{P}_2 = \rho(\mathcal{P}_1)$. Moreover, the drawings provided in Figures 1 and 2 for *depleted 5-cubes of Type* 1 (or *Type 2*, resp.) are also drawings of their images under ρ. It follows that, for the edge partition we describe, each plane can be drawn with at most 64 crossings implying that $cr_2(Q_8) \leq 128$ as desired.

A natural next step in this research is to determine whether or not this bound is sharp. The authors believe this to be the case; however, such a proof remains elusive. Alas, we leave the reader with the following conjecture.

Conjecture 1 $cr_2(Q_8) = 128$.

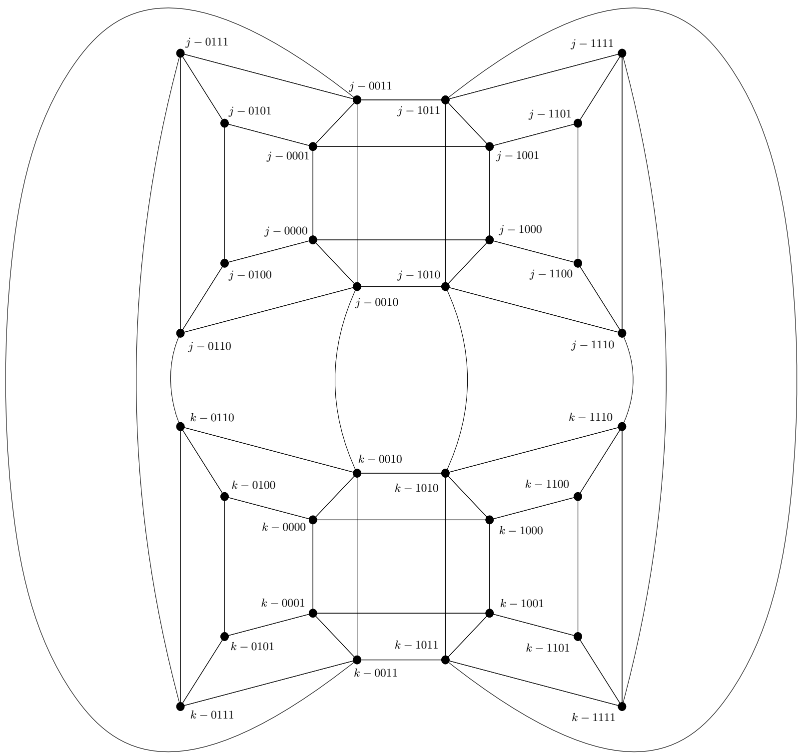

Figure 2: A drawing of $D_2(j, k)$ for $(j, k) \in P_2$ with eight crossings.

3 Lower bounds on *structurally-symmetric* k-planar crossing numbers for hypercubes

Notably, our bi-planar drawing of Q_8 satisfies $G_1 \cong G_2$. This is a rather special property and is termed *self-complementary* in [1]. It could be the case that there exists a non-isomorphic partition of $E(Q_8)$ which admits strictly fewer crossings. Yet, we wonder whether demanding that the G_i be isomorphic truly forces a suboptimal number of crossings for k-planar drawings. In particular, such symmetry would be expected when considering highly symmetric graphs like hyper-cubes.

To formalize this question, we introduce the following generalization of self-complementary edge partitions.

Definition 1 *For a finite graph $G = (V, E)$, let P denote an edge-partition $E = (E_1, E_2, ..., E_k)$ and define $G_i = (V, E_i)$ for all i. If for all pairs $(r, s) \in [k] \times [k]$ we have $G_r \cong G_s$, then P is a k-structurally-symmetric partition of G.*

Trivially, when $|E|$ is not a multiple of k, no *k-structurally-symmetric partition* of E exists.

Definition 2 *If there exists a k-structurally-symmetric partition for G that can be drawn with $cr_k(G)$ crossings then we say that the graph G is k-structurally-symmetric.*

It is unclear whether graphs exist for which any *k-structurally-symmetric partition* of E forces a sub-optimal k-planar drawing (which requires strictly more than $cr_k(G)$ crossings).

In particular, we leave the reader with the following question.

Question 1 *Is the d-dimensional hypercube 2-structurally-symmetric?*

This question motivates the following definition.

Definition 3 *Let $cr_{kss}(G)$ denote the minimum number of crossings required among all k-structurally symmetric partitions of G. We call cr_{kss}* the k-structurally-symmetric crossing number of G.

Trivially, $cr_{kss}(G) \geq cr_k(G)$. So, *k-structurally symmetric graphs* are precisely those graphs G that have $cr_k(G) = cr_{kss}(G)$. We conclude by presenting the reader questions concerning k-structurally-symmetric crossing numbers.

Question 2 *Characterize the set of all k-structurally-symmetric graphs. To this end, what structural properties ensure that a graph is k-structurally-symmteric or otherwise?*

Question 3 *Provide a graph for which the difference between $cr_{kss}(G)$ and $cr_k(G)$ is large (or even > 0). Further, is there an infinite family $(G_n)_{n\geq 1}$ such that $G_n \subseteq G_{n+1}$ and $(cr_{kss}(G_n) - cr_k(G_n))_{n\geq 1} \uparrow \infty$?*

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