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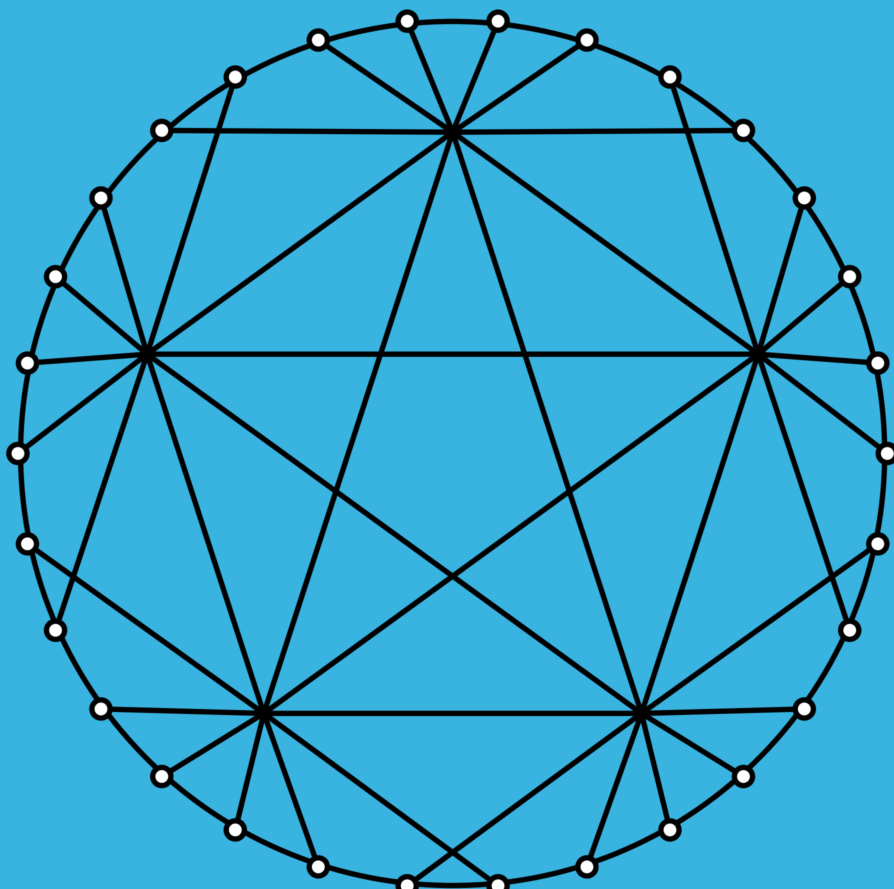

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A simple construction of 3-GDDs with block size 4 using SQS(v)

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Abstract: Recently, a 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$ was defined by extending the definitions of a group divisible design and a t-design. It was shown that the necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ except possibly when $n \equiv 1, 3 \pmod 6$, $n \neq 3, 7, 13$ and $\lambda_1 > \lambda_2$. In this short note we prove that the necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for $n \equiv 1, 7, 9 \pmod{12}$. The proof relies on a basic construction of a 3-GDD$(n, 2, 4, 3, 1)$. We also prove that for $n \equiv 3 \pmod{12}$, necessary conditions are sufficient except when $\lambda_1 \equiv 9 \pmod{12}$ and hence an open problem is to find a construction of a 3-GDD$(n, 2, 4, 9, 1)$ for $n \equiv 3 \pmod{12}$, $n \neq 3$.

1 Introduction

Definition 1.1. *A t-(v, k, λ) design, or a t-design, is a pair (X, B) where X is a v-set of points and B is a collection of k-subsets (blocks) of X with the property that every t-subset of X is contained in exactly λ blocks. The parameter λ is called the index of the design.*

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Definition 1.2. *A Steiner Quadruple System (SQS) is an ordered pair (V, B) where V is a finite set of v symbols and B is a collection of 4-subsets of V called blocks (quadruples) with the property that every 3-subset of V is a subset of exactly one quadruple B.*

A SQS is also denoted by 3-$(n, 4, 1)$ and it is known that the necessary conditions are sufficient for the existence of a 3-$(n, 4, \lambda)$ [1].

Definition 1.3. *[2] A 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$ is a set X of $2n$ elements partitioned into two parts of size n called groups together with a collection of k-subsets of X called blocks, such that*
(i) every 3-subset of each group occurs in λ_1 blocks and
(ii) every 3-subset where two elements are from one group and one element from the other group occurs in λ_2 blocks.

Lemma 1.4. *If a 3-$(2n, 4, \lambda_2)$, (i.e., a 3-GDD$(n, 2, 4, \lambda_2, \lambda_2)$) and a 3-$(n, 4, \lambda_1 - \lambda_2)$ exists, then a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ exists.*

Following necessary conditions (Table 1, where the values of λ_1 and λ_2 are given modulo 6) and the existence results of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ are given in [2].

λ_1/λ_2	0	1	2	3	4	5
0	all n	n even	all n	n even	all n	n even
1	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)
2	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)
3	n even	all n	n even	all n	n even	all n
4	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)
5	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)	$2, 4$ (mod 6)	$1, 2, 4, 5$ (mod 6)

Table 1

Lemma 1.5. *A necessary condition for the existence of a 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$ for odd n and k even is that λ_1 and λ_2 must be of the same parity.*

Theorem 1.6. *A 3-GDD$(n, 2, 4, 0, 1)$ exists for even n and a 3-GDD$(n, 2, 4, 0, 2)$ exists for all positive integers n.*

Lemma 1.7. *Given* $n \equiv 1, 2, 4, 5 \pmod 6$*, a* 3*-GDD*$(n, 2, 4, \lambda_1', \lambda_2')$ *exists for all even* λ_1' *and* λ_2' *if and only if a* 3*-GDD*$(n, 2, 4, \lambda_1, \lambda_2)$ *exists for all odd* λ_1 *and* λ_2*.*

Theorem 1.8. *Necessary conditions are sufficient for the existence of a* 3*-GDD*$(n, 2, 4, \lambda_1, \lambda_2)$ *for* $n \equiv 0, 2, 4, 5 \pmod 6$ *and* $n = 7$*.*

Theorem 1.9. *For* $n \equiv 1, 3 \pmod 6$*, the necessary conditions as described in Table 1 are sufficient for the existence of a* 3*-GDD*$(n, 2, 4, \lambda_1, \lambda_2)$ *when* $\lambda_1 \leq \lambda_2$*.*

In view of the above results, to prove that the necessary conditions are sufficient for the existence of 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$, we need the construction of 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for $n \equiv 1, 3 \pmod 6$ and $n \geq 9$ where $\lambda_1 > \lambda_2$.

2 Application of large sets and SQS(v)

2.1 A Construction of 3-GDD$(n, 2, 4, \lambda_1 = 3, \lambda_2 = 1)$ for $n \equiv 1, 3 \pmod 6$

Let us denote the groups for the required 3-GDD by $G_1 = \{a_1, a_2, \cdots, a_n\}$ and $G_2 = \{b_1, b_2, \cdots, b_n\}$. It is known that there exists a large set of STS(n)'s for $n \equiv 1, 3 \pmod 6$ and $n \neq 7$.
Hence, a large set, a partition of all 3-subsets of G_i into $n-2$ Steiner triple systems (STSs) on G_i exists, say $S_{i,1}, \cdots S_{i,n-2}$ for $i = 1, 2$. It is also well known that a SQS$(n+1)$ exists, as $n + 1 \equiv 2, 4 \pmod 6$.

We claim that the blocks of a SQS$(n+1)$ on $G_1 \bigcup \{b_{n-1}\}$, a SQS$(n+1)$ on $G_1 \bigcup \{b_n\}$, a SQS$(n+1)$ on $G_2 \bigcup \{a_{n-1}\}$, a SQS$(n+1)$ on $G_2 \bigcup \{a_n\}$, and the blocks obtained by taking union of the triples of $S_{1,j}$ with $\{b_j\}$ and by taking union of the triples of $S_{2,j}$ with $\{a_j\}$, for $j = 1, 2, \cdots n-2$, taken together give the blocks for a 3-GDD$(n, 2, 4, 3, 1)$.
We check the claim by counting the values of λ_1 and λ_2. Observe that in an STS on a group, say G_1, every pair (a_i, a_j) of distinct elements of the group comes only once. Hence, if we union its triples with an element, say b_t of the other group, triple $\{a_i, a_j, b_t\}$ occurs in exactly one block for $t = 1, 2, \cdots, n-2$. The triples $\{a_i, a_j, b_t\}$ for $t = n-1, n$ occur singly in the blocks of SQS$(n+1)$ on $G_1 \bigcup \{b_{n-1}\}$ and SQS$(n+1)$ on $G_1 \bigcup \{b_n\}$ respectively. Similarly, reversing the roles of G_1 and G_2, we see that λ_2 is

as required. Observe that a large set for each group contributes 1 towards λ_1 for the triples from the group and SQS$(n+1)$'s contribute the remaining 2 towards the λ_1 count.

Now recall that for $n \equiv 1, 3 \pmod 6$, a 3-GDD$(n, 2, 4, 0, 2)$ exists. Also for $n \equiv 1 \pmod 6$, a 3-$(2n, 4, 1)$ exists. Hence from Lemma 1.7, Lemma 1.5, Theorem 1.9 and Theorem 1.6 we have

Theorem 2.1. *Necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for $n \equiv 1 \pmod 6$.*

Proof. According to Lemmas 1.5 and 1.7 and Theorem 1.9, we only need to consider the case where both λ_1 and λ_2 are odd and $\lambda_1 > \lambda_2$.

For $n \equiv 1 \pmod 6$, a 3-$(n, 4, 4)$, a 3-GDD$(n, 2, 4, 3, 1)$, and a 3-GDD$(n, 2, 4, 0, 2)$ exist. Hence to construct a 3-GDD$(n, 2, 4, 2t+1, 2s+1)$ where $t > s$, we use $2s+1$ copies of a 3-$(2n, 4, 1)$ and $\frac{2t-2s}{4}$ copies of a 3-$(n, 4, 4)$ on each group, if $(2t - 2s) \equiv 0 \pmod 4$. If $(2t - 2s) \equiv 2 \pmod 4$, then we use one copy of a 3-GDD$(n, 2, 4, 3, 1)$, $2s$ copies of a 3-$(2n, 4, 1)$, and $\frac{2t-2s-2}{4}$ copies of a 3-$(n, 4, 4)$ on each group. □

Similarly, as for $n \equiv 3 \pmod 6$, a 3-$(2n, 4, 3)$ exists, we have the following result.

Theorem 2.2. *A 3-GDD$(n, 2, 4, \lambda_1 = 3t + 3s, \lambda_2 = t + 3s + 2m)$ exists for $n \equiv 3 \pmod 6$ and integers $t, s, m \geq 0$.*

Unlike $n \equiv 1 \pmod 6$, for $n \equiv 3 \pmod 6$, one needs to prove the existence for even λ_1 and λ_2 as well as for odd λ_1 and λ_2 as Theorem 1.7 is not applicable for $n \equiv 3 \pmod 6$. Also, recall that for $n \equiv 3 \pmod 6$, $\lambda_1 \equiv 0 \pmod 6$ (even) or $\lambda_1 \equiv 3 \pmod 6$ (odd). From Hanani [1], for $n \equiv 9 \pmod{12}$, a 3-$(n, 4, 6)$ exists, but for $n \equiv 3 \pmod{12}$, smallest λ for which a 3-$(n, 4, \lambda)$ exists is 12. Hence we have,

Theorem 2.3. *Necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ for $n \equiv 3 \pmod 6$ except when $\lambda_1 \equiv 9 \pmod{12}$ and $n \equiv 3 \pmod{12}$.*

Proof. Let λ_1 be even. Hence, as $n \equiv 3 \pmod 6$, $\lambda_1 = 6t$ for some nonnegative integer t. Two copies of a 3-GDD$(n, 2, 4, 3, 1)$ give a 3-GDD$(n, 2, 4, 6, 2)$. Also a 3-GDD$(n, 2, 4, 12, 2)$ can be obtained by a 3-$(n, 4, 12)$ and a 3-GDD$(n, 2, 4, 0, 2)$. Hence for any nonnegative integers t and s, when the necessary conditions are satisfied, a 3-GDD$(n, 2, 4, 6t, 2s)$ exists. (For $n \equiv 3 \pmod{12}$, a 3-GDD$(n, 2, 4, 6, 0)$ does not exists as necessary conditions are not satisfied.)

Let λ_1 be odd, hence $\lambda_1 = 6t + 3$ for some nonnegative integer t. For $n \equiv 9 \pmod{12}$, t copies of a 3-$(n, 4, 6)$ on each group, a 3-GDD$(n, 2, 4, 3, 1)$ and s copies of a 3-GDD$(n, 2, 4, 0, 2)$ provide us with a 3-GDD$(n, 2, 4, \lambda_1 = 6t+3, \lambda_2 = 2s+1)$ for any nonnegative integers t and s. Similarly, for $n \equiv 3 \pmod{12}$, we can construct a 3-GDD$(n, 2, 4, \lambda_1 = 12t + 3, \lambda_2 = 2s + 1)$ as a 3-$(n, 4, 12)$ on each group exists. □

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