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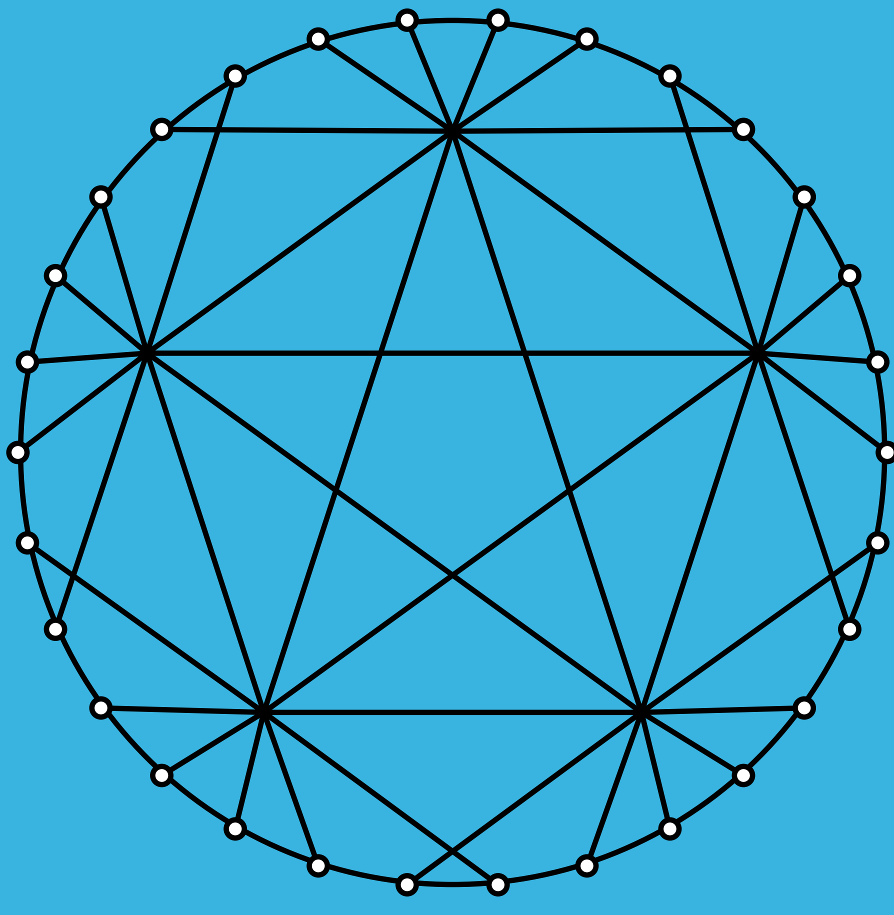

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Notes on spreads of degrees in graphs

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Abstract: Perhaps the very first elementary exercise one encounters in graph theory is the result that any graph on at least two vertices must have at least two vertices with the same degree. There are various ways in which this result can be non-trivially generalised. For example, one can interpret this result as saying that in any graph G on at least two vertices there is a set B of at least two vertices such that the difference between the largest and the smallest degrees (in G) of the vertices of B is zero. In this vein we make the following definition. For any $B \subset V(G)$, let the spread $sp(B)$ of B be defined to be the difference between the largest and the smallest of the degrees of the vertices in B. For any $k \geq 0$, let $sp(G, k)$ be the largest cardinality of a set of vertices B such that $sp(B) \leq k$. Therefore the first elementary result in graph theory says that, for any graph G on at least two vertices, $sp(G, 0) \geq 2$.

In this paper we first give a proof of a result of Erdös, Chen, Rousseau and Schelp which generalises the above to $sp(G, k) \geq k + 2$ for any graph on at least $k+2$ vertices. Our proof is short and elementary and does not use the famous Erdös-Gallai Theorem on vertex degrees. We then develop lower bounds for $sp(G, k)$ in terms of the order of G and its minimum, maximum and average degree. We then use these results to give lower bounds on $sp(G, k)$ for trees and maximal outerplanar graphs, most of which we show to be sharp.

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1 Introduction

One of the most fascinating aspects of combinatorics is that a trivial statement can be turned into a non-trivial result or even a very difficult problem by some very natural generalisation. Very often this involves the use of the pigeonhole principle. An application of this principle gives what we call the first elementary result in graph theory: any graph on at least two vertices has at least two vertices with the same degree. This result has been generalised in various directions, for example: a characterisation of those graphs which have only one repeated pair of degrees [1], and a characterisation of graphic sequences, that is, those sequences of positive integers which can be realised as the degree sequence of some graph [5, 6] .

In this paper we consider the following generalisation of the first elementary result in graph theory, introduced in [3]. Let $G = (V, E)$ be a graph. For B a subset of the vertex set V, we define the *spread* of B as $sp(B) = \{\max(deg(u)) - \min(deg(v)) : u, v \in B\}$, where the degrees are the degrees in graph G. We then let, for an integer $k \geq 0$, $sp(G, k)$ be $\max\{|B| : sp(B) \leq k\}$, namely the largest cardinality of a subset of vertices of G with spread at most k.

The first elementary result of graph theory therefore says that, if G has order at least 2, then $sp(G, 0) \geq 2$. The number $sp(G, k)$ is also a generalisation of the maximum occurrence of a value in the degree sequence of a graph, as defined in [2] and denoted by $rep(G)$, since $rep(G) = sp(G, 0)$.

The result $sp(G, 0) \geq 2$ was extended to general spreads in [3] where the following theorem was proved.

Theorem 1.1 (Erdös, Chen, Rousseau and Schelp). *Let G be a graph on $n \geq k + 2$ vertices, then $sp(G, k) \geq k + 2$.*

In this paper, in Section 2, we give a short and elementary proof of Theorem 1.1 avoiding the use of the Erdös-Gallai theorem. Then, in the same section, we develop a lower bound for $Sp(G, k)$ in terms of the parameters n, δ, d, Δ, which are respectively the number of vertices, the minimum degree, the average degree and the maximum degree of the graph G. Doing so we generalize the technique introduced in Lemma 2.1 in [2].

Then in Section 3, we consider the sharpness of the lower bounds obtained in Section 2 attained by trees and maximal outer-planar graphs (abbreviated to MOPs). We conclude in Section 4 with some concluding remarks and open problems.

Any standard graph theoretic terms and results not defined in this paper can be found in [8].

2 Bounds for $sp(G, k)$

The proof given in [3] of Theorem 1.1 uses the celebrated Erdös-Gallai characterization of graphic sequences [4]. Here we give a very short and elementary (avoiding Erdös-Gallai theorem) alternative proof.

Proof of Theorem 1.1.

Suppose, on the contrary, that G is a graph with $n = m(k+1)+r$ vertices, $m \geq 1$, $1 \leq r \leq k+1$ with $sp(G,k) \leq k+1$. Let the vertices of G be $\{v_1, \ldots, v_{m(k+1)}, v_{m(k+1)+1} = u_1, \ldots, , v_{m(k+1)+r} = u_r\}$. By assumption on G, $deg(v_{j+k+1}) \geq deg(v_j) + k + 1$ for $j = 1, \ldots, n-k-1$, as each interval has $k+2$ vertices and we assumed that $spG) \leq k+1$. Hence in particular,

$$deg(u_j) = deg(v_{m(k+1)+j}) \geq m(k+1) + deg(v_j) \tag{1}$$

for $j = 1, .., r$.

How many vertices among $v_1, \ldots, v_r$ can $u_1, \ldots, u_r$ be adjacent to?

Clearly each u_j can be adjacent among $v_{r+1}, \ldots, v_n$ to at most $n-r-1$ vertices and

$$n - r - 1 = m(k+1) + r - r - 1 = m(k+1) - 1. \tag{2}$$

Hence, using (1) and (2), u_j is adjacent to at least $d(u_j) - m(k+1) + 1 \geq m(k+1)) + deg(v_j) - m(k+1) + 1 = d(v_j) + 1$ vertices among $v_1, \ldots, v_r$. But then consider the bipartite graph H with $v_1, \ldots, v_r$ on one side and $u_1, \ldots, u_r$ on the other side. Clearly, if $deg_H(u_j)$ denotes the degree of vertex u_j in H, we obtain

$$\sum_{j=1}^{r} deg(v_j) \geq e(H) \geq \sum_{j=1}^{r} deg_H(u_j) \geq \sum_{j=1}^{r} deg(v_j) + 1,$$

a contradiction.

$\square$

Before stating our main results in this section , we observe that

$$sp(G,k) = sp(\overline{G},k),$$

since for any two vertices $u, v \in V(G)$,

$$\begin{aligned} deg_{\overline{G}}(u) - deg_{\overline{G}}(v) &= (n-1-deg_G(u)) - (n-1-deg_G(v)) \\ &= deg_G(v) - deg_G(u). \end{aligned}$$

Theorem 2.1. *Let G be a graph on n vertices average degree d, minimum degree δ and maximum degree Δ. Then:*

1. $sp(G,k) \geq \max\left\{\frac{n(k+1)}{2d-2\delta+k+1}, \frac{n(k+1)}{2\Delta-2d+k+1}\right\}$.
2. $sp(G,k) \leq (k+1)sp(G,0)$.

Proof. Let $r = sp(G,k)$ and set $n = rt + b$, where $0 \leq b \leq r-1$, and consider the intervals

$$I_1 = [\delta, \delta+k], I_2 = [\delta+k+1, \delta+2k+1], \ldots,$$

$$I_t = [\delta+(t-1)(k+1), \delta+t(k+1)-1], I_b = [\delta+t(k+1), \ldots, n-1].$$

Each interval I_j contains at most r vertices from $V(G)$ for otherwise $sp(G,k) \geq r+1$. There are t such intervals containing at most rt vertices altogether and at least b elements from the interval I_b so that the total number of vertices is $rt + b = n$. The smallest degree sum is achieved when we take exactly r elements in each interval I_j with value $\delta + (j-1)(k+1)$ and the b elements in I_b equals $\delta + t(k+1)$, so that the total sum of degrees is

$$\begin{aligned} 2e(G) &= dn \\ &\geq r[\delta + (\delta+k+1) + \ldots + (\delta+(t-1)(k+1)) + b(\delta+t(k+1))] \\ &= \frac{2rt\delta + rt(t-1)(k+1)}{2} + b(\delta+t(k+1)) \\ &= \frac{rt[2\delta+(t-1)(k+1)]}{2} + b(\delta+t(k+1)) \\ &= \frac{(n-b)[2\delta+(t-1)(k+1)]}{2} + b(\delta+t(k+1)) \\ &= n\delta + \frac{n(t-1)(k+1)}{2} - b\delta - \frac{b(t-1)(k+1)}{2} + b\delta + bt(k+1) \end{aligned}$$

$$
\begin{aligned}
&= n\delta + \frac{n(n-b)}{r-1)(k+1)} - \frac{bt(k+1)}{2} + \frac{b(k+1)}{2} + bt(k+1) \\
&= n\delta + \frac{n(k+1)(\frac{n}{r}-1)}{2} - \frac{nb(k+1)}{2r} + \frac{bt(k+1)}{2} + \frac{b(k+1)}{2} \\
&= n\delta + -\frac{nb(k+1)}{2r} + \frac{rbt(k+1)}{2r} + \frac{rb(k+1)}{2r} \\
&= n\delta + \frac{n(k+1)(\frac{n}{r}-1)}{2} - \frac{nb(k+1)}{2r} + \frac{(n-b)b(k+1)}{2r} + \frac{rb(k+1)}{2r} \\
&= n\delta + \frac{n(k+1)(\frac{n}{r}-1)}{2} - \frac{b^2(k+1)}{2r} + \frac{rb(k+1)}{2r} \\
&= n\delta + \frac{n(k+1)(\frac{n}{r}-1)}{2} + \frac{b(r-b)(k+1)}{2r} \qquad (3) \\
&\geq n\delta + \frac{n(k+1)(\frac{n}{r}-1)}{2}
\end{aligned}
$$

taking $b = 0$ in (3). Hence $dn \geq n\delta + \frac{n(k+1)(\frac{n}{r}-1)}{2}$ which after rearranging gives $r \geq \frac{n(k+1)}{2d-2\delta+k+1}$, the first expression.

Also since $sp(G,k) = sp(\overline{G},k)$ and using $\overline{d} = n-1-d$ and $\overline{\delta} = n-1-\Delta$, we get

$$sp(G,k) = sp(\overline{G},k) \geq \frac{n(k+1)}{2\Delta - 2d + k + 1}.$$

2. For $sp(G,k)$, the spread is determined by a set of vertices with degrees $p, p+1, \ldots, p+k$ respectively. Let S_i be the set of vertices of degree i in this set. Then $0 \leq |S_i| \leq sp(G,0)$, hence $sp(G,k) \leq (k+1)sp(G,0)$. □

Remark: Observe that in equation (3) we used $b = 0$, but if we substitute $b = n - rt$, then after some further algebra we get

$$r \geq \frac{2n(\delta - d + t(k+1))}{t(t+1)(k+1)} \qquad (4)$$

This will prove useful once we have a lower bound r^* on r using Theorem 2.1 and an upper bound t^* on t since from $n = rt + b$ we get $n \geq r^*t + b$ hence $\frac{n-b}{r^*} = t^* \geq t$.

Clearly r is at least the minimum in equation (4) over all t such that $1 \leq t \leq t^*$.

We shall use this remark several times in section 3.

3 Realisation of the lower bounds in certain families of graphs.

The characterisation of graphic sequences in general given in [4, 5, 6] is too wide to force restrictions on the degree sequence so that the bounds of Theorem 2.1 are attained. It is therefore interesting to investigate classes of graphs whose structure imposes such restrictions. In this section we show that trees and maximal outerplanar graphs come very close to having this required structure: for both classes, their average degree d which appears in the bound of Theorem 2.1, is known in terms of the number n of vertices, and their structure forces severe restrictions on the possible degrees which their vertices can have.

3.1 Trees

Theorem 3.1. *Let $k \geq 0$ and T be a tree on $n \geq k+2$ vertices. Then*

1. $sp(T,0) = rep(T) \geq \lceil \frac{n}{3} \rceil$ *which is sharp for* $n = 1 \pmod 3$*.*
2. *For* $k \geq 1$*,* $sp(T,k) \geq \frac{nk+2}{k+1}$ *and this is sharp.*

Proof. 1. The case $k = 0$ is from [2] and sharpness for $n = 1 \pmod 3$ is achieved by a tree made up of a path on $3k+2$ vertices, with a path of two edges attached to the vertices $v_3 \ldots v_{3k}$ to give a tree T on $3k+2+2(3k-2) = 9k-2$ vertices. This gives $3k$ vertices of degree 1, $3k$ vertices of degree 2 and $3k-2$ vertices of degree 3, and hence $sp(T,0) = 3k = \frac{n+2}{3}$.

1. For a tree, $\delta = 1$ and $d = \frac{2(n-1)}{n}$, and substituting into Theorem 2.1 with $k = 1$ gives

$$\frac{2n}{4 - \frac{4}{n} - 2 + 2} = \frac{2n^2}{4(n-1)} = \frac{n^2}{2(n-1)} > \frac{n+1}{2}.$$

Hence in $n = rt + b$ we just have $t = 1$ otherwise $rt > n$. Furthermore since $sp(T,k+1) \geq sp(T,k)$ we get that for all $k \geq 1$ we may assume $t = 1$.

For trees and $k \geq 1$ the lower bound (4) (with $t = 1$) gives

$$r \geq \frac{2n(\delta - d + t(k+1))}{t(t+1)(k+1)} = \frac{2n(12 + 2/n + k + 1)}{2(k+1)} = \frac{n(k + \frac{2}{n})}{k+1} = \frac{nk+2}{k+1}.$$

This is sharp for every $k \geq 1$, as can be seen with trees having degrees only 1 and $k+2$ using the following following equations with n_j being the number of vertices of degree j:

1. Vertex counting: $n_1 + n_{k+2} = n$
2. Edge counting: $n_1 + (k+2)n_{k+2} = 2n - 2$

Then solving for n_1 we get $n_1 = \frac{nk+2}{k+1} = sp(T,k)$ as required. □

3.2 Maximal outerplanar graphs

We now consider maximal outerplanar graphs. In general, for a maximal outerplanar graph G on n vertices, bound (1) gives

$$sp(G,k) \geq \frac{(k+1)n}{\frac{4(2n-3)}{n} - 4 + k + 1} \geq \frac{(k+1)n}{\frac{(5+k)n-12}{n}} \geq \frac{(k+1)n^2}{(5+k)n-12}$$
$$> \frac{(k+1)n}{5+k}.$$

We define

$$MOP(n,k) = \min\{sp(G,k){:}G \text{ is maximal outer-planar on } n \text{ vertices}\}.$$

We prove the following results.

Theorem 3.2. *For maximal outerplanar graphs*

1. $MOP(n,0) > \frac{n}{5}$.
2. $MOP(n,1) \geq \frac{n}{3} + 1$.
3. $MOP(n,2) \geq \frac{4n}{9}$.
4. *For* $k \geq 3$, $MOP(n,k) \geq \frac{(k-2)n}{k-1}$.

Bounds 1 and 2 and 4 are sharp up to small additive constants.

Proof.

1. $MOP(n, 0) = \min\{rep(G){:}G$ is maximal outer-planar on n vertices$\}$. Bound (1) gives $MOP(n, 0) > \frac{n}{5}$ which is the same as the lower bound given for $rep(G)$ in [2]. The construction given in [2] gives $rep(G) = \frac{n-4}{5} + 2 = \frac{n}{5} + \frac{6}{5}$ when $n = 4 \pmod{10}$.

2. For $k = 1$, the above observation gives $MOP(n, 1) > \frac{n}{3}$. Hence we may use $t = 1, 2$ and for $t = 2$ and $k = 1$ we get using bound (4)

$$MOP(n,1) \geq \frac{2n(2 - \frac{4n-6}{n} + 4)}{12} = \frac{n(2n+6)}{6n} = \frac{2n+6}{6} = \frac{n}{3} + 1.$$

   The following construction realises this bound up to a constant.

   Arrange three sets of vertices $U = \{u_1, \ldots, u_{p-1}\}$, $V = \{v_1, \ldots, v_p\}$ and $W = \{w_1, \ldots, w_{p-1}\}$. U will be the upper vertices, V will be in the middle vertices and W the bottom ones.

   Let u_i be connected to v_i and v_{i+1}; let w_i be connected to v_i and v_{i+1} and to w_{i-1} and w_{i+1}, except w_1 which is only connected to w_2, and w_{p-1} which is only connected to w_{p-2}. Let v_i be also connected to v_{i-1} and v_{i+1} (except the first and the last). Figure 1 shows this construction with $p = 5$.

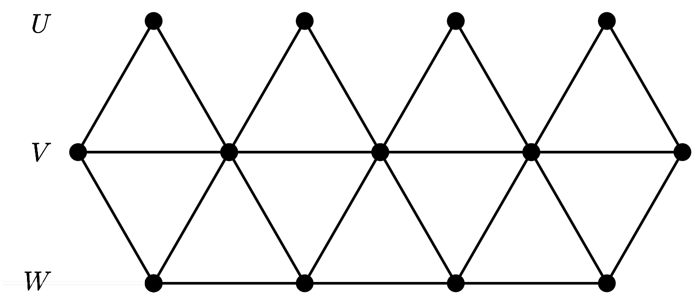

   Figure 1: The above construction for $p = 5$

   This is a maximal outerplanar graph with $p - 1$ vertices of degree 2, four vertices of degree 3, $p - 3$ vertices of degree 4 and $p - 2$ vertices of degree 6. So we have $3p - 2$ vertices and $sp(G, 1) = p + 3 = \frac{(3p-2)+11}{3} = \frac{n+11}{3}$, which differs from the lower bound by $\frac{8}{3}$.

3. For $MOP(n, 2)$, bound (1) gives $MOP(n, 2) \geq \frac{3n}{7}$ while bound (4) (with t=2) gives $MOP(n, 2) \geq \frac{4n}{9}$. We shall present a construction attaining the lower bound $\frac{5n+19}{11}$ later on.

4. In [7], the authors define $\beta_k(n)$ to be the maximum number of vertices of degree at least k amongst all maximal planar graph of order n. They show that for $k \geq 6$ and $n \geq k+2$,

$$\beta_k(n) \geq \left\lfloor \frac{n-6}{k-4} \right\rfloor .$$

Since $\delta = 2$ for any maximal outerplanar graph, it follows that

$$MOP(n,k) \geq n - \beta_{k+3}(n) \geq n - \left\lfloor \frac{n-6}{k-1} \right\rfloor \geq \frac{n(k-2)+6}{k-1} \geq \frac{(k-2)n}{k-1}.$$

We now give a construction taken [7] from which shows that this bound is sharp up to and additive decreasing function of k. Let B be the graph in Figure 2.

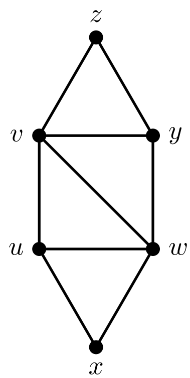

Figure 2: The graph B

The graph B' is obtained by replacing the edges ux and yz by paths P and Q respectively, containing $k-3$ internal vertices each. The vertex v is joined to every vertex on P and the vertex w is joined to every vertex in Q. We then create the graph F_k^t by taking the union of t copies of the graph B'. Figure 3 shows an example with $k=4$ and $t=3$. The graph F_k^t has $n = 2t(k-1)+6$ vertices. Such graphs have $2t+2$ vertices of degrees 2 and $2t(k-3)+2$ vertices of degree 3, two vertices of degree 4 and $2t$ vertices of degree $k+3$. This gives

$$MOP(n,k) = 2t(k-2)+6 = 2(k-2)\left(\frac{n-6}{2(k-1)}\right)+6 = \frac{(k-2)n+6}{k-1}.$$

Therefore

$$MOP(n,k) \geq \frac{(k-2)n}{k-1} + \frac{6}{k-1}$$

as required.

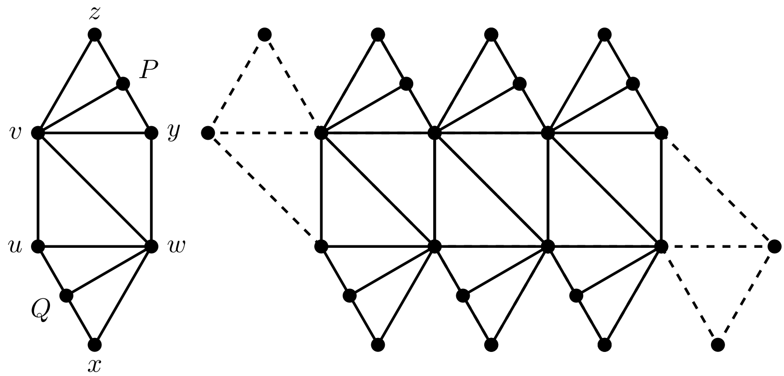

Figure 3: The graph F_4^3

□

Remarks: Although we do not have a construction which attains the bound in part 3 up to an additive constant, we show below the best constructions which we have which achieves the value of $MOP(n,k)$, $k=2$, closest to this bound.

The following construction shows that

$$MOP(n,2) \geq \frac{5n+19}{11} \text{for n= } 5 \pmod{11},$$

and hence for other values of $n \pmod{11}$ by adding at most 10 vertices. Hence $MOP(n,2) \geq \frac{4n}{9} + c(n,11)$ where $c(n,11)$ depends on $n \pmod{11}$.

Consider a path $V = v_1, v_2, \ldots, v_{5p+3}$, a path $u_1, \ldots, u_p$ above it and the vertices $w_1, \ldots, w_{5p+2}$ below the path. Let u_1 be adjacent to v_1 to v_7, and u_p adjacent to v_{5p-3} to v_{5p+3}, while for $1 \leq i \leq p-1$, u_i is adjacent to v_{5i-3} to v_{5i+2}. Vertex w_j for $1 \leq j \leq 5p+2$ is adjacent to v_j and v_{j+1}. This gives a total of $n = 11p+5$ vertices: p vertices of degree 8, $p-1$ vertices of degree 6, $4p+2$ vertices of degree 5, 2 vertices of degree 3 and $5p+2$ vertices of degree 2. Thus $Sp(n,2) = 5p+4 = \frac{5(n-5)}{11} + 4 = \frac{5n+19}{11}$. Figure 4 shows an example of this constuction.

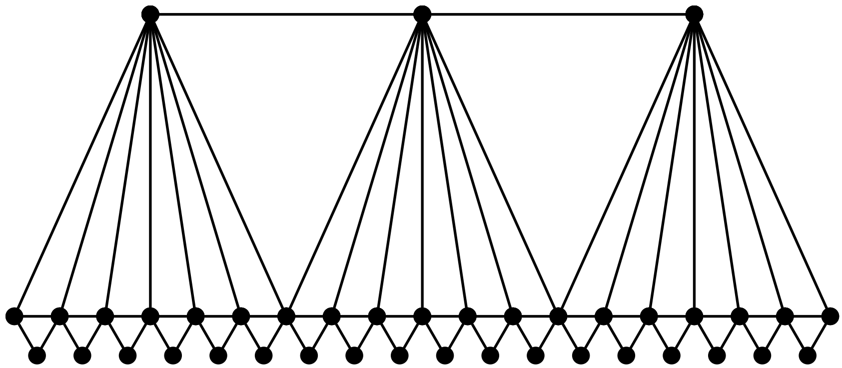

Figure 4: The above construction for $p = 3$

It might be useful to try and see at this point where applying Theorem 2.1 does not work even for trees and maximal outerplanar graphs. Let us elaborate on the simple observation we made in the introduction to this section. For trees, the phenomenon we described occurs for $k = 0$ because the proof of the Theorem would require degrees 1, 2, 3 with equal classes, but already for $k = 2$ with degrees 1 and 4 in equal classes the average degree would be 3 which is impossible for trees. Hence for $k \geq 2$ it all works out, with sharpness coming from $t = 1$ in (4) giving trees of degrees 1 and $k + 2$.

And again, for maximal outerplanar graphs, for $k = 2$ we should have degrees 2, 5, 8 with equal classes which will only give $n/3$ and $d = 5$, but this is too large as $d = 4$ for maximal outerplanars. So letting $b = 0$ in the proof of Theorem 2.1, which anyway would give $3n/7$, would force two big equal classes and the remainder. Using (4) of Theorem 2.1 with $t = 2$ gives $4n/9$ which would be possible if we could find maximal outerplanars with 4n/9 vertices of degree 2 and degrees 5 and $n/9$ vertices of degree 8, but we could not find such constructions yet.

4 Conclusion

The results presented in this paper naturally lead to an unanswered question and to the most likely next class for which one can investigate whether the spread attains the bounds of Theorem 2.1.

The obvious unanswered question is determining the best lower bound for $MOP(n,2)$, that is, the minimum spread $sp(G,2)$ among all maximal outerplanar graphs on n vertices. We know, by the general lower bound given by bound (4), that $sp(G,2)$ is at least $4n/9$. While for the other spreads we considered in Section 3 we could get close to the bound given by (4) up to small additive constants, for $MOP(n,2)$ the family of outerplanar graphs G on n vertices with lowest value for $sp(G,2)$ which we could find gave $sp(G,2)$ approaching $5n/11$.

Problem 1: Determine the correct order of magnitude of $MOP(n,2)$.

One can also consider maximal planar graphs. We define

$$MP(n,\delta,k) = \min\left\{ sp(G,k) : \begin{array}{l} G \text{ is a maximal planar on } n \text{ vertices} \\ \text{with mininmum degree } \delta \end{array} \right\}.$$

Problem 2: Determine $MP(n,\delta,k)$ for $\delta = 3,4,5$ and $k \geq 0$.

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