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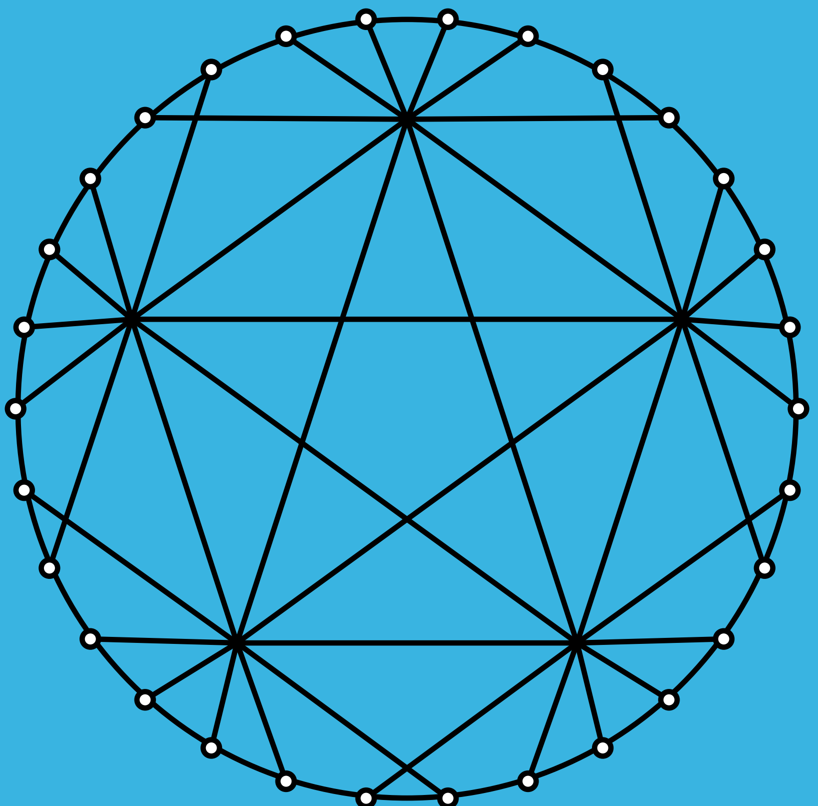

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The firefighter problem on orientations of the cubic grid

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Abstract: We investigate the firefighter problem on orientations of the grid whose edges correspond to a tiling of the plane with regular hexagons. It is proved that, for any orientation, a single fire can always be contained by a single firefighter.

1 Introduction

The *firefighter process* is the following discrete-time, dynamic process that takes place on a graph G. At time 0, a number $f \geq 1$ of vertices of G catch fire. At each subsequent time, $1, 2, \ldots$, a set of $d \geq 1$ firefighters each defend a vertex of G which is not on fire, and then the fire spreads from each burning vertex to all of its undefended neighbours. Once a vertex is burning (on fire) or defended, it remains so for all times. The process terminates when the fire can no longer spread.

When Hartnell introduced and studied the firefighter process [12] (the accompanying paper is [7]), the goal was to determine the n-vertex graphs such that, if a single fire breaks out at a random vertex and one vertex is defended per time step, then the expected number of vertices burned is minimized. The term "firefighter problem" has subsequently been commonly used to describe the various different aspects of the process that can be investigated. Many of these are surveyed in [9].

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The firefighter process can also take place on a directed graph. In that situation, the fire spreads from each burning vertex to all of its undefended out-neighbours. Any result for the firefighter problem on graphs can viewed as being for the *equivalent digraph* in which every edge is replaced by two oppositely oriented arcs. *Oriented graphs* – digraphs without directed cycles of length two – are implicitly considered in [5, 8] and explicitly considered in [2, 3, 13].

A question that has been of interest for infinite graphs is the minimum value of d such that a single fire can be *contained* by d firefighters, that is, the minimum value of d such that the firefighter process terminates in finite time. Infinite grids have received the most attention.

For the *infinite square grid of dimension 2*, the infinite planar graph whose faces can be drawn to form a tiling of the plane with squares, it is known that 2 firefighters are necessary and sufficient to contain any finite number of fires [17]. One firefighter can contain a single fire in some orientations of the infinite square grid, and two firefighters are needed for other orientations [3]. Develin and Hartke [4] show that $2n-1$ firefighters are necessary and sufficient to contain a single fire in a square grid of dimension $n \geq 3$, but for any such n there exists a number of fires which can not be contained by $2n-1$ firefighters.

Four firefighters are necessary and sufficient to contain any number of fires in the *strong grid of dimension 2*, which has vertex set $\mathbb{Z} \times \mathbb{Z}$ and edge set $\{(u,v)(x,y) : u-x = \pm 1 \text{ and } v-y = \pm 1\}$, [14].

The *infinite triangular grid* is the infinite planar graph whose faces can be drawn to form a tiling of the plane with equilateral triangles. It is known that 3 firefighters suffice to contain any finite number of fires on the infinite triangular grid [10, 15], but believed that 2 firefighters do not suffice to contain a single fire (see [11]).

Two firefighters suffice to contain any finite number of fires [14] on the *infinite cubic grid*, the infinite planar graph whose faces can be drawn to form a tiling of the plane with regular hexagons. However, it is widely believed (and conjectured in [11]) that 1 firefighter does not suffice to contain a single fire. Gavenčiak, Kratochvíl, and Prałat have shown that a single fire in the infinite cubic grid can be contained by a single firefighter, if an additional firefighter is available on any two not necessarily different time steps [11].

Other aspects of the firefighter process that have been considered for infinite graphs include bounding the proportion of vertices that a given number of firefighters can save from burning, results when the number of firefighters used per time step is not constant, and fractional or probabilistic versions; see [1, 6, 9, 10, 11, 14, 16, 18, 19].

We consider the firefighter on orientations of the infinite cubic grid, and prove the following:

Theorem 1.1. *A single firefighter is sufficient to contain a single fire on any orientation of the infinite cubic grid.*

2 Proof of Theorem 1.1

Our main result is proved by considering four cases depending on the out-degree of the vertex w where the fire breaks out. In the case where w has out-degree 3, we make use of the following lemma.

Lemma 2.1 ([11], Section 3.2)**.** *In the infinite cubic grid, if the fire is confined to an infinite strip of any width bounded by parallel rays and extending in only one direction, then it can be contained by a single firefighter.*

As noted in the introduction, Gavenčiak, Kratochvíl, and Prałat [11] have proved that a single fire in the cubic grid can be contained by a single firefighter, if an additional firefighter is available on any two not necessarily different time steps. The additional firefighters make it possible for the fire to be confined to an infinite strip or any width bounded by parallel rays and extending in only one direction. After this has happened, and the strip is allowed to become sufficiently long, it is possible for the firefighter to defend vertices in such a way that the fire is contained. See [11] for details.

Proposition 2.2. *If the fire starts at a vertex with out-degree* 0*, then it can not spread. If the fire starts at a vertex of out-degree* 1*, then it can be contained in* 1 *time step.*

Proposition 2.3. *If the fire starts at a vertex with out-degree* 2*, then the fire can be contained in at most* 4 *time steps.*

Proof. Suppose the fire starts at vertex w with two out-neighbours, x and y. Let H be the oriented hexagon containing x and y. Notice that H is not a directed 6-cycle.

In the first time step, defend y. The fire then spreads to x. If x has no out-neighbour, the fire can not spread and the process terminates. If x has exactly out-neighbour, then the process ends when this vertex is defended. Otherwise, x has exactly 2 out-neighbours. Defend the out-neighbour of x which does not belong to H. The fire then spreads to the out-neighbour of x which belongs to H, call it z. The process then ends in at most 2 more time steps. $\square$

An example illustrating a case where the vertex x has out-degree 2, and the unique out-neighbour z of x on H has out-degree 0 is shown in Figure 1. Here, and in all remaining figures, the (red) diamond indicates the vertex w where the fire starts, the blue squares represent vertices defended by the firefighter, and numbers indicate the time step at which vertices either catch fire or are defended.

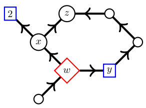

Figure 1: An example where the vertex x has out-degree 2 and the fire is contained in 2 steps.

We call two arcs in an orientation of the hexagonal grid *wall-adjacent* if they are opposite arcs of the same hexagon and oriented in the same direction. A *wall* is a sequence $e_1, e_2, \ldots, e_k$ of arcs such that every pair of consecutive arcs in the sequence are wall-adjacent. The number of arcs in a wall is its *length.*

An example of a wall is given in Figure 2. Edges shown without a direction can be oriented arbitrarily.

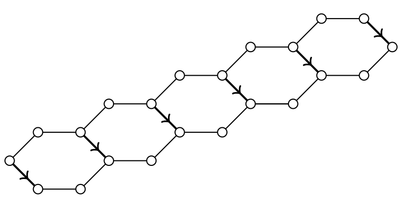

Figure 2: A wall of length 6; only part of the oriented cubic grid is shown.

Lemma 2.4. *If the fire starts at a vertex with out-degree 3, then it can be contained.*

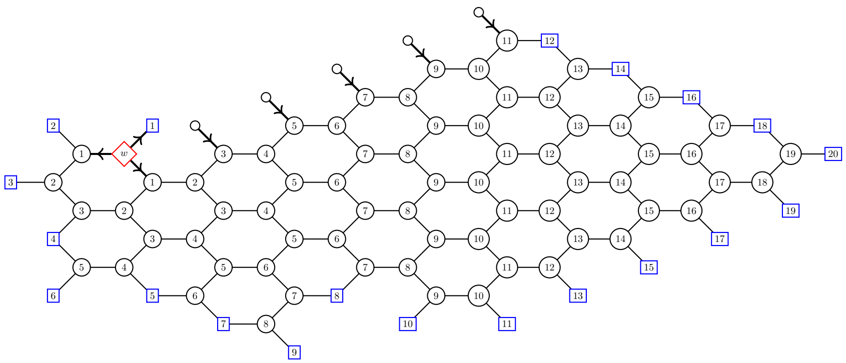

Figure 3: There is a wall of length 6 beginning at w.

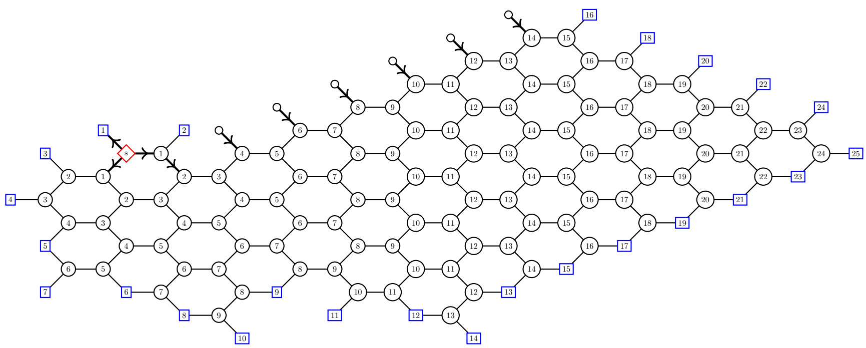

Figure 4: There is a wall of length 7 starting starting with an arc originating at x, y, or z.

Proof. Suppose the fire starts at vertex the w with out-neighbours x, y, and z. We consider three cases.

Case 1. There is a wall of length 6 starting with one of the arcs wx, wy, or wz. In this case, the fire is contained using the strategy shown in Figure 3. We account for all possible orientations by assuming that all edges other than those incident with w, and those in the wall, are undirected.

Case 2. There is a wall of length 7 starting with an arc originating at x, y, or z. In this case, the fire is contained using the strategy shown in Figure 4. We account for all possible orientations by assuming that all edges other than those incident with w, and those in the wall, are undirected.

Case 3. There are no walls of sufficient length starting at any of the arcs from w, x, y, or z. Therefore, walls that start with an arc originating at w have length at most 5, and walls (in any direction) that start with an arc originating at x, y, or z have length at most 6.

Choose two walls, one of which starts with an arc originating at w, and the other of which starts with an arc originating at one of x, y, or z. Now apply the strategy shown in Figure 5. We account for all possible orientations by assuming that all edges other than those incident with w, and those in the walls, are undirected.

After some time, the fire is confined to an infinite strip bounded by parallel rays and extending in only one direction. By Lemma 2.1, the fire can be contained. □

This completes the proof of Theorem 1.1. We remark that in Case 3 of the proof, the vertices incident with the arcs immediately following the end of the walls play the same role as those defended by the two additional firefighters in the theorem of Gavenčiak, Kratochvíl, and Prałat.

One may wonder about the conditions under which a similar result holds for a partially oriented graph (i.e. only some edges are oriented). The fire can be contained if one edge incident with the vertex w where the fire starts is oriented towards w. Can it be contained if exactly two edges are oriented against the "normal" flow of the fire? This statement is true if the edges are in the positions that arise in the proof of Lemma 2.4. A related question is whether there exists an integer k such that the fire can be contained if k edges are oriented (in some way).

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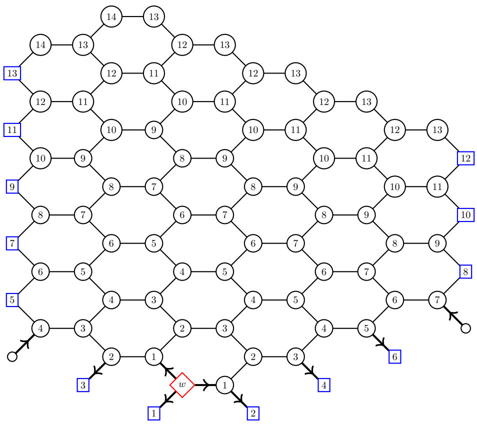

Figure 5: There is no long wall and the fire can be confined to a strip.

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