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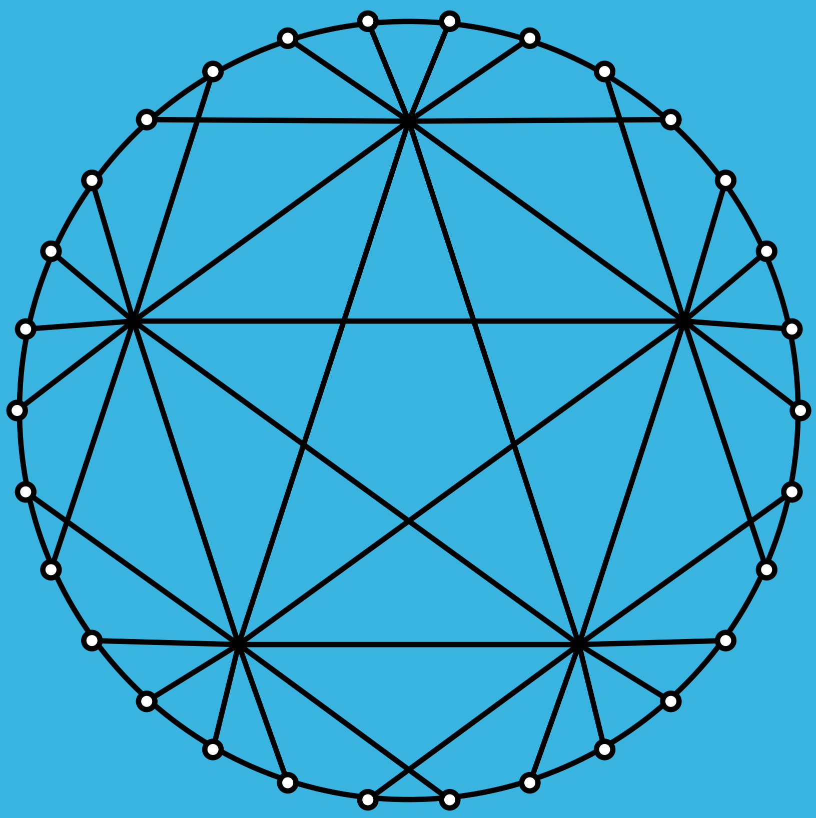

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A note on totally-omnitonal graphs

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Abstract: Let the edges of the complete graph K_n be coloured red or blue, and let G be a graph with $n \geq |V(G)|$. Then $\mathrm{ot}(n, G)$ is defined to be the minimum integer, if it exists, such that any colouring of K_n with at least $\mathrm{ot}(n, G)$ edges of each colour, contains a copy of G with r red edges and b blue edges for any $r, b \geq 0$ with $r + b = e(G)$. If $\mathrm{ot}(n, G)$ exists for every sufficiently large n, we say that G is *omnitonal*. Omnitonal graphs were introduced by Caro, Hansberg and Montejano [arXiv:1810.12375,2019]. Now let G_1, G_2 be two copies of G with their edges coloured red or blue. If there is a colour-preserving isomorphism from G_1 to G_2 we say that the 2-colourings of G are equivalent. Now we define $\mathrm{tot}(n, G)$ to be the minimum integer, if it exists, such that any colouring of K_n with at least $\mathrm{tot}(n, G)$ edges of each colour, contains all non-quivalent colourings of G with r red edges and b blue edges for any $r, b \geq 0$ with $r + b = e(G)$. If $\mathrm{tot}(n, G)$ exists for every sufficiently large n, we say that G is *totally-omnitonal*.

In this note we show that the only totally-omnitonal graphs are stars or star forests namely a forest all of whose components are stars.

1 Introduction

By a 2-colouring of the complete graph K_n we mean a function $f : E(K_n) \to \{red, blue\}$. The set of edges of K_n coloured red or blue is denoted by R

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or B, respectively. For short we also denote by R, B the subgraphs of K_n induced by these edge sets. If G is a subgraphs of such a 2-coloured K_n with r red edges b blue edges we say that K_n contains an (r, b)-coloured copy of G. We recall the definition of omnitonal graphs from [2]. For a given graph G, $\mathrm{ot}(n, G)$ is defined to be the minimum integer, if it exists, such that any 2-colouring of K_n, $n \geq |V(G)|$, with $\min\{|R|, |B|\} > \mathrm{ot}(n, G)$ contains an (r, b)-coloured copy of G for any $r \geq 0$ and $b \geq 0$ such that $r + b = e(G)$, where $e(G) = |E(G)|$. If $\mathrm{ot}(n, G)$ exists for every sufficiently large n, we say that G is *omnitonal*.

We now define totally-omnitonal graphs . Let G_1, G_2 be two (r, b)-coloured copies of G. Then if there is a colour preserving isomorphism $\phi : G_1 \to G_2$, we say that the two colourings G_1 and G_2 of G are equivalent. Otherwise the colourings are said to be non-equivalent. Now, for a given graph G, $\mathrm{tot}(n, G)$ is defined to be the minimum integer, if it exists, such that any 2-colouring of $E(K_n)$ with $\min\{|R|, |B|\} > \mathrm{tot}(n, G)$ contains every non-equivalent (r, b)-coloured copy of G for any $r \geq 0$ and $b \geq 0$ such that $r + b = e(G)$. If $\mathrm{tot}(n, G)$ exists for every sufficiently large n, we say that G is *totally-omnitonal*.

For other graph-theoterical terms we refer the reader to West [4]. We just recall that a star, denoted by $K_{1,p}$ is the graph consisting of one vertex joined to each of p other vertices. Therefore K_2 is the star $K_{1,1}$.

The main aim of this note is to show that a graph G is totally-omnitonal if and only if it is a star or a star forest, namely a forest all of whose components are stars.

In several places in this note we shall make use of the following result, which is part of Theorem 4.1 in [2].

Theorem A. *Let n and k be positive integers such that $n \geq 4k$. Then*

$$\mathrm{ot}(n, K_{1,k}) = \begin{cases} \lfloor (\frac{(k-1}{2})n \rfloor, & \textit{for } k \leq 3, \\ (k-2)n - \frac{k^2}{2} + \frac{3}{2}k - 1, & \textit{for } k \geq 4. \end{cases}$$

2 Results

2.1 A canonical colouring for K_n

In this subsection we define a canonical colouring of the edges of K_n which is essential to the proof of our results. Let A be a subset of $V(K_n)$ such that $2 \leq |A| < n-1$, and let B be $V(K_n) - A$. Colour red all $\binom{|A|}{2}$ edges joining vertices in A and colour blue all the remaining $\binom{|B|}{2} + |A| \cdot |B|$ edges of K_n. We observe that in such a colouring of K_n, there is no path P_4 with a *red* $-$ *blue* $-$ *red* colouring, that is, a colouring of the edges of P_4 such that the middle edge is coloured blue and the pendant edges are both coloured red. Also, there is no colouring of K_3 with two red and one blue edges. In order to use such a 2-colouring of K_n to show that P_4 and K_3 are not totally-omnitonal we shall need to show, for reasons which shall become clear below, that there is such a colouring for an infinite sequence of complete graphs in which the number of red edges is equal to the number of blue edges.

For this to hold, suppose $|A| = r$. We shall now use an argument employed in [3]. For the number of red edges to be equal to the number of blue edges, we require that

$$\frac{r(r-1)}{2} = \frac{n(n-1)}{4},$$

that is,

$$2(r^2 - r) = n^2 - n.$$

But this is equivalent to

$$(2n-1)^2 - 2(2r-1)^2 = -1,$$

and, if we let y and x be, respectively, the two odd integers $2n-1$ and $2r-1$, we obtain,

$$y^2 - 2x^2 = -1.$$

But this is Pell's equation which is known to have an infinite number of solutions for x and y [1].

So we define a *canonical colouring* of K_n, if it exists, to be a colouring in which the edges of a subclique are coloured red, while all the other edges are coloured blue, and the number of red edges is equal to the number of blue edges. We therefore have the following result from [3].

Theorem 2.1. *There is an infinite sequence of complete graphs K_n for which a canonical colouring exists.*

2.2 P_4 and K_3 are not totally-omnitonal

Suppose that P_4 (or K_3) is totally-omnitonal. Therefore there exists an integer $t(n) = \text{tot}(n, P_4)$ (or $t(n) = \text{tot}(n, K_3)$), such that any 2-colouring of K_n with $\min\{|R|, |B|\} > t(n)$ contains an $red - blue - red$ colouring of P_4 (or, a $(2,1)$-colouring of K_3) for n sufficiently large. Note that $t(n)$ must be less than $n(n-1)/4$. However, we have seen that for any N there is a K_n with $n > N$ which has a canonical colouring. In this colouring, $|R| = |B| = n(n-1)/4$, and K_n does not contain an $red - blue - red$ colouring of P_4 (nor, a $(2,1)$-colouring of K_3). We have therefore proved the following.

Lemma 2.2. *Both P_4 and K_3 are not totally-omnitonal.*

We now use this lemma to characterise connected omnitonal graphs.

Theorem 2.3. *Let G be a connected graph on at least three vertices which is not a star. Then G is not totally-omnitonal.*

Proof. If G is not a star or K_3 which we already proved to be non-omnitonal, then it is well-known that it must have a pair of independent edges. Let e_1, e_2 be the closest pair of independent edges. They must therefore be joined by an edge e_3. Now colour e_3 blue and all the other edges of G red.

This is a specific $(e(G)1, 1)$-colouring of G, but as we have shown in Lemma 2.2, for infinitely many values of n there is a canonical colouring of K_n which does not contain this colouring of G due to the specific colouring of the P_4 subgraph of G. Therefore G cannot be totally-omnitonal. □

However we do have the following.

Lemma 2.4. *Stars are totally-omnitonal.*

Proof. It is known from Theorem A in [2] stated above, that $K_{1,p}$ is omnitonal. Therefore there is a number $\text{ot}(n, K_{1,p})$ such that, for any positive integers r, b with $r + b = p$, and for all n sufficiently large, if $E(K_n)$ is two-coloured with $\min\{|R|, |B|\} > \text{ot}(n, K_{1,p})$, then it contains an (r, b)-coloured copy of $K_{1,p}$. However, by the symmetry of the edges of $K_{1,p}$, any two (r, b)-coloured copies of $K_{1,p}$ with $r + b = p$ are equivalent. Therefore $K_{1,p}$ is totally-omnitonal with $\text{tot}(n, K_{1,p}) = \text{ot}(n, K_{1,p})$. □

From all the above we can conclude the following.

Corollary 2.5. *A connected graph G is totally-omnitonal if and only if it is a star.*

2.2.1 Disconnected totally-omnitonal graphs

And so we come to our main theorem. By a star-forest we shall mean a graph all of whose connected components are stars.

Theorem 2.6. *A graph G is totally omnitonal if and only if it is a star forest.*

Proof. If G is connected then we are done since we have already shown that the only connected totally-omnitonal graphs are stars. Therefore suppose G is disconnected.

If even one component of G is not totally-omnitonal, then G is not totally-omnitonal. Therefore if even one component of G is not a star then G is not totally-omnitonal. This is because:

1. If there is a K_3 component then we colour $E(K_3)$ with one blue edge and two red edges, and all the other edges of G are coloured red. This is a specific $(e(G)-1,1)$-coloured pattern of G, which requires that K_3 will be coloured in a $(2,1)$-colouring, which is impossible as we have shown in Lemma 2.2.

2. If there is no K_3 but there is a component which is not a star, then this component must contain P_4, and we colour the middle edge of this P_4 blue and all the other edges of G red. This is a specific $(e(G)-1,1)$coloured pattern of G, but we have shown the canonical coloring cannot contain any $red - blue - red$ coloured P_4, hence all components must be stars.

Conversely, suppose each component of G is totally-omnitonal, that is, each component is a star. We need to show that G is totally-omnitonal.

We know from Theorem A that $\mathrm{ot}(n, K_{1,k}) < (k-1)n$ for $n \geq 4k$. Let $G = \cup K_{1,p_j} for j = 1, \ldots, q$ and $p_1 \geq p_2 \geq \ldots \geq p_q$.

Let $n \geq 4(p_1 + p_2 + \ldots + p_q + q - 1)$ then we claim that $\mathrm{tot}(n, G) < M(p_1, \ldots, p_q, n) := (p_1 + p_2 + ..p_q + q - 2)n$. We have to show first that the conditions on n and M are feasible for every $q \geq 1$, namely that for $n \geq 4(p_1+p_2+\ldots+p_q+q-1)$ a colouring with $\min\{|R|, |B|\} \geq M(p_1, \ldots, p_q, n)$ exists. Hence we have to show that $2(p_1 + \ldots + p_q + q - 2)n \leq n(n-1)/2$. But this is equivalent to $4(p_1+\ldots+p_q+q-2) = 4(p_1+\ldots+p_q+q-1)-4 \leq n-4 < n-1$. Hence the condition is satisfied with $n \geq 4(p_1+\ldots+p_q+q-1)$ and $M(p_1, \ldots, p_q, n)$ as defined above.

We now prove the theorem by induction on q. For $q = 1$ the result is true by Theorem A. Assume result is true for $q - 1$ components and assume G has q components.

Let f be any colouring of the edges of G. We wish to show that in any 2-colouring $g : E(K_n) \to \{red, blue\}$ with $\min\{|R|, |B|\} \geq (p_1 + \ldots + p_q + q - 2)n$ there is a copy of G on which g restricted to the edges of G is equivalent to the colouring f of G.

We have already shown above that such a colouring g for K_n exists for $n \geq 4(p_1+p_2+\ldots+p_q+q-1)$. So suppose $n \geq 4(p_1+p_2+\ldots+p_q+q-1)$. Let $G^* = G \backslash K_{1,p_q}$. Since $p_1 \geq p_q$, $n > 4p_q$ and $\min\{|R|, |B|\} > n(p_q - 1)$, it follows from Theorem A that there is a copy of K_{1,p_q} which is precisely f-coloured, namely the colouring of K_{1,p_q} is equivalent to the colouring induced by f.

Remove the vertices of this f-coloured K_{1,p_q} from $V(K_n)$ and remove the edges incident with at least one vertex of $V(K_{1,p_q})$. We are left with n^* vertices where $n^* = n - (p_q + 1) \geq 4(p_1 + \ldots + p_{q-1} + p_q + q - 1) - (p_q + 1) > 4(p_1 + \ldots + p_{q-1} + q - 2)$ and with $\min\{|R|, |B|\} \geq (p_1 + \ldots + p_q + q2)n - \binom{p_q+1}{2} - (p_q + 1)(np_q - 1) > (p_1 + \ldots + p_q + q - 2)n - n(p_q + 1) = (p_1 + \ldots + p_{q-1} + q - 3)n > (p_1 + \ldots + p_{q-1} + q - 3)n^*$.

So the conditions for $q1$ components are satisfied.

Therefore by the induction hypothesis, K_{n^*} contains an f-coloured copy of G^* which together with the deleted f-coloured copy of K_{1,p_q}, giving the required f-coloured copy of G in $E(K_n)$. $\square$

The bounds we have proved can now be stated as a corollary.

Corollary 2.7. *Let G be a star-forest with components $K_{1,p_1} \cup K_{1,p_2} \cup \ldots \cup K_{1,p_q}$ with $p_1 \geq p_2 \geq \ldots \geq p_q$. Then, for $n \geq 4(p_1 + \ldots + p_q + q - 1)$, any 2-colouring of $E(K_n)$ with* $\min\{|R|, |B|\} > (p_1 + \ldots + p_q + q - 2)n$

contains every non-equivalent (r, b)*-coloured copy of* G*, for any* $r, b \geq 0$ *and* $r + b = e(G)$*.*

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