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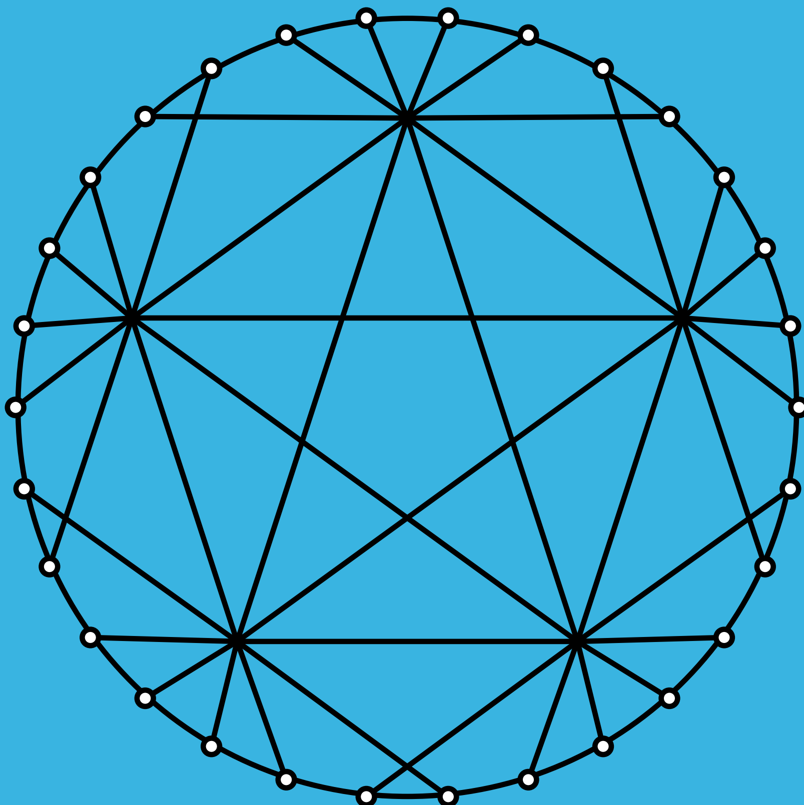

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Morphisms of Skew Hadamard Matrices

Philip Heikoop, Guillermo Nuñez Ponasso*, Padraig Ó Catháin, and John Pugmire

Worcester Polytechnic Institute, Worcester, MA 01609, USA
ptheikoop@wpi.edu, gcnunez@wpi.edu,
pocathain@wpi.edu and pugmire@ucsb.edu

Abstract

Quaternary unit Hadamard (QUH) matrices were introduced by Fender, Kharaghani and Suda along with a method to construct them at prime power orders. We present a novel construction of real Hadamard matrices from QUH matrices. Our construction recovers the result by Mukhopadhyay on the existence of real Hadamard matrices of order $q^n + q^{n-1}$ for each prime power $q \equiv 3 \mod 4$, and $n \geq 1$. Furthermore we provide nonexistence conditions for QUH matrices.

1 Introduction

A celebrated theorem of Hadamard characterises the complex matrices with entries of norm at most one which have maximal determinant: they are precisely the solutions to the matrix equation $HH^* = nI_n$ satisfying $|h_{ij}| = 1$ for all $1 \leq i, j \leq n$. Equivalently, all entries of H have unit norm, and all rows are mutually orthogonal under the Hermitian inner product, [7]. Real

*Corresponding author.

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Hadamard matrices, having entries in $\{\pm 1\}$, have been extensively studied for a century, though the existence problem is far from settled. We refer the reader to the recent monographs of Horadam and of de Launey and Flannery for extensive discussion of Hadamard matrices, [8, 3].

In this paper we will study the problem of constructing real Hadamard matrices from complex Hadamard matrices (CHM). Suppose that X is a set of complex numbers of modulus 1. We define $\mathcal{H}(n, X)$ to be the set of $n \times n$ Hadamard matrices with entries drawn from X. In the special case that X is the set of k^{th} roots of unity, a CHM is called a *Butson Hadamard matrix*; the set of such matrices is denoted $\mathcal{BH}(n, k)$. Examples of Butson Hadamard matrices are furnished by the character tables of abelian groups of order n and exponent k. Cohn and Turyn proved independently that the existence of $H \in \mathcal{BH}(n, 4)$ implies the existence of a real Hadamard matrix of order $2n$, [1, 14]. More recently, Compton, Craigen and de Launey proved that an $n \times n$ matrix with entries in the *unreal* sixth roots of unity $\{\omega_6, \omega_6^2, \omega_6^4, \omega_6^5\}$ can be used to construct a real Hadamard matrix of order $4n$, [2].

A general construction for mappings between sets of Butson Hadamard matrices is described by Egan and one of the present authors, [4]. A key ingredient in the construction is a matrix $H \in \mathcal{BH}(n, k)$ with minimal polynomial $\Phi_t(x)$ for some integer t. The construction of such matrices was considered further in collaboration with Eric Swartz, [5]. In all the examples considered previously, matrix entries are roots of unity, and all fields considered are cyclotomic. In this paper, we consider a family of complex Hadamard matrices with entries in the biquadratic extension $\mathbb{Q}[\sqrt{-q}, \sqrt{q+1}]$. When the matrix entries are all in the set $X_q = \{\frac{\pm 1 \pm \sqrt{-q}}{\sqrt{q+1}}\}$, such a matrix is called a *Quaternary Unit Hadamard matrix*, abbreviated QUH. Such matrices were first considered by Fender, Kharaghani and Suda, [6].

We will construct a morphism from QUH matrices onto real Hadamard matrices, using skew-Hadamard matrices. This provides a new construction for a family of Hadamard matrices of order $q^n + q^{n-1}$ for each prime power $q \equiv 3 \mod 4$ and each $n \geq 1$, previously constructed by Mukhopadhyay and studied further by Seberry, [12, 13]. We conclude the paper by studying the decomposition of prime ideals in the field $\mathbb{Q}[\sqrt{-q}, \sqrt{q+1}]$ to obtain non-existence results for QUH matrices in the style of Winterhof [15].

2 Morphisms of QUH matrices

In this section we construct an isomorphism of fields, which we lift to an isomorphism of matrix algebras. We prove that this isomorphism carries a QUH matrix in the set $\mathcal{H}(n, X_m)$ to a real Hadamard matrix of order $n(m+1)$; that is, the isomorphism is a *morphism* of complex Hadamard matrices. We will require some standard results in algebra, as discussed in, e.g., Chapters 17–19 of Isaacs' *Graduate Algebra*, [10]. An *extension field* k of $\mathbb{Q}$ is a field containing $\mathbb{Q}$ as a subfield; in this case k is a $\mathbb{Q}$-vector space and its *degree* is its dimension as a vector space. The degree of k over $\mathbb{Q}$ is denoted by $[k : \mathbb{Q}]$. In the ring $\mathbb{Q}[x]$ every ideal contains a unique monic polynomial of minimal degree, this polynomial is irreducible if and only if the ideal is maximal. For a polynomial $p(x)$ the quotient $\mathbb{Q}[x]/\left(p(x)\right)$ is a field if and only if the polynomial $p(x)$ is irreducible. An extension field k is the *splitting field* of a polynomial $p(x) \in \mathbb{Q}[x]$ if k is a field of minimal degree over $\mathbb{Q}$ which contains all the roots of $p(x)$. We apply these results to the polynomial $\mathfrak{m}(x) = x^4 + \frac{2(m-1)}{m+1}x^2 + 1$. By abuse of notation, a Hadamard matrix is *skew* if $H - I$ is a skew-symmetric matrix.

Proposition 2.1. *Let H be a skew-Hadamard matrix of order $m+1$, where $m+1$ is not a perfect square. The minimal polynomial of $\alpha_m = \frac{1+\sqrt{-m}}{\sqrt{m+1}}$ and the minimal polynomial of $\frac{1}{\sqrt{m+1}}H$ are both equal to*

$$\mathfrak{m}(x) = x^4 + \frac{2(m-1)}{m+1}x^2 + 1\,.$$

Proof. It is easily checked that α_m is a root of $\mathfrak{m}(x)$. Since $\mathfrak{m}(x)$ is even, $-\alpha_m$ is also a root. The coefficients of $\mathfrak{m}(x)$ are real, thus α_m^* and $-\alpha_m^*$ are roots. From the fact that $\mathfrak{m}(x)$ has degree 4, we conclude that these are all the possible roots. Therefore we obtain the factorisation

$$\mathfrak{m}(x) = (x - \alpha_m)(x - \alpha_m^*)(x + \alpha_m)(x + \alpha_m^*)\,.$$

Clearly $\mathfrak{m}(x)$ has no linear factors in $\mathbb{Q}[x]$. The only possible quadratic factors with real entries are $(x - \alpha_m)(x - \alpha_m^*) = x^2 - 2x/\sqrt{m+1} + 1$ and $(x + \alpha_m)(x + \alpha_m^*) = x^2 + 2x/\sqrt{m+1} + 1$. By hypothesis, $m+1$ is not a perfect square so these factors are not in $\mathbb{Q}[x]$. We have shown that $\mathfrak{m}(x)$ is irreducible. The field extension $\mathbb{Q}[\alpha_m]$ contains $\alpha^{-1} = \alpha_m^*$ and $-\alpha_m$, so it is the splitting field of $\mathfrak{m}(x)$.

Since H is skew-Hadamard we have both $HH^\top = (m+1)I_{m+1}$ and $H^\top = 2I - H$. It follows that $H(2I - H) = (m+1)I$, or $H^2 = 2H - (m+1)I$.

Hence,

$$\left(\frac{1}{\sqrt{m+1}}H\right)^2 = \frac{2}{m+1}H - I\,.$$

We also compute that

$$\begin{aligned}\left(\frac{1}{\sqrt{m+1}}H\right)^4 &= \frac{4}{(m+1)}\left(\frac{1}{\sqrt{m+1}}H\right)^2 - \frac{4}{m+1}H + I \\ &= \frac{4}{(m+1)}\left(\frac{1}{\sqrt{m+1}}H\right)^2 - 2\left(\frac{2}{m+1}H - I\right) - I \\ &= \frac{4}{(m+1)}\left(\frac{1}{\sqrt{m+1}}H\right)^2 - 2\left(\frac{1}{\sqrt{m+1}}H\right)^2 - I \\ &= \frac{2-2m}{m+1}\left(\frac{1}{\sqrt{m+1}}H\right)^2 - I\end{aligned}$$

We conclude that the unitary matrix $\frac{1}{\sqrt{m+1}}H$ is a root of polynomial $\mathfrak{m}(x)$, which must be the minimal polynomial of $\frac{1}{\sqrt{m+1}}H$ by irreducibility. □

When $m+1$ is a perfect square, the polynomial $\mathfrak{m}(x)$ factors into two irreducible quadratic factors in $\mathbb{Q}[x]$, which correspond to the distinct minimal polynomials of α_m and $-\alpha_m$. In this case, the minimal polynomials of α_m and $\frac{1}{\sqrt{m+1}}H$ coincide, and also the minimal polynomials of $-\alpha_m$ and $\frac{1}{\sqrt{m+1}}(H-2I)$ coincide. The case that $m+1$ is a perfect square will be discussed after the proof of Theorem 2.4. From Proposition 2.1, the next result is immediate.

Proposition 2.2. *If H is a skew-Hadamard matrix of order m, then all of the following $\mathbb{Q}$-algebras are isomorphic:*

$$\mathbb{Q}[x]/(\mathfrak{m}(x)) \simeq \mathbb{Q}\left[\frac{1}{\sqrt{m+1}}H\right] \simeq \mathbb{Q}[\alpha_m]. \tag{1}$$

Definition 2.3. *A Quaternary Unit Hadamard (QUH) matrix is an element of $\mathcal{H}(n, X_m)$, where*

$$X_m = \left\{\frac{\pm 1 \pm \sqrt{-m}}{\sqrt{m+1}}\right\}.$$

Now we give the main result of this section.

Theorem 2.4. *If there exists a skew-Hadamard matrix H of order $m+1$, where $m+1$ is not a perfect square, there exists a morphism*

$$\mathcal{H}(n, X_m) \to BH(nm+n, 2).$$

Proof. Assume that there exists $M \in \mathcal{H}(n, X_m)$, since otherwise the claim is vacuous. By Proposition 2.2 that there exists an isomorphism $\mathbb{Q}(\alpha_m) \to \mathbb{Q}(\frac{1}{\sqrt{m+1}}H)$. We make this explicit:

$$\varphi : \alpha_m \mapsto \frac{1}{\sqrt{m+1}}H$$

and since α_m is a generator of $\mathbb{Q}(\alpha_m)$ the function φ extends uniquely to the whole field. Recalling that H is skew, we obtain

$$\varphi(-\alpha_m) = \frac{-1}{\sqrt{m+1}}H = \frac{1}{\sqrt{m+1}}(H - 2I)^\top, \;\; \varphi(\alpha_m^*) = \frac{1}{\sqrt{m+1}}H^\top .$$

Define M^φ to be the block matrix obtained from M by applying φ entry-wise. Then M^φ is a real matrix of size $n(m+1) \times n(m+1)$ with entries in the set $\{\pm 1/\sqrt{m+1}\}$. Since $M \in \mathcal{H}(n, X_m)$ the (Hermitian) inner product of columns c_i and c_j of M is $\langle c_i, c_j \rangle = n\delta_i^j$, where δ_i^j is the Kronecker δ function. Since φ is an isomorphism of $\mathbb{Q}$-algebras, $\varphi(0) = \mathbf{0}_{m+1}$ and $\varphi(1) = I_{m+1}$. It follows that

$$\begin{aligned}\sum_k \varphi(c_{i,k})\varphi(c_{j,k})^\top &= \sum_k \varphi(c_{i,k})\varphi(c_{j,k}^*) \\ &= \varphi\left(\sum_k c_{i,k}c_{j,k}^*\right) \\ &= \varphi(\langle c_i, c_j \rangle) \\ &= n\delta_i^j I_{m+1} .\end{aligned}$$

This shows that $M^\varphi \left(M^\varphi\right)^\top = nI_{n(m+1)}$. The entries of M^φ are in the set $\{\pm 1/\sqrt{m+1}\}$, so the entries of $\sqrt{m+1}M^\varphi$ are in the set $\{\pm 1\}$. We have shown that

$$\sqrt{m+1}M^\varphi \left(\sqrt{m+1}M^\varphi\right)^\top = n(m+1)I_{n(m+1)} ,$$

which establishes the theorem. □

A less technical method to prove the above theorem without assumptions on $m+1$ is as follows: Let $H \in \mathcal{H}(n, X_m)$, and let

$$H = \frac{1}{\sqrt{m+1}}A + \frac{\sqrt{-m}}{\sqrt{m+1}}B,$$

where A and B are ± 1 matrices of order n. Then

$$AB^\top = BA^\top \text{ and } AA^\top + BB^\top = n(m+1)I_n.$$

Let M be a skew Hadamard matrix of order $m+1$. Substituting A for the diagonal entries of M and $\pm B$ for the off-diagonal entries ± 1 of M, it can be verified that the resulting matrix will be a Hadamard matrix of order $n(m+1)$. Although this proof is simpler than that of Theorem 2.4, the morphism method gives additional insights into existence and non-existence of QUH matrices, as demonstrated in Section 3.

Let q be an odd prime power and $\mathbb{F}_q$ be a finite field with q elements. The element $a \in \mathbb{F}_q$ is a *quadratic residue* if there exists $y \in \mathbb{F}_q$ such that $y^2 = a$. Otherwise, a is a non-residue. The *quadratic character* is defined to be $\chi_q(a) = 1$ if $a \in \mathbb{F}_q^* = \mathbb{F}_q - \{0\}$ is a quadratic residue in $\mathbb{F}_q$, $\chi_q(a) = -1$ if $a \in \mathbb{F}_q^*$ is a quadratic non-residue in $\mathbb{F}_q$ and $\chi_q(0) = 0$. In the case where $q = p$ is a prime number, the quadratic character $\chi_p(a)$ on $\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}$ can be identified with the *Legendre symbol* and is denoted (a/p). Later we will use the fact that for a fixed prime p and for every $a, b \in \mathbb{Z}$, $(ab/p) = (a/p)\,(b/p)$, [9, Proposition 5.1.2]. Let $\{g_0 = 0, g_1, \ldots, g_{q-1}\}$ be an enumeration of $\mathbb{F}_q$ then $Q = [\chi_q(g_i - g_j)]_{0 \le i,j \le q-1}$ is the *Jacobsthal matrix* of order q.

Theorem 2.5 (Section 3, [6])**.** *Let q be an odd prime power with $q \equiv 3 \pmod 4$. Define 1×1 matrices $A_0 = B_0 = 1$, let Q be the $q \times q$ Jacobsthal matrix and J_q the $q \times q$ all-ones matrix. For each $t \ge 1$, define*

$$A_t = J_q \otimes B_{t-1}, \;\; B_t = I_q \otimes A_{t-1} + Q \otimes B_{t-1}\,.$$

Then for each t the matrix $\frac{1}{\sqrt{q+1}} A_t + i\frac{\sqrt{q}}{\sqrt{q+1}} B_t$ is a matrix in $\mathcal{H}(q^t, X_q)$.

Hence there exist $\mathcal{H}(q^t, X_q)$ matrices for all prime powers $q \equiv 3 \mod 4$. Since the Paley matrix of order $q+1$ is skew, we can apply Theorem 2.4 to obtain the following result.

Corollary 2.6. *Let $q \equiv 3 \mod 4$ be a prime power. For any integer $n \ge 1$ there exists a (real) Hadamard matrix of order $q^n + q^{n-1}$.*

This result was first discovered by Mukhopadhyay, and later clarified and elaborated by Seberry, [12, 13]. Of course, it would be interesting to develop constructions of Hadamard matrices at previously unknown orders. As a first contribution in this direction, we investigate the non-existence of QUH matrices in the next section.

3 Nonexistence of quaternary unit Hadamard matrices

The *Galois group* of an irreducible polynomial $p(x)$ is the group of field automorphisms of a splitting field of $p(x)$. Over $\mathbb{Q}$, the order of the Galois group and the degree of the splitting field coincide. The Galois correspondence gives an inclusion-reversing bijection between the lattice of subfields of $\mathbb{Q}[x]/\left(p(x)\right)$ and the subgroups of the Galois group.

An element $x \in \mathbb{C}$ is an *algebraic integer* if it is a root of a monic polynomial in $\mathbb{Z}[x]$. The ring of integers of a number field $k \subseteq \mathbb{C}$ is the largest subring of the algebraic integers contained in k, usually denoted $\mathcal{O}_k$. In the ring of integers of a number field, ideals factorise uniquely as a product of *prime ideals*, [11, Theorem 14]. Studying prime factorisations related to the determinant of a putative complex Hadamard matrix can sometimes yield nonexistence results. This argument is similar to one given by Winterhof for certain Butson Hadamard matrices, [15].

First, we introduce terminology for the factorisation of a prime ideal of $\mathbb{Z}$ in $\mathcal{O}_k$ for a number field k. As is customary we will denote prime ideals in k by the gothic letters $\mathfrak{p}$ and $\mathfrak{q}$ and rational primes by p and q.

Definition 3.1. *Let k be the splitting field of an irreducible polynomial, and q be a rational prime.*

- *q is inert in $\mathcal{O}_k$ if (q) is a prime ideal in $\mathcal{O}_k$.*
- *If q is not inert then it splits in $\mathcal{O}_k$. Let $(q) = \prod \mathfrak{q}_i^e$ be the prime ideal decomposition of (q). If $e \geq 1$ then q is ramified, otherwise it splits completely.*

The *discriminant* of a number field is an integer valued invariant that controls the factorisation of rational primes in that field. The following result is a special case of a more general result on the splitting of rational primes on number fields, see Theorems 21, 23 and 24 of Marcus' *Number Fields* for details, [11].

Theorem 3.2. *Let k be a number field. If a rational prime q is ramified in $\mathcal{O}_k$, then $q \mid \operatorname{disc}(k)$. Let k be the splitting field of some irreducible polynomial, where the degree of k over $\mathbb{Q}$ is $n = [k : \mathbb{Q}]$. If q is a rational prime such that $q \nmid \operatorname{disc}(k)$, then*

$$(q) = \mathfrak{q}_1 \dots \mathfrak{q}_r,$$

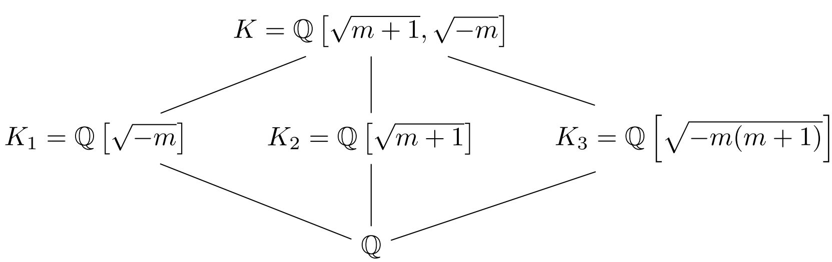
where $r|n$. Furthermore the action of the Galois group on $\{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$ is transitive.

In a quadratic extension of $\mathbb{Q}$, the Legendre symbol controls the splitting of prime ideals.

Theorem 3.3 (p.24, Theorem 25, [11])**.** *Let $k = \mathbb{Q}[\sqrt{d}]$ where d is a square-free integer. Then* $\operatorname{disc}(k) = d$ *if* $d \equiv 1 \mod 4$ *and* $\operatorname{disc}(k) = 4d$ *if* $d \equiv 2, 3 \mod 4$. *Suppose that q is an odd rational prime and* $q \nmid \operatorname{disc}(k)$. *Then*

- *q is inert in $\mathcal{O}_k$ if $(d/q) = -1$.*
- *q splits into distinct prime ideals in $\mathcal{O}_k$ if $(d/q) = 1$.*

We will study these concepts for the field $K = \mathbb{Q}[\alpha]$, which by Proposition 2.1 is the splitting field of $\mathfrak{m}(x)$. Since $2/(\alpha_m + \alpha_m^*) = \sqrt{m+1}$ and $(\sqrt{m+1})\alpha_m - 1 = \sqrt{-m}$ we have isomorphism $\mathbb{Q}[\alpha_m] \simeq \mathbb{Q}[\sqrt{-m}, \sqrt{m+1}]$. There are three intermediate subfields of K, as illustrated.



The lattice of subfields of K.

The discriminant of a biquadratic extension is given as an exercise by Marcus.

Proposition 3.4 (p.36-37, [11]). *The discriminant of a biquadratic extension* $k = \mathbb{Q}[\sqrt{a}, \sqrt{b}]$ *where* $\gcd(a, b) = 1$ *is*

$$\mathrm{disc}(k) = \mathrm{disc}(k_1)\mathrm{disc}(k_2)\mathrm{disc}(k_3),$$

where $k_1 = \mathbb{Q}[\sqrt{a}]$, $k_2 = \mathbb{Q}[\sqrt{b}]$ *and* $k_3 = \mathbb{Q}[\sqrt{ab}]$.

Let $G = \mathrm{Gal}(K/\mathbb{Q})$ be the Galois group the splitting field of $\mathfrak{m}(x)$. By the Galois correspondence G has order 4, and has three distinct subgroups of order 2. So G is elementary abelian, generated by $\sigma : \sqrt{m+1} \mapsto -\sqrt{m+1}$ and $\tau : \sqrt{-m} \mapsto -\sqrt{-m}$. We identify τ with complex conjugation. Note that $K_1 = \mathrm{Fix}(\sigma)$ is the fixed field of σ, that $K_2 = \mathrm{Fix}(\tau)$ is the fixed field of τ and $K_3 = \mathrm{Fix}(\sigma\tau)$ is the fixed field of $\sigma\tau$.

From now on, let $m = p$ be a prime congruent to 3 modulo 4, and write s for the squarefree part of $p + 1$. Then $K \simeq \mathbb{Q}[\sqrt{-p}, \sqrt{s}]$, and applying Proposition 3.4 we have

$$\mathrm{disc}(K) = \begin{cases} s^2p^2 & \text{if } s \equiv 1 \mod 4 \\ 16s^2p^2 & \text{if } s \equiv 2, 3 \mod 4 \end{cases}.$$

Let q be a prime number. By Theorem 3.2, the prime q ramifies in $\mathcal{O}_K$ only if $q = p$ or $q|s$. Next we describe which non-ramified primes split in $\mathcal{O}_K$.

Proposition 3.5. *Let* q *be a rational prime not dividing* $\mathrm{disc}(k)$. *Then one of the following holds:*

- $(q) = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4$ *in* $\mathcal{O}_K$ *and* q *splits in every subfield of* K.
- $(q) = \mathfrak{q}_1\mathfrak{q}_2$ *in* $\mathcal{O}_K$ *and* q *splits in one proper subfield of* K, *being inert in the other two.*

Proof. By Theorem 3.3, the prime q splits in K_1 if and only if $(-p/q) = 1$, and q splits in K_2 if and only if $(s/q) = 1$. Suppose that $(-p/q) = (s/q) = -1$. Then $(-ps/q) = (-p/q)\,(s/q) = 1$, so q splits in K_3. We conclude that no rational prime is inert in K.

Since by assumption q does not ramify, Theorem 3.2 tells us that q splits in $\mathcal{O}_K$ into two or four prime ideals. Suppose that $(q) = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4$. Then up to a relabeling of the primes $\mathfrak{q}_i$ we can assume that

$$\begin{aligned} \mathfrak{q}_1^\sigma &= \mathfrak{q}_2, & \mathfrak{q}_3^\sigma &= \mathfrak{q}_4 \\ \mathfrak{q}_1^\tau &= \mathfrak{q}_3, & \mathfrak{q}_2^\tau &= \mathfrak{q}_4 \\ \mathfrak{q}_1^{\sigma\tau} &= \mathfrak{q}_4, & \mathfrak{q}_2^{\sigma\tau} &= \mathfrak{q}_3 \end{aligned}$$

This implies that $(\mathfrak{q}_1\mathfrak{q}_2)^\sigma = \mathfrak{q}_1\mathfrak{q}_2$ and $(\mathfrak{q}_3\mathfrak{q}_4)^\sigma = \mathfrak{q}_3\mathfrak{q}_4$, therefore $\mathfrak{q}_1\mathfrak{q}_2$ and $\mathfrak{q}_3\mathfrak{q}_4$ are ideals in the fixed field K_1 of σ and thus q splits as $(q) = (\mathfrak{q}_1\mathfrak{q}_2)(\mathfrak{q}_3\mathfrak{q}_4)$ in K_1. We can show analogously that q splits in K_2 and K_3. Suppose next that q splits in K as $\mathfrak{q}_1\mathfrak{q}_2$. Then the Galois group acts as in one of the following possibilities.

$\mathfrak{q}_1^\sigma$	$\mathfrak{q}_1^\tau$	$\mathfrak{q}_1^{\sigma\tau}$	Subfield containing $\mathfrak{q}_1$ and $\mathfrak{q}_2$
$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_2$	$K_1 = \mathrm{Fix}(\sigma)$
$\mathfrak{q}_2$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$K_2 = \mathrm{Fix}(\tau)$
$\mathfrak{q}_2$	$\mathfrak{q}_2$	$\mathfrak{q}_1$	$K_3 = \mathrm{Fix}(\sigma\tau)$

In each case, there is exactly one non-identity element $g \in G$ fixing both $\mathfrak{q}_1$ and $\mathfrak{q}_2$. So q splits in the fixed field of g, and is inert in the other two intermediate subfields. □

In our application to QUH matrices, we will require the following special case of Proposition 3.5.

Corollary 3.6. *Let q be an odd rational prime q, coprime to both p and s. In $\mathcal{O}_K$, we have $(q) = \mathfrak{q}_1\mathfrak{q}_2$ with $\mathfrak{q}_1^\tau = \mathfrak{q}_1$ and $\mathfrak{q}_2^\tau = \mathfrak{q}_2$ if and only if $(-p/q) = -1$ and $(s/q) = 1$.*

Proof. Since $\mathfrak{q}_1^\tau = \mathfrak{q}_1$ it must be the case that $\mathfrak{q}_1^\sigma = \mathfrak{q}_2$ and, by Proposition 3.5, q splits in K_2 as $\mathfrak{q}_1\mathfrak{q}_2$. So by Theorem 3.3, we must have $(s/q) = 1$. Furthermore, (q) must be inert in K_1, from which we obtain $(-p/q) = -1$ as required. The converse follows from Theorem 3.3 and Proposition 3.5. □

Recall that the action of τ on K corresponds to the action of complex conjugation on K. Therefore the case above is equivalent to $(q) = \mathfrak{q}_1\mathfrak{q}_2$ with $\mathfrak{q}_1^* = \mathfrak{q}_1$ and $\mathfrak{q}_2^* = \mathfrak{q}_2$. We can now formulate our main nonexistence theorem.

Theorem 3.7. *Let n be an odd integer, with squarefree part t. Let $p \equiv 3 \mod 4$ be a prime number such that the squarefree part of $p+1$ is $s > 1$. If there exists an odd prime q such that*

- *q divides t,*
- *$(s/q) = 1$, and*
- *$(-p/q) = -1$,*

then $\mathcal{H}(n, X_p)$ is empty.

Proof. Let $M \in \mathcal{H}(n, X_p)$ and set $D = (p+1)^n \det M$. Then $D \in \mathcal{O}_K$, since $(p+1)\alpha \in \mathcal{O}_K$ for every $\alpha \in X_p$. The matrix H is complex Hadamard, therefore $DD^* = (p+1)^{2n} n^n = a^2 t^n$, for some $a \in \mathbb{Z}$. By Corollary 3.6, $(q) = \mathfrak{q}_1 \mathfrak{q}_2$ in $\mathcal{O}_K$ with $\mathfrak{q}_1 = \mathfrak{q}_1^*$. We have that $q|t$, so since n is odd the prime ideal $\mathfrak{q}_1$ appears with odd multiplicity in the decomposition of $(p+1)^{2n} n^n$ in $\mathcal{O}_K$. Since $\mathfrak{q}_1$ is prime and divides the product $(D)(D^*)$, it divides one of the factors; without loss of generality, suppose that $\mathfrak{q}_1$ divides (D). So (D) factors into prime ideals uniquely as

$$(D) = \mathfrak{q}_1^{\ell} \prod_j \mathfrak{p}_j^{\ell_j},$$

Then $(D^*) = (D)^* = \mathfrak{q}_1^{\ell} \prod_j (\mathfrak{p}_j^*)^{\ell_j}$. But implies that $\mathfrak{q}_1$ appears with even multiplicity in $(D)(D^*)$, contradicting its odd multiplicity in $(p+1)^{2n} n^n$. □

The only prime of the form $n^2 - 1$ is 3. In this case the matrices $\mathcal{H}(n, X_3)$ coincide with the unreal $BH(n, 6)$ matrices of Compton, Craigen and de Launey. The set $\mathcal{H}(n, X_3)$ is empty if there exists a prime $q \equiv 5 \pmod 6$ which divides the square-free part of n (see Theorem 2 of [2] or Theorem 5 of [15] for a proof).

We conclude this paper by discussing some consequences of Theorem 3.7. Suppose first that $p = 7$. Then a prime q satisfying both $(q/7) = -1$ and $(2/q) = 1$ cannot divide the square-free part of n. By quadratic reciprocity, these are the primes which satisfy both $q \equiv 3, 5, 6 \mod 7$ and $q \equiv 1, 7 \mod 8$. By Dirichlet's Theorem on primes in arithmetic progressions, there are infinitely many such primes. Similar results hold for each prime p, as illustrated in the table below.

p	n
7	$17, 31, 41, 47, 51, 73, 85, 89, 93, 97, 103, 119, 123, 141, \ldots$
11	$13, 39, 61, 65, 73, 83, 91, 107, 109, 117, 131, 143, 167, \ldots$
19	$29, 31, 41, 59, 71, 79, 87, 89, 93, 109, 123, 145, 151, \ldots$
23	$5, 15, 19, 35, 43, 45, 53, 55, 57, 65, 67, 85, 95, 97, 105, \ldots$
31	$17, 23, 51, 69, 73, 79, 85, 89, 115, 119, 127, 137, 151, \ldots$
43	$5, 7, 15, 19, 21, 35, 37, 45, 55, 57, 63, 65, 77, 85, 89, 91, \ldots$

Pairs (n, p) such that $\mathcal{H}(n, p)$ is empty.

In fact, it is a consequence of the Chebotarev Density Theorem that the proportion of primes $q \leq N$ to which the conditions of Theorem 3.7 apply tends to 1/4 as N tends to infinity. In particular, there are infinitely many primes which obstruct the existence of matrices in $\mathcal{H}(n, X_p)$ for any fixed p.

To illustrate Theorem 3.7 in a case where not all ideals are principal, we consider $p = 43$ and $q = 5$, then $s = 11$. We have $(5/43) = -1$, thus the prime 5 should be inert in $\mathcal{O}_{K_1}$. By Proposition 3.5, (5) splits in $\mathcal{O}_K$ as the product of two prime ideals in $\mathcal{O}_{K_2}$, indeed $(5) = (5, 1+\sqrt{11})(5, 1-\sqrt{11})$ in $\mathcal{O}_K$. If there exists $H \in \mathcal{H}(5, X_{43})$ then $D = 11^5 \det H$ and $DD^* = 11^{10} \cdot 5^5$. Thus in $\mathcal{O}_K$ this means

$$(D)(D)^* = (11^5)^2(5, 1 + \sqrt{11})^5(5, 1 - \sqrt{11})^5.$$

The ideal $(5, 1+\sqrt{11}) = (5, 1+\sqrt{11})^*$ appears with even multiplicity on the left hand side and odd multiplicity on the right hand side. Hence $\mathcal{H}(5, X_{43})$ is empty.

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