Uniformly resolvable decompositions of $K_n$ into 1-factors and $P_k$-factors

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Abstract

Let $P_n$ and $K_n$ respectively denote a path and a complete graph on $n$ vertices. In this paper, it is shown that the necessary conditions are sufficient for the existence of uniformly resolvable decomposition of $K_n$ into $r$ parallel classes containing $K_2$-factors and $s$ parallel classes containing $P_k$-factors for any even $k \geq 4$ and $r, s \geq 0$.

1 Introduction

In this paper, the vertex set and edge set of graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. Let $P_n$, $K_n$ and $I_n$ respectively denote a path, a
Uniformly resolvable decompositions

A complete graph and an independent set on \( n \) vertices. Given a collection of graphs \( \mathcal{H} \), an \( \mathcal{H} \)-decomposition of a graph \( G \) is a set of subgraphs (blocks) of \( G \) whose edge sets partition \( E(G) \), and each subgraph is isomorphic to a graph from \( \mathcal{H} \). A parallel class of a graph \( G \) is a set of subgraphs whose vertex sets partition \( V(G) \). A parallel class is called uniform if each block of the parallel class is isomorphic to the same graph. An \( \mathcal{H} \)-decomposition of a graph \( G \) is called (uniformly) resolvable if the blocks can be partitioned into (uniform) parallel classes. A resolvable \( \mathcal{H} \)-decomposition of \( G \) is also referred as \( \mathcal{H} \)-factorization of \( G \). We write \( G = H_1 \oplus H_2 \oplus \ldots \oplus H_k \), if \( H_1, H_2, \ldots, H_k \) are edge-disjoint subgraphs of \( G \) and \( E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_k) \).

For two graphs \( G \) and \( H \) their wreath product \( G \otimes H \) has the vertex set \( V(G) \times V(H) \) and their edge set \( E(G \otimes H) = \{(v, w) \mid (g, g') \in E(G) \text{ or } g = g' \text{, and } (h, h') \in E(H) \} \). An \( r \)-factor of \( G \) is an \( r \)-regular spanning subgraph of \( G \). A near 1-factor of \( G \) is a 1-regular subgraph which contains all but one vertex of \( G \). Let \( K_{k,k} \) be the complete bipartite graph with bipartition \( (X,Y) \), where \( X = Y = \{0, 1, \ldots, k-1\} \). The 1-factor of distance \( t \) consists of the edges \( \{(i, i+t) : 0 \leq i \leq k-1\} \), where the addition is taken modulo \( k \).

Rees \[16\], obtained the necessary and sufficient conditions for the existence of uniformly resolvable \((K_2, K_3)\)-designs of order \( n \). Horton \[10\], has proved the existence of resolvable \( P_k \)-designs of order \( n \) for \( k = 3 \) and Bermond et.al \[2\], have proved it for \( k = 4 \). Many other results on uniformly resolvable decomposition of \( K_n \) into distinct subgraphs have been obtained in \[4, 3, 13, 17, 5, 8, 15, 11, 12\]. Recently \[6, 7\] Mario Gionfriddo and Salvatore Milici have investigated the existence of uniformly resolvable \( \mathcal{H} \)-designs with \( \mathcal{H} = \{P_3, P_4\} \) and \( \{K_2, P_k\} \) for \( k = 3, 4 \).

- We denote the existence of uniformly resolvable decomposition of \( G \) into \( r \) parallel classes consisting of \( K_2 \)-factors and \( s \) parallel classes consisting of \( P_k \)-factors by \((K_2, P_k)\-URD(G; r, s)\).

- Let \( I_1(n) \) (resp., \( I_2(n) \)) denote the set of possible pairs \((r, s)\) for which \((K_2, P_k)\-URD(K_n; r, s)\) exists when \( k \) is even (resp., \( k \) is odd).

For all even \( k \geq 4 \) and \( n \equiv 0 \pmod{k} \), if \( n \equiv 0 \pmod{k(k-1)} \) we define

\[
I_1(n) = \left\{ (n - 1 - (k-1)x, \frac{k}{2}x) : x = 0, 1, \ldots, \frac{n}{k(k-1)} \right\} \quad (1)
\]
and if \( n \equiv a \pmod{k(k-1)} \), when \( 0 \leq a \equiv 0 \pmod{k} \leq k(k-2) \), we define

\[
I_1(n) = \left\{ (n - 1 - (k-1)x, \frac{k}{2}x) : x = 0, 1, \ldots, \frac{n-a}{k} \right\}. \tag{2}
\]

For all odd \( k \geq 3 \) and \( n \equiv 0 \pmod{2k} \), if \( n \equiv 0 \pmod{2k(k-1)} \) we define

\[
I_2(n) = \left\{ ((n-1)-2(k-1)x, kx) : x = 0, 1, \ldots, \frac{n-2(k-1)}{2(k-1)} \right\}. \tag{3}
\]

and if \( n \equiv a \pmod{2k(k-1)} \), when \( 0 \leq a \equiv 0 \pmod{k} \leq 2k(k-2) \), we define

\[
I_2(n) = \left\{ ((n-1)-2(k-1)x, kx) : x = 0, 1, \ldots, \frac{n-a}{2(k-1)} \right\}. \tag{4}
\]

In this paper, we prove that the necessary conditions are sufficient for the existence of \((K_2, P_k)-URD(K_n; r, s)\) for all even \( k \geq 4 \). Further, we give necessary conditions for the existence of \((K_2, P_k)-URD(K_n; r, s)\) for all odd \( k \geq 3 \).

## 2 Preliminary results

In this section, we present some known results required to prove our main results.

**Theorem 2.1.** ([1] Walecki’s Construction).

1. For all odd \( n \geq 3 \), the graph \( K_n \) has a Hamilton cycle decomposition.
2. For all even \( n \geq 4 \), the graph \( K_n - I \) has a Hamilton cycle decomposition with prescribed cycles \( \{C, \sigma(C), \sigma^2(C), \ldots, \sigma^{\frac{n-4}{2}}(C)\} \). where \( \sigma = (0)(12\ldots n-1) \) is a permutation, \( C = (01\ldots n-1) \) is a Hamilton cycle and \( I = \left\{ (0, \frac{n}{2}), (i, n-i) : 1 \leq i \leq \frac{n}{2} - 1 \right\} \) is a 1-factor of \( K_n \).

**Theorem 2.2.** [14, 9]

1. There exist a 1-factorization (resp., a near 1-factorization) of \( K_n \) if and only if \( n \) is even (resp., \( n \) is odd).
Every regular bipartite graph is 1-factorable.

**Theorem 2.3.** [18] For all even \( k \), the graph \( K_n \) has a \( P_k \)-factorization if and only if \( n \equiv k \pmod{k(k-1)} \).

**Lemma 2.1.** [18, 19] If \( k \) is even, then the graph \( K_{k,k} \) can be decomposed into one 1-factor and \( \frac{k}{2} P_k \)-factors.

### 3 Necessary conditions

In this section, we give necessary conditions for the existence of \( (K_2, P_k) \)-URD\( (K_n; r, s) \) for all \( k \geq 3 \).

**Lemma 3.1.** For all even \( k \geq 4 \), if \( (K_2, P_k) \)-URD\( (K_n; r, s) \) exists, then \( n \equiv 0 \pmod{k} \) and \( (r, s) \in I_1(n) \).

**Proof.** The condition \( n \equiv 0 \pmod{k} \) is trivial. Let \( \mathcal{D} \) be an arbitrary \( (K_2, P_k) \)-URD\( (K_n; r, s) \). By resolvability, we have

\[
\frac{r}{2} n + s \frac{n}{k} (k-1) = \frac{n(n-1)}{2}
\]

Hence

\[
rk + 2s(k-1) = k(n-1) \quad (5)
\]

Now (5) gives

\[
rk \equiv k(n-1) \pmod{2(k-1)} \quad \text{and} \quad 2s(k-1) \equiv k(n-1) \pmod{k} \quad (6)
\]

If \( k \) is even, then (6) implies the following:

Now letting \( s = \frac{k}{2} x \), Equation (5) gives \( r = (n-1) - (k-1)x \). Since \( r \) and \( s \) cannot be negative, and \( x \) is an integer, the value of \( x \) must be in the range for \( I_1(n) \). (See Equations 1 and 2.)

**Lemma 3.2.** For all odd \( k \geq 3 \), if \( (K_2, P_k) \)-URD\( (K_n; r, s) \) exists, then \( n \equiv 0 \pmod{2k} \) and \( (r, s) \in I_2(n) \).

**Proof.** The condition \( n \equiv 0 \pmod{2k} \) is trivial. Let \( \mathcal{D} \) be an arbitrary \( (K_2, P_k) \)-URD\( (K_n; r, s) \). By resolvability, we have

\[
\frac{r}{2} n + s \frac{n}{k} (k-1) = \frac{n(n-1)}{2}
\]
Table 1: For even $k$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$s$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(k-2) \pmod{(k-1)}$</td>
<td>$0 \pmod{\frac{k}{2}}$</td>
<td>$0 \pmod{k(k-1)}$</td>
</tr>
<tr>
<td>$0 \pmod{(k-1)}$</td>
<td>$0 \pmod{\frac{k}{2}}$</td>
<td>$k \pmod{k(k-1)}$</td>
</tr>
<tr>
<td>$1 \pmod{(k-1)}$</td>
<td>$0 \pmod{\frac{k}{2}}$</td>
<td>$2k \pmod{k(k-1)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(k-3) \pmod{(k-1)}$</td>
<td>$0 \pmod{\frac{k}{2}}$</td>
<td>$(k-2)k \pmod{k(k-1)}$</td>
</tr>
</tbody>
</table>

Hence

$$rk + 2s(k-1) = k(n-1)$$  \hspace{1cm} (7)

Now (7) gives

$$rk \equiv k(n-1) \pmod{2(k-1)} \text{ and } 2s(k-1) \equiv k(n-1) \pmod{k}$$  \hspace{1cm} (8)

If $k$ is odd, then (8) implies the following:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$s$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2k-3) \pmod{2(k-1)}$</td>
<td>$0 \pmod{k}$</td>
<td>$0 \pmod{2k(k-1)}$</td>
</tr>
<tr>
<td>$1 \pmod{2(k-1)}$</td>
<td>$0 \pmod{k}$</td>
<td>$2k \pmod{2k(k-1)}$</td>
</tr>
<tr>
<td>$3 \pmod{2(k-1)}$</td>
<td>$0 \pmod{k}$</td>
<td>$4k \pmod{2k(k-1)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(2k-5) \pmod{2(k-1)}$</td>
<td>$0 \pmod{k}$</td>
<td>$2k(k-2) \pmod{2k(k-1)}$</td>
</tr>
</tbody>
</table>

Now letting $s = kx$, Equation (7) gives $r = (n-1) - 2(k-1)x$. Since $r$ and $s$ cannot be negative, and $x$ is an integer, the value of $x$ must be in the range for $I_2(n)$. (See Equations 3 and 4.)

4 Base construction

We present some definitions and results which are required to prove our main result.
Definition 4.1. For each Hamilton cycle $h_i$ of $K_l$, we define $N_i$ to be the graph with vertex set $V(N_i) = V(K_l \times I_k)$ and edge set $E(N_i)$, where

- $V(K_l) = \{ x : 0 \leq x \leq l - 1 \}$,
- $V(I_k) = \{ j : 1 \leq j \leq k \}$ and
- $E(N_i) = \{ (x, j), (y, j + 1) : (x, y) \in E(h_i), 1 \leq j \leq k \}$.

(Addition taken modulo $k$, i.e., $1, 2, \ldots, k$). See Figure 1.

![Figure 1](image)

Figure 1: The graph $N_i$, $i = 1, 2$.

Definition 4.2. Let $M$ be a graph with $V(M) = V(N_i)$ and the edge set

$$E(M) = \begin{cases} \bigcup_{i=1}^{i=\frac{l-1}{2}} E(N_i), & \text{when } l \text{ is odd} \\ \bigcup_{i=1}^{i=\frac{l-2}{2}} E(N_i) \cup F, & \text{when } l \text{ is even} \end{cases}$$

where $F$ is a 1-factor of $M$ (which correspond to the 1-factor of $K_l$) (see Figure 2) as follows: $F = \{ ((0, a), (\frac{l}{2}, a + 1)), ((i, a), (l - i, a + 1)) | 0 \leq a \leq k - 1, 1 \leq i \leq \frac{l}{2} - 1 \}$.

Remark. Clearly the graph $M$ defined in Definition 4.2 has an $N$-decomposition, $N_i \cong N$. 
Definition 4.3. Let \((r_1, s_1)\) and \((r_2, s_2)\) be two pairs of non-negative integers. Then we define \((r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)\). Usually positive integers are denoted as \(\mathbb{Z}_+\). If \(A = \{(r_1, s_1) \mid r_1, s_1 \in \mathbb{Z}_+\}\); \(B = \{(r_2, s_2) \mid r_2, s_2 \in \mathbb{Z}_+\}\) and \(h \in \mathbb{Z}_+\), then \(A + B = \{(r_1, s_1) + (r_2, s_2) \mid (r_1, s_1) \in A, (r_2, s_2) \in B\}\) and \(h \ast A\) denotes the set of all pairs of non-negative integers which can be obtained by adding any \(h\) elements of \(A\) together (repetitions of elements of \(A\) are allowed).

Now, let us define the following subgraphs in \(M\) for our convenience as follows:

\[
P = \bigcup_{i=1}^{\frac{k}{2}} N_i = \frac{k}{2} N \quad \text{and} \quad Q = \bigcup_{i=1}^{k-1} N_i = (k-1)N
\]

Lemma 4.1. For all even \(k \geq 4\), there exists a \((K_2, P_k)\)-URD\((N; r, s)\) with \((r, s) = (2, 0)\).

Proof. For any \(i\), \(0 \leq i \leq \frac{k-2}{2}\), we define subsets of \(V(N)\) as follows:
\[
X_i^1 = \{(x, 2i) \mid 0 \leq x \leq l-1\}, \quad X_i^2 = \{(x, 2i + 1) \mid 0 \leq x \leq l-1\}, \quad Y_i^1 = \{(x, 2i + 1) \mid 0 \leq x \leq l-1\} \quad \text{and} \quad Y_i^2 = \{(x, 2i + 2) \mid 0 \leq x \leq l-1\},
\]
where the addition is taken modulo \(k\). Then the edges between the vertex sets

\[
\text{(a) When } \ell \text{ is odd} \quad \text{(b) When } \ell \text{ is even}
\]

Figure 2: The graph \(M\)
Lemma 4.2. For all even $k \geq 4$, there exists a $(K_2, P_k)$-URD($P; r, s$) with each $(r, s) \in \{(1, \frac{k}{2}), (k, 0)\}$. 

Proof. We prove in two cases.

Case 1. $(1, \frac{k}{2})$.
We first construct one $P_k$-factor from each $N_j$, $1 \leq j \leq \frac{k}{2}$ as follows: For any fixed $j$, $1 \leq j \leq \frac{k}{2}$, we define the subsets of $V(N_j)$ as $X^j = \{(x, 2(j - 1)) | 0 \leq x \leq l - 1\}$ and $Y^j = \{(x, 2(j - 1) + 1) | 0 \leq x \leq l - 1\}$, where the addition is taken modulo $k$. Now keep the edges between the subsets $X^j$ and $Y^j$ for future purpose. The remaining graph will form one $P_k$-factor in $N_j$. By repeating the process for each $N_j$, we obtain $\frac{k}{2} P_k$-factors in $P$. Now the edges between the sets $X^j$ and $Y^j$ from each $N_j$ together gives one 1-factor in $P$. Therefore, we get the required uniform resolvable decomposition.

Case 2. $(k, 0)$.
Each $N_j$, $1 \leq j \leq \frac{k}{2}$ can be decomposed into two 1-factors, by Lemma 4.1. Hence, we obtain the required resolvable decomposition of $P$. 

Lemma 4.3. For all even $k \geq 4$, there exists a $(K_2, P_k)$-URD($Q; r, s$) with each $(r, s) \in \{(2(k - 1), 0), (k - 1, \frac{k}{2}), (0, k)\}$. 

Proof. We prove in three cases.

Case 1. $(2(k - 1), 0)$.
Clearly the graph $Q = (k - 1)N$ has a $2(k - 1)$ 1-factors, by Lemma 4.1.

Case 2. $(k - 1, \frac{k}{2})$.
Take $Q = (k - 1)N = (\frac{k-2}{2}) N + (\frac{k}{2}) N = X + Y$. By Lemmas 4.1 and 4.2, the graphs $X$ and $Y$ have $(k-2)$ 1-factors and one 1-factor and $\frac{k}{2} P_k$-factors respectively. Hence, we obtain $(k - 1)$ 1-factors and $\frac{k}{2}$ $P_k$-factors in $Q$.

Case 3. $(0, k)$.
We first construct one $P_k$-factor from each $N_j$, $1 \leq j \leq k - 1$ as follows: For any fixed $j$, $1 \leq j \leq k - 1$, we define the subsets of $V(N_j)$ as $X^j = \{(x, j - 1) | 0 \leq x \leq l - 1\}$ and $Y^j = \{(x, j) | 0 \leq x \leq l - 1\}$. Now keep the edges between the subsets $X^j$ and $Y^j$ for future purpose. The remaining
The order (number of vertices) of the graph \( M \) (defined in Definition 4.2) be denoted as \( \Theta \). For all even \( k \geq 4 \) and \( \Theta \equiv 0 \) (mod \( k \)), if \( \Theta \equiv 0 \) (mod \( k(k-1) \)), we define

\[
I(\Theta) = \left\{ \left( l - 1 - (k-1)x, \frac{k}{2}x \right) : x = 0, 1, \ldots, \frac{l - (k-1)}{(k-1)} \right\}
\]

(9)

and if \( \Theta \equiv a \) (mod \( k(k-1) \)), when \( 0 < a \equiv 0 \) (mod \( k \)) \leq k(k-2), we define

\[
I(\Theta) = \left\{ \left( l - 1 - (k-1)x, \frac{k}{2}x \right) : x = 0, 1, \ldots, \frac{l - a}{k(k-1)} \right\}.
\]

(10)

**Lemma 4.4.** For all even \( k \geq 4 \), if \((K_2, P_k)-URD(M; r, s)\) exists, then \( \Theta \equiv 0 \) (mod \( k \)) and \((r, s) \in I(\Theta)\).

**Proof.** The condition \( \Theta \equiv 0 \) (mod \( k \)) is trivial and hence \( \Theta = kl, l \in \mathbb{Z}_+ \).

Let \( D \) be an arbitrary \((K_2, P_k)-URD(M; r, s)\). By resolvability, we have

\[
\frac{kl}{2} + s\frac{kl}{k} = \frac{kl(l - 1)}{2}
\]

Hence

\[
rt + 2sk(l - 1) = k(l - 1)
\]

(11)

Letting \( s = \frac{k}{2}x \), Equation (11) gives \( r = l - 1 - (k-1)x \). Since \( r \) and \( s \) cannot be negative, and \( x \) is an integer, the value of \( x \) must be in the range for \( I(\Theta) \). (See Equations 9 and 10.)

**Lemma 4.5.** For any \( \Theta \equiv 0 \) (mod \( 4 \)), there exists \((K_2, P_4)-URD(M; r, s)\).

**Proof.** Let \( \Theta \equiv 0 \) (mod \( 4 \)), we have a \( \Theta \equiv a \) (mod \( 12 \)) with \( a = 0, 4, 8 \). We prove in three cases.

**Case 1.** For \( \Theta \equiv 0 \) (mod \( 12 \)), we have a \( \Theta = 12x = 4(3x) \), where \( x \geq 1 \).

**Subcase 1.** If \( x \) is odd, then the graph

\[
M = \left( \frac{3x - 1}{2} \right)N = \left( \frac{3x - 3}{2} \right)N \cup N = \left( \frac{x - 1}{2} \right)Q \cup N.
\]
Hence, by Lemmas 4.1 and 4.3, we get the required URDs.

**Subcase 2.** If \( x \) is even, then the graph
\[
M = \left(\frac{3x-2}{2}\right)N \cup F = \left(\frac{3x-6}{2}\right)N \cup 2N \cup F = \left(\frac{x-2}{2}\right)Q \cup P \cup F.
\]
Hence, by Lemmas 4.2 and 4.3 along with \( F \), we get the required URDs.

**Case 2.** For \( \Theta \equiv 4 \pmod{12} \), we have \( \Theta = 12x + 4 = 4(3x + 1) \), where \( x \geq 0 \).

**Subcase 1.** If \( x \) is odd, then the graph
\[
M = \left(\frac{3x-1}{2}\right)N \cup F = \left(\frac{3x-3}{2}\right)N \cup N \cup F = \left(\frac{x-1}{2}\right)Q \cup N \cup F.
\]
Hence, by Lemmas 4.1 and 4.3 along with \( F \), we get the required URDs.

**Subcase 2.** If \( x \) is even, then the graph
\[
M = \left(\frac{3x}{2}\right)N = \left(\frac{x}{2}\right)Q.
\]
Hence, by Lemma 4.3, we get the required URDs.

**Case 3.** For \( \Theta \equiv 8 \pmod{12} \), we have \( \Theta = 12x + 8 = 4(3x + 2) \), where \( x \geq 0 \).

**Subcase 1.** If \( x \) is odd, then the graph
\[
M = \left(\frac{3x+1}{2}\right)N = \left(\frac{3x-3}{2}\right)N \cup 2N = \left(\frac{x-1}{2}\right)Q \cup P.
\]
Hence, by Lemmas 4.2 and 4.3, we get the required URDs.

**Subcase 2.** If \( x \) is even, then the graph
\[
M = \left(\frac{3x}{2}\right)N \cup F = \left(\frac{x}{2}\right)Q \cup F.
\]
Hence, by Lemma 4.3 along with \( F \), we get the required URDs.

**Lemma 4.6.** For even \( k \geq 6 \) and \( \Theta \equiv 0 \pmod{k} \), \((K_2, P_k)\)-URD\((M; r, s)\) exists.

**Proof.** Let \( \Theta \equiv 0 \pmod{k} \), we have a \( \Theta \equiv a \pmod{k(k-1)} \) with \( 0 \leq a \equiv 0 \pmod{k} \leq k(k-2) \). We prove in six cases.
Case 1. For $\Theta \equiv 0 \pmod{k(k-1)}$, we have a $\Theta = k(k-1)x$, where $x \geq 1$.

Subcase 1. If $x$ is odd, then the graph

$$M = \left(\frac{(k-1)x-1}{2}\right)N = \left(\frac{(k-1)x-(k-2)-1}{2}\right)N \cup \left(\frac{k-2}{2}\right)N$$

$$= \left(\frac{x-1}{2}\right)Q \cup \left(\frac{k-2}{2}\right)N.$$

Hence, by Lemmas 4.1 and 4.3, we get the required URDs.

Subcase 2. If $x$ is even, then the graph

$$M = \left(\frac{(k-1)x-2}{2}\right)N \cup F = \left(\frac{(k-1)(x-2)}{2}\right)N \cup (k-2)N \cup F$$

$$= \left(\frac{x}{2}\right)Q \cup P \cup N \cup F.$$

Hence, by Lemmas 4.1 to 4.3 along with $F$, we get the required URDs.

Case 2. For $\Theta \equiv k \pmod{k(k-1)}$, we have a $\Theta = k(k-1)x + k = k((k-1)x + 1)$, where $x \geq 0$.

Subcase 1. If $x$ is odd, then the graph

$$M = \left(\frac{(k-1)x}{2}\right)N \cup F = \left(\frac{(k-2)(x-1)-1}{2}\right)N \cup (k-2)N \cup F$$

$$= \left(\frac{x-1}{2}\right)Q \cup \left(\frac{k-2}{2}\right)N \cup F.$$

Hence, by Lemmas 4.1 and 4.3 along with $F$, we get the required URDs.

Subcase 2. If $x$ is even, then the graph

$$M = \left(\frac{(k-1)x}{2}\right)N = \left(\frac{x}{2}\right)Q.$$

Hence, by Lemma 4.3, we get the required URDs.

Case 3. For $\Theta \equiv 2k \pmod{k(k-1)}$, we have a $\Theta = k(k-1)x + 2k = k((k-1)x + 2)$, where $x \geq 0$.

Subcase 1. If $x$ is odd, then the graph

$$M = \left(\frac{(k-1)x+1}{2}\right)N = \left(\frac{(k-1)x+1-k}{2}\right)N \cup \left(\frac{k}{2}\right)N$$

$$= \left(\frac{x-1}{2}\right)Q \cup P.$$
Hence, by Lemmas 4.2 and 4.3, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$M = \left(\frac{(k-1)x}{2}\right)N \cup F = \left(\frac{x}{2}\right)Q \cup F.$$ 

Hence, by Lemma 4.3 along with $F$, we get the required URDs.

**Case 4.** For $\Theta \equiv 3k \ (\text{mod} \ k(k-1))$, we have a $\Theta = k(k-1)x + 3k = k((k-1)x+3)$, where $x \geq 0$.

**Subcase 1.** If $x$ is odd, then the graph

$$M = \left(\frac{(k-1)x+1}{2}\right)N \cup F = \left(\frac{(k-1)x+1-k}{2}\right)N \cup \left(\frac{k}{2}\right)N \cup F$$

$$= \left(\frac{x-1}{2}\right)Q \cup P \cup F.$$ 

Hence, by Lemmas 4.2 and 4.3 along with $F$, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$M = \left(\frac{(k-1)x+2}{2}\right)N = \left(\frac{(k-1)x+1}{2}\right)N \cup N$$

$$= \left(\frac{x}{2}\right)Q \cup N.$$ 

Hence, by Lemmas 4.1 and 4.3, we get the required URDs.

**Case 5.** For $\Theta \equiv a \ (\text{mod} \ k(k-1))$ with $3k < a \equiv 0 \ (\text{mod} \ k) < k(k-2)$, we have a $\Theta = k(k-1)y + a = k(k-1)y + kx = k((k-1)y + x)$, where $y \geq 0$ and $4 \leq x \leq k-3$.

**Subcase 1.** Let $x = 2z + 2$, where $1 \leq z \leq \frac{k-6}{2}$ and even $y \geq 0$, then the graph

$$M = \left(\frac{(k-1)y+x-2}{2}\right)N \cup F = \left(\frac{(k-1)y+2z}{2}\right)N \cup F$$

$$= \left(\frac{(k-1)y}{2}\right)N \cup zN \cup F$$

$$= \left(\frac{y}{2}\right)Q \cup zN \cup F.$$ 

Hence, by Lemmas 4.1 and 4.3 along with $F$, we get the required URDs.
Subcase 2. Let \( x = 2z + 2 \), where \( 1 \leq z \leq \frac{k-6}{2} \) and odd \( y \geq 1 \), then the graph
\[
M = \left( \frac{(k-1)y + x - 1}{2} \right) N = \left( \frac{(k-1)y + 2z + 1}{2} \right) N = \left( \frac{(k-1)y - k + 1}{2} \right) N \cup \left( \frac{k}{2} \right) N \cup zN
\]
\[
= \left( \frac{y-1}{2} \right) Q \cup P \cup zN.
\]
Hence, by Lemmas 4.1 to 4.3, we get the required URDs.

Subcase 3. Let \( x = 2z + 3 \), where \( 1 \leq z \leq \frac{k-6}{2} \) and odd \( y \geq 1 \), then the graph
\[
M = \left( \frac{(k-1)y + x - 2}{2} \right) N \cup F = \left( \frac{(k-1)y + 2z + 1}{2} \right) N \cup F
\]
\[
= \left( \frac{(k-1)y - k + 1}{2} \right) N \cup \left( \frac{k}{2} \right) N \cup zN \cup F
\]
\[
= \left( \frac{y-1}{2} \right) Q \cup P \cup zN \cup F.
\]
Hence, by Lemmas 4.1 to 4.3 along with \( F \), we get the required URDs.

Subcase 4. Let \( x = 2z + 3 \), where \( 1 \leq z \leq \frac{k-6}{2} \) and even \( y \geq 0 \), then the graph
\[
M = \left( \frac{(k-1)y + x - 1}{2} \right) N = \left( \frac{(k-1)y + 2z + 2}{2} \right) N
\]
\[
= \left( \frac{(k-1)y}{2} \right) N \cup (z + 1)N
\]
\[
= \left( \frac{y}{2} \right) Q \cup (z + 1)N.
\]
Hence, by Lemmas 4.1 and 4.3, we get the required URDs.

Case 6. For \( \Theta \equiv k(k-2) \) (mod \( k(k-1) \)), we have a \( \Theta = k(k-1)x + k(k-2) = k((k-1)x + (k-2)) \), where \( x \geq 0 \).

Subcase 1. If \( x \) is odd, then the graph
\[
M = \left( \frac{(k-1)x + k - 3}{2} \right) N = \left( \frac{(x-1) - k + 1}{2} \right) N \cup (k-2)N
\]
\[
= \left( \frac{x-1}{2} \right) Q \cup \left( \frac{k}{2} \right) N \cup \left( \frac{k-4}{2} \right) N
\]
\[
= \left( \frac{x-1}{2} \right) Q \cup P \cup \left( \frac{k-4}{2} \right) N.
\]
Hence, by Lemmas 4.1 to 4.3, we get the required URDs.

**Subcase 2.** If \(x\) is even, then the graph
\[
M = \left(\frac{(k-1)x+k-4}{2}\right)N \cup F = \left(\frac{(k-1)x}{2}\right)N \cup \left(\frac{k-4}{2}\right)N \cup F
\]

\[
= \left(\frac{x}{2}\right)Q \cup \left(\frac{k-4}{2}\right)N \cup F.
\]

Hence, by Lemmas 4.1 and 4.3 along with \(F\), we get the required URDs.

**Theorem 4.1.** For all even \(k \geq 4\), if \((K_2, P_k)\)-URD \((M; r, s)\) if and only if \(\Theta \equiv 0 \mod k\) and \((r, s) \in I(\Theta)\).

**Proof.** Follows from Lemmas 4.4 to 4.6.

\[\square\]

## 5 Sufficient conditions

In this section, we prove that the necessary conditions are sufficient for the existence of uniformly resolvable decomposition of \(K_n\) into \(r\) parallel classes containing \(K_2\)-factors and \(s\) parallel classes containing \(P_k\)-factors for any even \(k \geq 4\) and \(r, s \geq 0\).

**Lemma 5.1.** For all even \(k \geq 4\) and \(n \equiv 0 \mod k\), there exists \((K_2, P_k)\)-URD \((M; r, s)\).

**Proof.** As \(n \equiv 0 \mod k\), let \(n = kl, l \in \mathbb{Z}_+\).

**Case 1.** \(l\) is odd. For \(l = 1\), there exists a required uniform resolvable decomposition, by Theorems 2.2 and 2.3. For \(l \geq 3\), let \(V(K_{kl}) = \bigcup_{x=0}^{l-1} A_x\), where \(A_x = \{(x, kx+i) : 0 \leq i \leq k-1\} \ mod k\). We obtain a new graph \(A\) from \(K_{kl}\), by identifying each \(A_x\) with a single vertex \(a_x\) and joint \(a_x\) and \(a_y\) if there exists a complete bipartite graph \(K_{|A_x|,|A_y|}\) between \(A_x\) and \(A_y\) in \(K_{kl}\). Then the new graph \(A \cong K_l\). By Theorem 2.2, the graph \(K_l\) has \(l\) near 1-factors say \(F_x, 0 \leq x \leq l-1\) with the missing vertex \(x\). Corresponding to each \(F_x\) with a missing vertex \(x\) of \(K_l\), we have a \(\left(\frac{l-1}{2}\right)K_{k,k}\) in \(K_{kl}\) and corresponding to \(A_x\) in \(K_{kl}\), we have a \(K_{|A_x|}\) in \(K_k\). By Theorem 2.2, the graphs \(K_{k,k}, K_k\) have \(k, (k-1)\) 1-factors, respectively and by Lemma 2.1, the graph \(K_{k,k}\) has a one 1-factor and \(\frac{k}{2}\) \(P_k\)-factors. Also by Theorem 2.3, the graph \(K_k\) has a \(\frac{k}{2}\) \(P_k\)-factors.
First we use \((k-1)\) 1-factors corresponding to each \(F_x\) from each \(K_{k,k}\) and \(K_k\) to get \((r,s) = (k-1,0)\). Finally, we are left with a 1-factor in each \(K_{k,k}\) and \(k\) isolated vertices in each \(K_k\). Similarly when we use \(\frac{k}{2}\) \(P_k\)-factors we get \((r,s) = (0,\frac{k}{2})\). By repeating this process for all \(l\) near 1-factors of \(K_l\), we obtain \((r,s) \in l \ast \{(k-1,0),\left(0,\frac{k}{2}\right)\}\) and a new graph \(M\) (defined in Definition 4.2) in which there is only one 1-factor between each pair of \(A_x\) and \(A_y\) in \(K_{kl}\). By Theorem 4.1, the graph \(M\) has a \((K_2, P_k)-URD(r, s)\) with \((r,s) \in I(\Theta)\). Therefore, it is easy to see that

\[
I_1(n) \subseteq l \ast \{(k-1,0),\left(0,\frac{k}{2}\right)\} + I(\Theta).
\]

**Case 2.** \(l\) is even. For \(l = 2\), we have \(K_{2k} \equiv K_{k,k} \oplus 2K_k\). Applying Theorems 2.2 and 2.3 and Lemma 2.1, it is easy to obtain \(\{(k,0),\left(1,\frac{k}{2}\right)\}\) + \(\{(k-1,0),\left(0,\frac{k}{2}\right)\}\) \(\supseteq I_1(2k)\). For \(l \geq 4\), we have

\[
K_{kl} \equiv (K_l \otimes I_k) \oplus l \ K_k
= ((F_0 \oplus F_1 \oplus \cdots \oplus F_{l-2}) \otimes I_k) \oplus l \ K_k
= ((F_0 \otimes I_k) \oplus (F_1 \otimes I_k) \oplus \cdots \oplus (F_{l-2} \otimes I_k)) \oplus l \ K_k.
\]

By Theorem 2.2, \(K_l\) has a \((l-1)\) 1-factors say \(F_x, \ 0 \leq x \leq l-2\). Each \(F_x\) of \(K_l\) will gives rise to \(\frac{l}{2}K_{k,k}\) in \(K_{kl}\). By Theorem 2.2 and Lemma 2.1, the graph \(K_{k,k}\) has a \(k\) 1-factors, and a 1-factor and \(\frac{k}{2}\) \(P_k\)-factors respectively. First we use \((k-1)\) 1-factors corresponding to each \(F_x\) from each \(K_{k,k}\) to get \((r,s) = (k-1,0)\). Similarly we use \(\frac{k}{2}\) \(P_k\)-factors to get \((r,s) = (0,\frac{k}{2})\).

Finally, we are left with a 1-factor in each \(K_{k,k}\). Repeating this process for all \((l-1)\) 1-factors of \(K_l\), we obtain \((r,s) \in (l-1) \ast \{(k-1,0),\left(0,\frac{k}{2}\right)\}\) and a new graph \(M\) (defined in Definition 4.2) which is a subgraph of \(K_l \otimes I_k\). By Theorems 2.2 and 2.3, the graph \(K_k\) has a \((k-1)\) 1-factor and \(\frac{k}{2}\) \(P_k\)-factors.

Hence \(lK_k\) has a \((K_2, P_k)-URD(r, s)\) with \((r,s) \in \{(k-1,0),\left(0,\frac{k}{2}\right)\}\). Therefore, it is easy to see that

\[
I_1(n) \subseteq (l-1) \ast \{(k-1,0),\left(0,\frac{k}{2}\right)\} + I(\Theta) + \{(k-1,0),\left(0,\frac{k}{2}\right)\}.
\]

\[\square\]

**6 Main result**

Lemmas 3.1 and 5.1 together give our main result.
Theorem 6.1. For all even $k \geq 4$, there exists a $(K_2, P_k)$-$URD(K_n; r, s)$ if and only if $n \equiv 0 \pmod{k}$ and $(r, s) \in I_1(n)$.

Remark. In this paper, we completely solved the existence of a uniformly resolvable decomposition of $K_n$ into $r$ classes containing only copies of $K_2$-factors and $s$ classes containing only copies of $P_k$-factors when $k$ is even. Further we proved that the necessary conditions for odd $k$. Finding sufficient conditions for odd $k$ is still open.

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