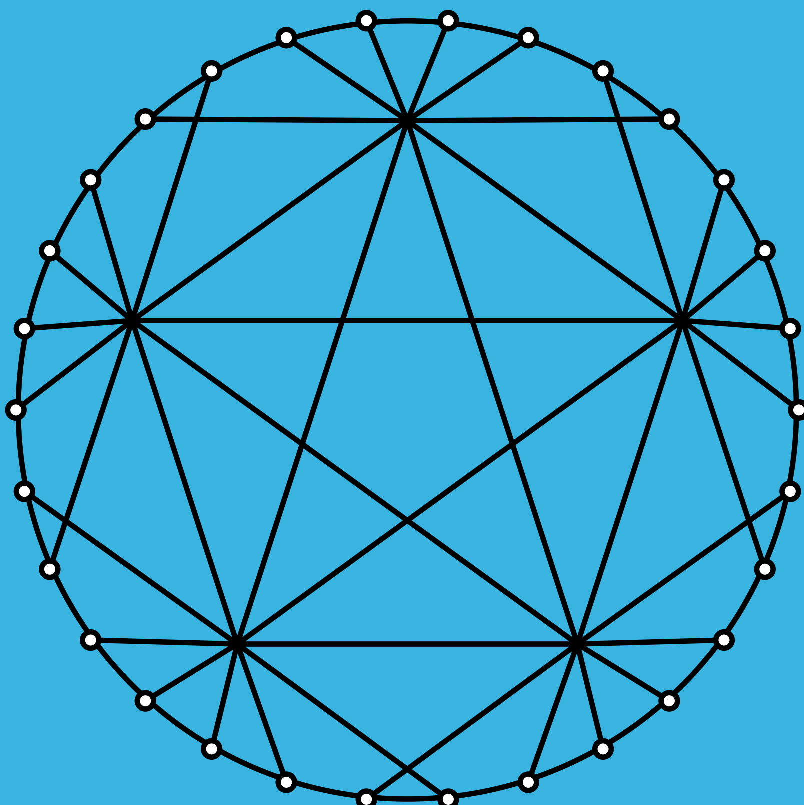
# BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 93
October 2021





Boca Raton, FL, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume 93 (2021), 126–142

# Uniformly resolvable decompositions of $K_n$ into 1-factors and $P_k$-factors

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**Abstract**

Let $P_n$ and $K_n$ respectively denote a path and a complete graph on $n$ vertices. In this paper, it is shown that the necessary conditions are sufficient for the existence of uniformly resolvable decomposition of $K_n$ into $r$ parallel classes containing $K_2$-factors and $s$ parallel classes containing $P_k$-factors for any even $k \geq 4$ and $r, s \geq 0$.

## 1 Introduction

In this paper, the vertex set and edge set of graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. Let $P_n, K_n$ and $I_n$ respectively denote a path, a

[*]Corresponding author.
**AMS (MOS) Subject Classifications:** 05B30, 05C38.
**Key words and phrases:** Parallel class, Path, Resolvable decomposition.

Received: 23 March 2020
Accepted: 15 March 2021

complete graph and an independent set on $n$ vertices. Given a collection of graphs $\mathcal{H}$, an *$\mathcal{H}$-decomposition* of a graph $G$ is a set of subgraphs (blocks) of $G$ whose edge sets partition $E(G)$, and each subgraph is isomorphic to a graph from $\mathcal{H}$. A *parallel class* of a graph $G$ is a set of subgraphs whose vertex sets partition $V(G)$. A parallel class is called *uniform* if each blocks of the parallel class is isomorphic to the same graph. An $\mathcal{H}$-decomposition of a graph $G$ is called (uniformly) *resolvable* if the blocks can be partitioned into (uniform) parallel classes. A resolvable $\mathcal{H}$-decomposition of $G$ is also referred as *$\mathcal{H}$-factorization* of $G$. We write $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$, if $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of $G$ and $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_k)$.

For two graphs $G$ and $H$ their *wreath product* $G \otimes H$ has the vertex set $V(G) \times V(H)$ and their edge set $E(G \otimes H) = \big\{\big((g,h),(g^{'},h^{'})\big)\big|(g,g^{'}) \in E(G)\,\text{or}\, g = g^{'}, \text{and}\ (h,h^{'}) \in E(H)\big\}$. An *$r$-factor* of $G$ is an $r$-regular spanning subgraph of $G$. A *near 1-factor* of $G$ is a 1-regular subgraph which contains all but one vertex of $G$. Let $K_{k,k}$ be the *complete bipartite* graph with bipartition $(X,Y)$, where $X = Y = \{0,1,\ldots,k-1\}$. The 1-factor of distance $t$ consists of the edges $\{(i,i+t) : 0 \le i \le k-1\}$, where the addition is taken modulo $k$.

Rees [16], obtained the necessary and sufficient conditions for the existence of uniformly resolvable $(K_2, K_3)$-designs of order $n$. Horton [10], has proved the existence of resolvable $P_k$-designs of order $n$ for $k = 3$ and Bermond et.al [2], have proved it for $k \ge 4$. Many other results on uniformly resolvable decomposition of $K_n$ into distinct subgraphs have been obtained in [4, 3, 13, 17, 5, 8, 15, 11, 12]. Recently [6, 7] Mario Gionfriddo and Salvatore Milici have investigated the existence of uniformly resolvable $\mathcal{H}$-designs with $\mathcal{H} = \{P_3, P_4\}$ and $\{K_2, P_k\}$ for $k = 3, 4$.

- We denote the existence of uniformly resolvable decomposition of $G$ into $r$ parallel classes consisting of $K_2$-factors and $s$ parallel classes consisting of $P_k$-factors by $(K_2, P_k)$-$URD(G; r, s)$.
- Let $I_1(n)$ (resp., $I_2(n)$) denote the set of possible pairs $(r, s)$ for which $(K_2, P_k)$-$URD(K_n; r, s)$ exists when $k$ is even (resp.,$k$ is odd).

For all even $k \ge 4$ and $n \equiv 0 \pmod{k}$, if $n \equiv 0 \pmod{k(k-1)}$ we define

$$I_1(n) = \left\{ \big(n - 1 - (k-1)x, \frac{k}{2}x\big) : x = 0, 1, \ldots, \frac{n-(k-1)}{(k-1)} \right\} \qquad (1)$$

and if $n \equiv a \pmod{k(k-1)}$, when $0 \le a \equiv 0 \pmod{k} \le k(k-2)$, we define

$$I_1(n) = \left\{ \left(n-1-(k-1)x, \frac{k}{2}x\right) : x = 0, 1, \ldots, \frac{n-\frac{a}{k}}{(k-1)} \right\}. \tag{2}$$

For all odd $k \ge 3$ and $n \equiv 0 \pmod{2k}$, if $n \equiv 0 \pmod{2k(k-1)}$ we define

$$I_2(n) = \left\{ ((n-1)-2(k-1)x, kx) : x = 0, 1, \ldots, \frac{n-2(k-1)}{2(k-1)} \right\} \tag{3}$$

and if $n \equiv a \pmod{2k(k-1)}$, when $0 \le a \equiv 0 \pmod{k} \le 2k(k-2)$, we define

$$I_2(n) = \left\{ ((n-1)-2(k-1)x, kx) : x = 0, 1, \ldots, \frac{n-\frac{a}{k}}{2(k-1)} \right\}. \tag{4}$$

In this paper, we prove that the necessary conditions are sufficient for the existence of $(K_2, P_k)$-$URD(K_n; r, s)$ for all even $k \ge 4$. Further, we give necessary conditions for the existence of $(K_2, P_k)$-$URD(K_n; r, s)$ for all odd $k \ge 3$.

## 2 Preliminary results

In this section, we present some known results required to prove our main results.

**Theorem 2.1.** ([1] Walecki's Construction).

1. *For all odd $n \ge 3$, the graph $K_n$ has a Hamilton cycle decomposition.*
2. *For all even $n \ge 4$, the graph $K_n - I$ has a Hamilton cycle decomposition with prescribed cycles $\{C, \sigma(C), \sigma^2(C), \ldots, \sigma^{\frac{n-4}{2}}(C)\}$. where $\sigma = (0)(12\ldots n-1)$ is a permutation, $C = (01\ldots n-1)$ is a Hamilton cycle and $I = \left\{ \left(0, \frac{n}{2}\right), \left(i, n-i\right) \middle| 1 \le i \le \frac{n}{2} - 1 \right\}$ is a 1-factor of $K_n$.*

**Theorem 2.2.** [14, 9]

1. *There exist a 1-factorization (resp., a near 1-factorization) of $K_n$ if and only if $n$ is even (resp., $n$ is odd).*

*2. Every regular bipartite graph is 1-factorable.*

**Theorem 2.3.** [18] *For all even $k$, the graph $K_n$ has a $P_k$-factorization if and only if $n \equiv k \pmod{k(k-1)}$.*

**Lemma 2.1.** [18, 19] *If $k$ is even, then the graph $K_{k,k}$ can be decomposed into one 1-factor and $\frac{k}{2} P_k$-factors.*

# 3 Necessary conditions

In this section, we give necessary conditions for the existence of

$$(K_2, P_k)\text{-}URD(K_n; r, s)$$

for all $k \geq 3$.

**Lemma 3.1.** *For all even $k \geq 4$, if $(K_2, P_k)$-$URD(K_n; r, s)$ exists, then $n \equiv 0 \pmod{k}$ and $(r, s) \in I_1(n)$.*

*Proof.* The condition $n \equiv 0 \pmod{k}$ is trivial. Let $\mathcal{D}$ be an arbitrary $(K_2, P_k)$-$URD(K_n; r, s)$. By resolvability, we have

$$r\frac{n}{2} + s\frac{n}{k}(k-1) = \frac{n(n-1)}{2}$$

Hence

$$rk + 2s(k-1) = k(n-1) \tag{5}$$

Now (5) gives

$$rk \equiv k(n-1) \pmod{2(k-1)} \text{ and } 2s(k-1) \equiv k(n-1) \pmod{k} \tag{6}$$

If $k$ is even, then (6) implies the following:
Now letting $s = \frac{k}{2}x$, Equation (5) gives $r = (n-1) - (k-1)x$. Since $r$ and $s$ cannot be negative, and $x$ is an integer, the value of $x$ must be in the range for $I_1(n)$. (See Equations 1 and 2.) □

**Lemma 3.2.** *For all odd $k \geq 3$, if $(K_2, P_k)$-$URD(K_n; r, s)$ exists, then $n \equiv 0 \pmod{2k}$ and $(r, s) \in I_2(n)$.*

*Proof.* The condition $n \equiv 0 \pmod{2k}$ is trivial. Let $\mathcal{D}$ be an arbitrary $(K_2, P_k)$-$URD(K_n; r, s)$. By resolvability, we have

$$r\frac{n}{2} + s\frac{n}{k}(k-1) = \frac{n(n-1)}{2}$$

| $r$ | $s$ | $n$ |
|---|---|---|
| $(k-2) \pmod{(k-1)}$ | $0 \pmod{\frac{k}{2}}$ | $0 \pmod{k(k-1)}$ |
| $0 \pmod{(k-1)}$ | $0 \pmod{\frac{k}{2}}$ | $k \pmod{k(k-1)}$ |
| $1 \pmod{(k-1)}$ | $0 \pmod{\frac{k}{2}}$ | $2k \pmod{k(k-1)}$ |
| . | . | . |
| . | . | . |
| . | . | . |
| $(k-3) \pmod{(k-1)}$ | $0 \pmod{\frac{k}{2}}$ | $(k-2)k \pmod{k(k-1)}$ |

Table 1: For even $k$

Hence

$$rk + 2s(k-1) = k(n-1) \tag{7}$$

Now (7) gives

$$rk \equiv k(n-1) \pmod{2(k-1)} \text{ and } 2s(k-1) \equiv k(n-1) \pmod{k} \tag{8}$$

If $k$ is odd, then (8) implies the following:

| $r$ | $s$ | $n$ |
|---|---|---|
| $(2k-3) \pmod{2(k-1)}$ | $0 \pmod{k}$ | $0 \pmod{2k(k-1)}$ |
| $1 \pmod{2(k-1)}$ | $0 \pmod{k}$ | $2k \pmod{2k(k-1)}$ |
| $3 \pmod{2(k-1)}$ | $0 \pmod{k}$ | $4k \pmod{2k(k-1)}$ |
| . | . | . |
| . | . | . |
| . | . | . |
| $(2k-5) \pmod{2(k-1)}$ | $0 \pmod{k}$ | $2k(k-2) \pmod{2k(k-1)}$ |

Now letting $s = kx$, Equation (7) gives $r = (n-1) - 2(k-1)x$. Since $r$ and $s$ cannot be negative, and $x$ is an integer, the value of $x$ must be in the range for $I_2(n)$. (See Equations 3 and 4.) □

# 4 Base construction

We present some definitions and results which are required to prove our main result.

**Definition 4.1.** *For each Hamilton cycle $h_i$ of $K_l$, we define $N_i$ to be the graph with vertex set $V(N_i) = V(K_l \times I_k)$ and edge set $E(N_i)$, where*

$$V(K_l) = \{x : 0 \leq x \leq l-1\},$$
$$V(I_k) = \{j : 1 \leq j \leq k\} \text{ and}$$
$$E(N_i) = \{\big((x,j),(y,j+1)\big) : (x,y) \in E(h_i),\ 1 \leq j \leq k\}.$$

*(Addition taken modulo $k$ ,i.e., $1, 2, \ldots, k$). See Figure 1.*

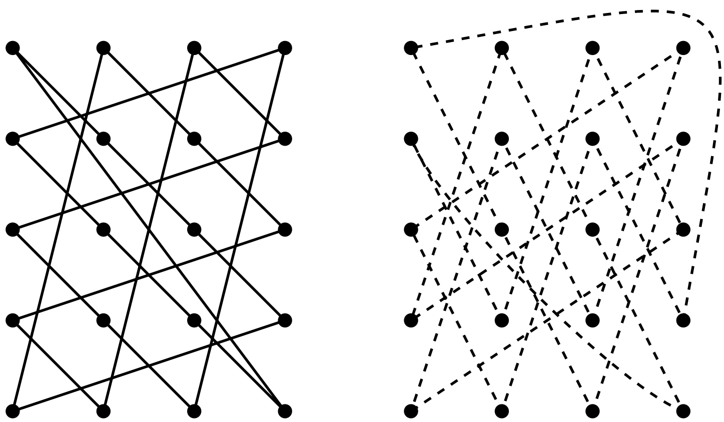


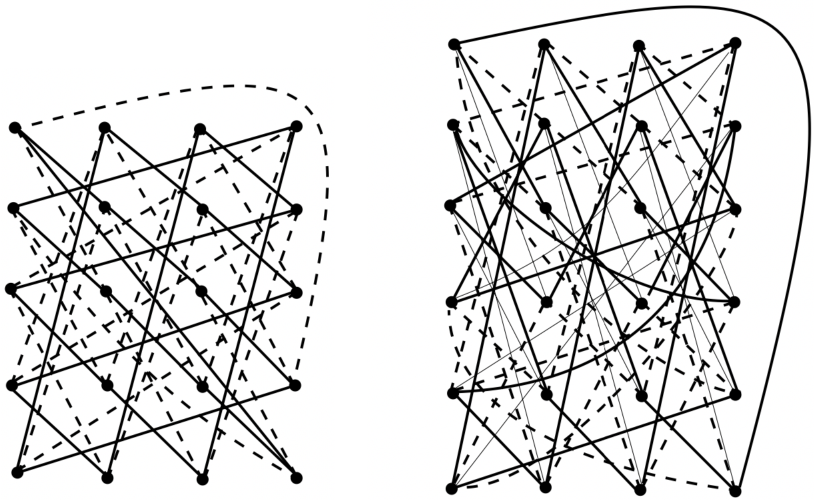

Figure 1: The graph $N_i, i = 1, 2$.

**Definition 4.2.** *Let $M$ be a graph with $V(M) = V(N_i)$ and the edge set*

$$E(M) = \begin{cases} \bigcup\limits_{i=1}^{\frac{l-1}{2}} E(N_i), & \text{when } l \text{ is odd} \\ \bigcup\limits_{i=1}^{\frac{l-2}{2}} E(N_i) \bigcup F, & \text{when } l \text{ is even} \end{cases}$$

*where $F$ is a 1-factor of $M$ (which correspond to the 1-factor of $K_l$) (see Figure 2) as follows: $F = \Big\{\big((0,a),(\frac{l}{2},a+1)\big), \big((i,a),(l-i,a+1)\big) \big| 0 \leq a \leq k-1,\ 1 \leq i \leq \frac{l}{2}-1\Big\}$.*

**Remark.** Clearly the graph $M$ defined in Definition 4.2 has an $N$-decomposition, $N_i \cong N$.



(a) When $\ell$ is odd (b) When $\ell$ is even

Figure 2: The graph $M$

**Definition 4.3.** *Let $(r_1, s_1)$ and $(r_2, s_2)$ be two pairs of non-negative integers. Then we define $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ . Usually positive integers are denoted as $\mathbb{Z}_+$. If $A = \{(r_1, s_1)|\ r_1, s_1 \in \mathbb{Z}_+\}$; $B = \{(r_2, s_2)|\ r_2, s_2 \in \mathbb{Z}_+\}$ and $h \in \mathbb{Z}_+$, then $A + B = \{(r_1, s_1) + (r_2, s_2)|(r_1, s_1) \in A,\ (r_2, s_2) \in B\}$ and $h * A$ denotes the set of all pairs of non-negative integers which can be obtained by adding any $h$ elements of $A$ together (repetitions of elements of $A$ are allowed).*

Now, let us define the following subgraphs in $M$ for our convenience as follows:

$$P = \bigcup_{i=1}^{\frac{k}{2}} N_i = \frac{k}{2}N \text{ and } Q = \bigcup_{i=1}^{k-1} N_i = (k-1)N$$

**Lemma 4.1.** *For all even $k \geq 4$, there exists a $(K_2, P_k)$-$URD(N; r, s)$ with $(r, s) = (2, 0)$.*

*Proof.* For any $i$, $0 \leq i \leq \frac{k-2}{2}$, we define subsets of $V(N)$ as follows: $X_1^i = \{(x, 2i)|\ 0 \leq x \leq l-1\}$, $X_2^i = \{(x, 2i+1)|\ 0 \leq x \leq l-1\}$, $Y_1^i = \{(x, 2i+1)|\ 0 \leq x \leq l-1\}$ and $Y_2^i = \{(x, 2i+2)|\ 0 \leq x \leq l-1\}$, where the addition is taken modulo $k$. Then the edges between the vertex sets

$X_1^i$ and $X_2^i$ will form one 1-factor in $N$. Similarly the sets $Y_1^i$ and $Y_2^i$ will form one more 1-factor in $N$. Hence, we obtain the required resolvable decomposition. □

**Lemma 4.2.** *For all even $k \geq 4$, there exists a $(K_2, P_k)$-URD$(P; r, s)$ with each $(r, s) \in \{(1, \frac{k}{2}), (k, 0)\}$.*

*Proof.* We prove in two cases.

**Case 1.** $(1, \frac{k}{2})$.
We first construct one $P_k$-factor from each $N_j$, $1 \leq j \leq \frac{k}{2}$ as follows: For any fixed $j$, $1 \leq j \leq \frac{k}{2}$, we define the subsets of $V(N_j)$ as $X^j = \{(x, 2(j-1)) \mid 0 \leq x \leq l-1\}$ and $Y^j = \{(x, 2(j-1)+1) \mid 0 \leq x \leq l-1\}$, where the addition is taken modulo $k$. Now keep the edges between the subsets $X^j$ and $Y^j$ for future purpose. The remaining graph will form one $P_k$-factor in $N_j$. By repeating the process for each $N_j$, we obtain $\frac{k}{2}$ $P_k$-factors in $P$. Now the edges between the sets $X^j$ and $Y^j$ from each $N_j$ together gives one 1-factor in $P$. Therefore, we get the required uniform resolvable decomposition.

**Case 2.** $(k, 0)$.
Each $N_j$, $1 \leq j \leq \frac{k}{2}$ can be decomposed into two 1-factors, by Lemma 4.1. Hence, we obtain the required resolvable decomposition of $P$. □

**Lemma 4.3.** *For all even $k \geq 4$, there exists a $(K_2, P_k)$-URD$(Q; r, s)$ with each $(r, s) \in \{(2(k-1), 0), (k-1, \frac{k}{2}), (0, k)\}$.*

*Proof.* We prove in three cases.

**Case 1.** $(2(k-1), 0)$.
Clearly the graph $Q = (k-1)N$ has a $2(k-1)$ 1-factors, by Lemma 4.1.

**Case 2.** $(k-1, \frac{k}{2})$.
Take $Q = (k-1)N = \left(\frac{k-2}{2}\right)N + \left(\frac{k}{2}\right)N = X + Y$. By Lemmas 4.1 and 4.2, the graphs $X$ and $Y$ have $(k-2)$ 1-factors and one 1-factor and $\frac{k}{2}$ $P_k$-factors respectively. Hence, we obtain $(k-1)$ 1-factors and $\frac{k}{2}$ $P_k$-factors in $Q$.

**Case 3.** $(0, k)$.
We first construct one $P_k$-factor from each $N_j$, $1 \leq j \leq k-1$ as follows: For any fixed $j$, $1 \leq j \leq k-1$, we define the subsets of $V(N_j)$ as $X^j = \{(x, j-1) | 0 \leq x \leq l-1\}$ and $Y^j = \{(x, j) | 0 \leq x \leq l-1\}$. Now keep the edges between the subsets $X^j$ and $Y^j$ for future purpose. The remaining

graph will form one $P_k$-factor in $N_j$. By repeating the process for each $N_j$, we obtain $(k-1)$ $P_k$-factors in $Q$. Now the edges between the sets $X^j$ and $Y^j$ from each $N_j$ which were kept aside together gives one $P_k$-factor in $Q$. Therefore, we get the required resolvable decomposition. $\square$

The order (number of vertices) of the graph $M$ (defined in Definition 4.2) be denoted as $\Theta$. For all even $k \geq 4$ and $\Theta \equiv 0 \pmod{k}$, if $\Theta \equiv 0 \pmod{k(k-1)}$, we define

$$I(\Theta) = \left\{ \left(l - 1 - (k-1)x, \frac{k}{2}x\right) : x = 0, 1, \ldots, \frac{l-(k-1)}{(k-1)} \right\} \tag{9}$$

and if $\Theta \equiv a \pmod{k(k-1)}$, when $0 < a \equiv 0 \pmod{k} \leq k(k-2)$, we define

$$I(\Theta) = \left\{ \left(l - 1 - (k-1)x, \frac{k}{2}x\right) : x = 0, 1, \ldots, \frac{l-\frac{a}{k}}{(k-1)} \right\}. \tag{10}$$

**Lemma 4.4.** *For all even $k \geq 4$, if $(K_2, P_k)$-URD$(M; r, s)$ exists, then $\Theta \equiv 0 \pmod{k}$ and $(r, s) \in I(\Theta)$.*

*Proof.* The condition $\Theta \equiv 0 \pmod{k}$ is trivial and hence $\Theta = kl$, $l \in \mathbb{Z}_+$. Let $\mathcal{D}$ be an arbitrary $(K_2, P_k)$-URD$(M; r, s)$. By resolvability, we have

$$r\frac{kl}{2} + s\frac{kl}{k}(k-1) = \frac{kl(l-1)}{2}$$

Hence

$$rk + 2s(k-1) = k(l-1) \tag{11}$$

Letting $s = \frac{k}{2}x$, Equation (11) gives $r = (l-1) - (k-1)x$. Since $r$ and $s$ cannot be negative, and $x$ is an integer, the value of $x$ must be in the range for $I(\Theta)$. (See Equations 9 and 10.) $\square$

**Lemma 4.5.** *For any $\Theta \equiv 0 \pmod{4}$, there exists $(K_2, P_4)$-URD$(M; r, s)$.*

*Proof.* Let $\Theta \equiv 0 \pmod{4}$, we have a $\Theta \equiv a \pmod{12}$ with $a = 0, 4, 8$. We prove in three cases.

**Case 1.** For $\Theta \equiv 0 \pmod{12}$, we have a $\Theta = 12x = 4(3x)$, where $x \geq 1$.

**Subcase 1.** If $x$ is odd, then the graph

$$M = \left(\frac{3x-1}{2}\right)N = \left(\frac{3x-3}{2}\right)N \cup N = \left(\frac{x-1}{2}\right)Q \cup N.$$

Hence, by Lemmas 4.1 and 4.3, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$M = \left(\frac{3x-2}{2}\right)N \cup F = \left(\frac{3x-6}{2}\right)N \cup 2N \cup F = \left(\frac{x-2}{2}\right)Q \cup P \cup F.$$

Hence, by Lemmas 4.2 and 4.3 along with $F$, we get the required URDs.

**Case 2.** For $\Theta \equiv 4 \pmod{12}$, we have a $\Theta = 12x + 4 = 4(3x + 1)$, where $x \geq 0$.

**Subcase 1.** If $x$ is odd, then the graph

$$M = \left(\frac{3x-1}{2}\right)N \cup F = \left(\frac{3x-3}{2}\right)N \cup N \cup F = \left(\frac{x-1}{2}\right)Q \cup N \cup F.$$

Hence, by Lemmas 4.1 and 4.3 along with $F$, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$M = \left(\frac{3x}{2}\right)N = \left(\frac{x}{2}\right)Q.$$

Hence, by Lemma 4.3, we get the required URDs.

**Case 3.** For $\Theta \equiv 8 \pmod{12}$, we have a $\Theta = 12x + 8 = 4(3x + 2)$, where $x \geq 0$.

**Subcase 1.** If $x$ is odd, then the graph

$$M = \left(\frac{3x+1}{2}\right)N = \left(\frac{3x-3}{2}\right)N \cup 2N = \left(\frac{x-1}{2}\right)Q \cup P.$$

Hence, by Lemmas 4.2 and 4.3, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$M = \left(\frac{3x}{2}\right)N \cup F = \left(\frac{x}{2}\right)Q \cup F.$$

Hence, by Lemma 4.3 along with $F$, we get the required URDs. □

**Lemma 4.6.** *For even $k \geq 6$ and $\Theta \equiv 0 \pmod{k}$, $(K_2, P_k)$-URD$(M; r, s)$ exists.*

*Proof.* Let $\Theta \equiv 0 \pmod{k}$, we have a $\Theta \equiv a \pmod{k(k-1)}$ with $0 \leq a \equiv 0 \pmod{k} \leq k(k-2)$. We prove in six cases.

**Case 1.** For $\Theta \equiv 0 \pmod{k(k-1)}$, we have a $\Theta = k(k-1)x$, where $x \geq 1$.

**Subcase 1.** If $x$ is odd, then the graph

$$M = \Big(\frac{(k-1)x-1}{2}\Big)N = \Big(\frac{(k-1)x-(k-2)-1}{2}\Big)N \cup \Big(\frac{k-2}{2}\Big)N = \Big(\frac{x-1}{2}\Big)Q \cup \Big(\frac{k-2}{2}\Big)N.$$

Hence, by Lemmas 4.1 and 4.3, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$M = \Big(\frac{(k-1)x-2}{2}\Big)N \cup F = \Big(\frac{(k-1)(x-2)}{2}\Big)N \cup (k-2)N \cup F = \Big(\frac{x}{2}\Big)Q \cup P \cup N \cup F.$$

Hence, by Lemmas 4.1 to 4.3 along with $F$, we get the required URDs.

**Case 2.** For $\Theta \equiv k \pmod{k(k-1)}$, we have a $\Theta = k(k-1)x + k = k\big((k-1)x+1\big)$, where $x \geq 0$.

**Subcase 1.** If $x$ is odd, then the graph

$$M = \Big(\frac{(k-1)x-1}{2}\Big)N \cup F = \Big(\frac{(k-2)(x-1)-1}{2}\Big)N \cup \Big(\frac{k-2}{2}\Big)N \cup F = \Big(\frac{x-1}{2}\Big)Q \cup \Big(\frac{k-2}{2}\Big)N \cup F.$$

Hence, by Lemmas 4.1 and 4.3 along with $F$, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$M = \Big(\frac{(k-1)x}{2}\Big)N = \Big(\frac{x}{2}\Big)Q.$$

Hence, by Lemma 4.3, we get the required URDs.

**Case 3.** For $\Theta \equiv 2k \pmod{k(k-1)}$, we have a $\Theta = k(k-1)x + 2k = k\big((k-1)x+2\big)$, where $x \geq 0$.

**Subcase 1.** If $x$ is odd, then the graph

$$M = \Big(\frac{(k-1)x+1}{2}\Big)N = \Big(\frac{(k-1)x+1-k}{2}\Big)N \cup \Big(\frac{k}{2}\Big)N = \Big(\frac{x-1}{2}\Big)Q \cup P.$$

Hence, by Lemmas 4.2 and 4.3, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$M = \Big(\frac{(k-1)x}{2}\Big)N \cup F = \Big(\frac{x}{2}\Big)Q \cup F.$$

Hence, by Lemma 4.3 along with $F$, we get the required URDs.

**Case 4.** For $\Theta \equiv 3k \pmod{k(k-1)}$, we have a $\Theta = k(k-1)x + 3k = k\big((k-1)x+3\big)$, where $x \geq 0$.

**Subcase 1.** If $x$ is odd, then the graph

$$\begin{aligned} M = \Big(\frac{(k-1)x+1}{2}\Big)N \cup F &= \Big(\frac{(k-1)x+1-k}{2}\Big)N \cup \Big(\frac{k}{2}\Big)N \cup F \\ &= \Big(\frac{x-1}{2}\Big)Q \cup P \cup F. \end{aligned}$$

Hence, by Lemmas 4.2 and 4.3 along with $F$, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$\begin{aligned} M = \Big(\frac{(k-1)x+2}{2}\Big)N &= \Big(\frac{(k-1)x+1}{2}\Big)N \cup N \\ &= \Big(\frac{x}{2}\Big)Q \cup N. \end{aligned}$$

Hence, by Lemmas 4.1 and 4.3, we get the required URDs.

**Case 5.** For $\Theta \equiv a \pmod{k(k-1)}$ with $3k < a \equiv 0 \pmod{k} < k(k-2)$, we have a $\Theta = k(k-1)y + a = k(k-1)y + kx = k\big((k-1)y + x\big)$, where $y \geq 0$ and $4 \leq x \leq k-3$.

**Subcase 1.** Let $x = 2z+2$, where $1 \leq z \leq \frac{k-6}{2}$ and even $y \geq 0$, then the graph

$$\begin{aligned} M = \Big(\frac{(k-1)y+x-2}{2}\Big)N \cup F &= \Big(\frac{(k-1)y+2z}{2}\Big)N \cup F \\ &= \Big(\frac{(k-1)y}{2}\Big)N \cup zN \cup F \\ &= \Big(\frac{y}{2}\Big)Q \cup zN \cup F. \end{aligned}$$

Hence, by Lemmas 4.1 and 4.3 along with $F$, we get the required URDs.

**Subcase 2.** Let $x = 2z + 2$, where $1 \leq z \leq \frac{k-6}{2}$ and odd $y \geq 1$, then the graph

$$\begin{aligned} M = \Big(\frac{(k-1)y + x - 1}{2}\Big)N &= \Big(\frac{(k-1)y + 2z + 1}{2}\Big)N \\ &= \Big(\frac{(k-1)y - k + 1}{2}\Big)N \cup \Big(\frac{k}{2}\Big)N \cup zN \\ &= \Big(\frac{y-1}{2}\Big)Q \cup P \cup zN. \end{aligned}$$

Hence, by Lemmas 4.1 to 4.3, we get the required URDs.

**Subcase 3.** Let $x = 2z + 3$, where $1 \leq z \leq \frac{k-6}{2}$ and odd $y \geq 1$, then the graph

$$\begin{aligned} M = \Big(\frac{(k-1)y + x - 2}{2}\Big)N \cup F &= \Big(\frac{(k-1)y + 2z + 1}{2}\Big)N \cup F \\ &= \Big(\frac{(k-1)y - k + 1}{2}\Big)N \cup \Big(\frac{k}{2}\Big)N \cup zN \cup F \\ &= \Big(\frac{y-1}{2}\Big)Q \cup P \cup zN \cup F. \end{aligned}$$

Hence, by Lemmas 4.1 to 4.3 along with $F$, we get the required URDs.

**Subcase 4.** Let $x = 2z + 3$, where $1 \leq z \leq \frac{k-6}{2}$ and even $y \geq 0$, then the graph

$$\begin{aligned} M = \Big(\frac{(k-1)y + x - 1}{2}\Big)N &= \Big(\frac{(k-1)y + 2z + 2}{2}\Big)N \\ &= \Big(\frac{(k-1)y}{2}\Big)N \cup (z+1)N \\ &= \Big(\frac{y}{2}\Big)Q \cup (z+1)N. \end{aligned}$$

Hence, by Lemmas 4.1 and 4.3, we get the required URDs.

**Case 6.** For $\Theta \equiv k(k-2) \pmod{k(k-1)}$, we have a $\Theta = k(k-1)x + k(k-2) = k\big((k-1)x + (k-2)\big)$, where $x \geq 0$.

**Subcase 1.** If $x$ is odd, then the graph

$$\begin{aligned} M = \Big(\frac{(k-1)x + k - 3}{2}\Big)N &= \Big(\frac{(x-1) - k + 1}{2}\Big)N \cup (k-2)N \\ &= \Big(\frac{x-1}{2}\Big)Q \cup \Big(\frac{k}{2}\Big)N \cup \Big(\frac{k-4}{2}\Big)N \\ &= \Big(\frac{x-1}{2}\Big)Q \cup P \cup \Big(\frac{k-4}{2}\Big)N. \end{aligned}$$

Hence, by Lemmas 4.1 to 4.3, we get the required URDs.

**Subcase 2.** If $x$ is even, then the graph

$$\begin{aligned} M = \Big(\frac{(k-1)x+k-4}{2}\Big)N \cup F &= \Big(\frac{(k-1)x}{2}\Big)N \cup \Big(\frac{k-4}{2}\Big)N \cup F \\ &= \Big(\frac{x}{2}\Big)Q \cup \Big(\frac{k-4}{2}\Big)N \cup F. \end{aligned}$$

Hence, by Lemmas 4.1 and 4.3 along with $F$, we get the required URDs. □

**Theorem 4.1.** *For all even $k \geq 4$, if $(K_2, P_k)$-URD$(M; r, s)$ if and only if $\Theta \equiv 0 \pmod k$ and $(r, s) \in I(\Theta)$.*

*Proof.* Follows from Lemmas 4.4 to 4.6. □

# 5 Sufficient conditions

In this section, we prove that the necessary conditions are sufficient for the existence of uniformly resolvable decomposition of $K_n$ into $r$ parallel classes containing $K_2$-factors and $s$ parallel classes containing $P_k$-factors for any even $k \geq 4$ and $r, s \geq 0$.

**Lemma 5.1.** *For all even $k \geq 4$ and $n \equiv 0 \pmod k$, there exists*

$$(K_2, P_k)\text{-}URD(M; r, s).$$

*Proof.* As $n \equiv 0 \pmod k$, let $n = kl$, $l \in \mathbb{Z}_+$.
**Case 1.** $l$ is odd. For $l = 1$, there exists a required uniform resolvable decomposition, by Theorems 2.2 and 2.3. For $l \geq 3$, let $V(K_{kl}) = \bigcup_{x=0}^{l-1} A_x$, where $A_x = \{(x, kx+i) : 0 \leq i \leq k-1$ and the addition is taken modulo $k\}$. We obtain a new graph $A$ from $K_{kl}$, by identifying each $A_x$ with a single vertex $a_x$ and joint $a_x$ and $a_y$ if there exists a complete bipartite graph $K_{|A_x|,|A_y|}$ between $A_x$ and $A_y$ in $K_{kl}$. Then the new graph $A \cong K_l$. By Theorem 2.2, the graph $K_l$ has $l$ near 1-factors say $F_x$, $0 \leq x \leq l-1$ with the missing vertex $x$. Corresponding to each $F_x$ with a missing vertex $x$ of $K_l$, we have a $\big(\frac{l-1}{2}\big)K_{k,k}$ in $K_{kl}$ and corresponding to $A_x$ in $K_{kl}$, we have a $K_{|A_x|} \cong K_k$. By Theorem 2.2, the graphs $K_{k,k}, K_k$ have $k, (k-1)$ 1-factors, respectively and by Lemma 2.1, the graph $K_{k,k}$ has a one 1-factor and $\frac{k}{2}$ $P_k$-factors. Also by Theorem 2.3, the graph $K_k$ has a $\frac{k}{2}$ $P_k$-factors.

First we use $(k-1)$ 1-factors corresponding to each $F_x$ from each $K_{k,k}$ and $K_k$ to get $(r,s)=(k-1,0)$. Finally, we are left with a 1-factor in each $K_{k,k}$ and $k$ isolated vertices in each $K_k$. Similarly when we use $\frac{k}{2}$ $P_k$-factors we get $(r,s)=(0,\frac{k}{2})$. By repeating this process for all $l$ near 1-factors of $K_l$, we obtain $(r,s)\in l*\left\{(k-1,0),\left(0,\frac{k}{2}\right)\right\}$ and a new graph $M$ (defined in Definition 4.2) in which there is only one 1-factor between each pair of $A_x$ and $A_y$ in $K_{kl}$. By Theorem 4.1, the graph $M$ has a $(K_2,P_k)$-$URD(r,s)$ with $(r,s)\in I(\Theta)$. Therefore, it is easy to see that

$$I_1(n)\subseteq l*\left\{(k-1,0),\left(0,\frac{k}{2}\right)\right\}+I(\Theta).$$

**Case 2.** $l$ is even. For $l=2$, we have $K_{2k}\equiv K_{k,k}\oplus 2K_k$. Applying Theorems 2.2 and 2.3 and Lemma 2.1, it is easy to obtain $\left\{(k,0),\left(1,\frac{k}{2}\right)\right\}+\left\{(k-1,0),\left(0,\frac{k}{2}\right)\right\}\supseteq I_1(2k)$. For $l\geq 4$, we have

$$\begin{aligned}K_{kl}&\equiv(K_l\otimes I_k)\oplus l\ K_k\\&=\big((F_0\oplus F_1\oplus\cdots\oplus F_{l-2})\otimes I_k\big)\oplus l\ K_k\\&=\big((F_0\otimes I_k)\oplus(F_1\otimes I_k)\oplus\cdots\oplus(F_{l-2}\otimes I_k)\big)\oplus l\ K_k.\end{aligned}$$

By Theorem 2.2, $K_l$ has a $(l-1)$ 1-factors say $F_x,\ 0\leq x\leq l-2$ . Each $F_x$ of $K_l$ will gives rise to $\frac{l}{2}K_{k,k}$ in $K_{kl}$. By Theorem 2.2 and Lemma 2.1, the graph $K_{k,k}$ has a $k$ 1-factors, and a 1-factor and $\frac{k}{2}$ $P_k$-factors respectively. First we use $(k-1)$ 1-factors corresponding to each $F_x$ from each $K_{k,k}$ to get $(r,s)=(k-1,0)$. Similarly we use $\frac{k}{2}$ $P_k$-factors to get $(r,s)=(0,\frac{k}{2})$. Finally, we are left with a 1-factor in each $K_{k,k}$. Repeating this process for all $(l-1)$ 1-factors of $K_l$, we obtain $(r,s)\in(l-1)*\left\{(k-1,0),\left(0,\frac{k}{2}\right)\right\}$ and a new graph $M$ (defined in Definition 4.2) which is a subgraph of $K_l\otimes I_k$. By Theorems 2.2 and 2.3, the graph $K_k$ has a $(k-1)$ 1-factor and $\frac{k}{2}$ $P_k$-factors. Hence $lK_k$ has a $(K_2,P_k)$-$URD(r,s)$ with $(r,s)\in\left\{(k-1,0),\left(0,\frac{k}{2}\right)\right\}$. Therefore, it is easy to see that

$$I_1(n)\subseteq(l-1)*\left\{(k-1,0),\left(0,\frac{k}{2}\right)\right\}+I(\Theta)+\left\{(k-1,0),\left(0,\frac{k}{2}\right)\right\}.$$

□

# 6 Main result

Lemmas 3.1 and 5.1 together give our main result.

**Theorem 6.1.** *For all even $k \geq 4$, there exists a $(K_2, P_k)$-URD$(K_n; r, s)$ if and only if $n \equiv 0 \pmod{k}$ and $(r, s) \in I_1(n)$.*

**Remark.** In this paper, we completely solved the existence of a uniformly resolvable decomposition of $K_n$ into $r$ classes containing only copies of $K_2$-factors and $s$ classes containing only copies of $P_k$-factors when $k$ is even. Further we proved that the necessary conditions for odd $k$. Finding sufficient conditions for odd $k$ is still open.


# Acknowledgments

Authors thank the University Grant Commission, Government of India, New Delhi for its support through the Grant No.F.510/7/DRS-I/2016(SAP-I). The second author thank the Department of Atomic Energy, Government of India, Mumbai for its support through the Grant No.DAE No.2/40(22)/2016/R&D-11/15245.


# References


[1] B. Alspach, J.C. Bermond and D. Sotteau, *Decomposition into cycles I: Hamilton decompositions*, in: "Cycles and Rays", Hahn, Gena, Sabidussi, Gert, Woodrow and Robert, Eds., Kluwer Academic Publisher, 1990.

[2] J.C. Bermond, K. Heinrich and M.L. Yu, Existence of resolvable path designs, *Europ. J. Combin.*, **11** (1990), 205–211.

[3] F. Chen and H. Cao, Uniformly resolvable decompositions of $K_n$ into $K_2$ and $K_{1,3}$ graphs, *Discrete Math.*, **339** (2016), 2056–2062.

[4] J.H. Dinitz, A.C.H. Ling and P. Danziger, Maximum uniformly resolvable designs with block sizes 2 and 4, *Discrete Math.*, **309** (2009) 4716–4721.

[5] M. Gionfriddo and S. Milici, On the existence of uniformly resolvable decompositions of $K_n$ and into paths and kites, *Discrete Math.*, **313** (2013), 2830–2834.

[6] M. Gionfriddo and S. Milici, Uniformly resolvable $\mathcal{H}$-designs with $\{P_3, P_4\}$, *Australas J. Combin.*, **60** (2014), 325–332.

[7] M. Gionfriddo and S. Milici, On uniformly resolvable $\{K_2, P_k\}$-designs with $k = 3, 4$, *Contr. Discrete Math.*, **10** (2015), 126–133.

[8] M. Gionfriddo, S. Kucukcifci, S. Milici and E. Yazici, Uniformly resolvable $(C_4, K_{1,3})$-designs of index 2, *Contr. Discrete Math.*, **13** (2018), 23–34.

[9] F. Harary, *Graph Theory*, Addison-Wesley, 1969.

[10] J.D. Horton, Resolvable path designs, *J. Combin. Theory Ser. A*, **39** (1985), 117–131.

[11] S. Kucukcifci, G. Lo Faro, S. Milici and A. Tripodi, Resolvable 3-star designs, *Discrete Math.*, **338** (2014), 608–614.

[12] S. Kucukcifci, S. Milici and Z. Tuza, Maximum uniformly resolvable decompositions of $K_n$ and $K_n - I$ into 3-stars and 3-cycles, *Discrete Math.*, **338** (2015), 1667–1673.

[13] G. Lo Faro, S. Milici and A. Tripodi, Uniformly resolvable decompositions of $K_n$ into paths on two, three and four vertices, *Discrete Math.*, **338** (2015), 2212–2219.

[14] E. Mendelsohn and A. Rosa, One-Factorizations of the complete graph a survey, *J. Graph Theory*, **9** (1985), 43–65.

[15] S. Milici and Z. Tuza, Uniformly resolvable decompositions of $K_n$ into $P_3$ and $K_3$ graphs, *Discrete Math.*, **331** (2014), 137–141.

[16] R. Rees, Uniformly resolvable pairwise balanced designs with block sizes two and three, *J. Combin. Theory Ser. A*, **45** (1987), 207–225.

[17] H. Wei and G. Ge, Some more uniformly resolvable designs with block sizes 2 and 4, *Discrete Math.*, **340** (2017), 2243–2249.

[18] M.L. Yu, *Resolvable path designs of complete graphs*, B.Sc. Thesis, Simon Fraser University, 1987.

[19] M.L. Yu, On path factorizations of complete multipartite graphs, *Discrete Math.*, **122** (1993), 325–333.