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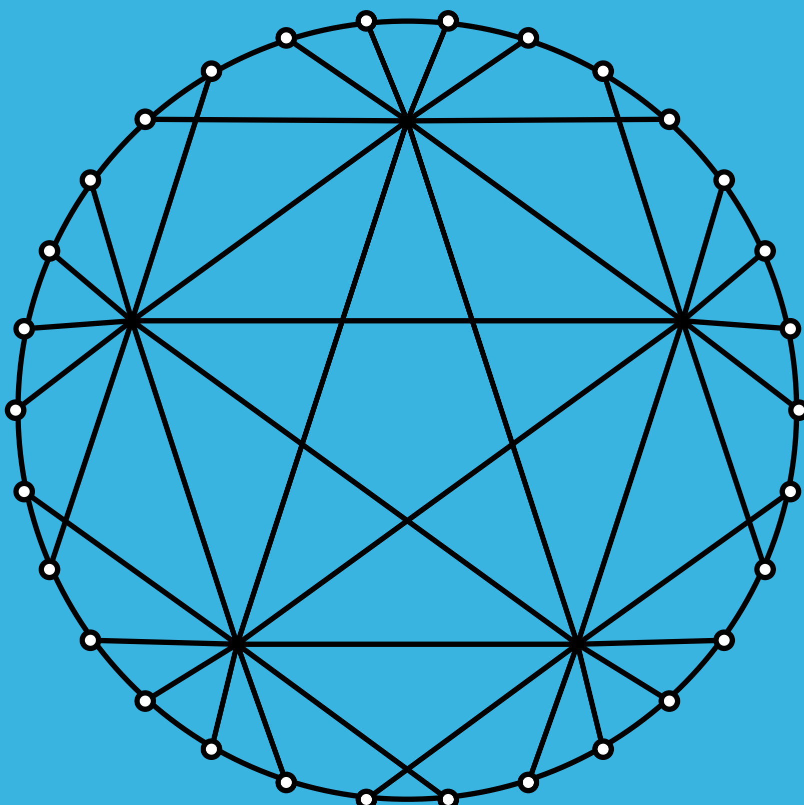

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Face-magic labelings of type (a, b, c) from edge-magic labelings of type (α, β)

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Abstract

We introduce a generalization of edge-magic total (EMT) labeling which allows multiple labels on the vertices or edges of a graph. Then we use this new labeling as a tool to construct face-magic labelings of some infinite families of graphs. We take the following novel approach to investigating face-magic labelings. Given a graph G, we ask: For which $a, b, c \in \{0, 1\}$ does G admit a face-magic labeling of type (a, b, c)? We completely answer this question for two families of chained cycles, ladders and subdivided ladders, fans and subdivided fans, and wheels and subdivided wheels.

1 Introduction

Let $G = (V, E)$ be a simple graph and $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ be a bijection. If there exists an integer k such that $f(u) + f(uv) + f(v) = k$ for every edge $uv \in E$, then f is an *edge-magic total labeling* (EMT) of G. If in

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addition, $f(v) \in \{1, 2, \ldots, |V|\}$ for all $v \in V$, we call f an *super edge-magic total labeling* (SEMT). If a graph G admits an (S)EMT labeling, we say the graph is an (S)EMT graph.

The notion of edge-magic total labelings was first introduced 50 years ago by Kotzig and Rosa in [7]. See Chapter 2 in [10] for more information on these labelings, and [5] for a survey of results including open problems.

We introduce the following variation of edge-magic total labeling. Let $G = (V, E)$ be a simple graph, α and β be nonnegative integers, and $f : V \cup E \to \{1, 2, \ldots, \alpha|V| + \beta|E|\}$ be an assignment such that every vertex receives exactly α labels, every edge receives exactly β labels, and no label is repeated. For every edge $uv \in E$, define the *weight*, $w(uv)$ of the edge as the sum of all the labels in $f(u) \cup f(uv) \cup f(v)$. If there exists an integer k, called the *magic constant*, such that $w(e) = k$ for every edge $e \in E$, then we say that f is an *edge-magic labeling of type* (α, β) (EMT(α, β)) of G. As with EMT labelings, if $f(v) \subseteq \{1, 2, \ldots, \alpha|V|\}$ for all $v \in V$, we call f a *super edge-magic labeling of type* (α, β) (SEMT(α, β)). If a graph G admits an (S)EMT(α, β) labeling, we say the graph is an (S)EMT(α, β) graph.

In Sections 3 and 4, we will generalize the following three results in terms of (super) edge-magic type (α, β) labelings.

Theorem 1.1. [2] *The path P_n has an super edge-magic total labeling if $n \geq 2$.*

Theorem 1.2. [7] *The cycle C_n has an edge-magic total labeling if $n \geq 3$.*

Theorem 1.3. [4] *The cycle C_n has a super edge-magic total labeling if and only if $n \geq 3$ is odd.*

Some 20 years after the introduction of edge-magic total labelings, a magic-type labeling in which one sums over the faces of a graph was introduced by Lih [8]. Let $G = (V, E, F)$ be a graph and $a, b, c \in \{0, 1\}$. An assignment f of the labels $\{1, 2, \ldots, a|V| + b|E| + c|F|\}$ to the vertices, edges, and faces of the graph that gives exactly a labels to every vertex, exactly b labels to every edge, exactly c labels to every face, and no label is repeated is called a *labeling of type* (a, b, c). Define the *weight* of a face as the sum of the label of the face itself (when present), along with the labels of the vertices and edges surrounding that face (when present). If the weight of every s-sided face is equal to the same number $\mu(s)$ called the *magic constant*, we call f a *face-magic labeling of type* (a, b, c). If $f(v) \leq a|V|$ for all $v \in V$, we call f a *super face-magic labeling of type* (a, b, c).

In Section 5, we use edge-magic type (α, β) labelings to solve some open problems in face-magic labeling. Given a graph G, the spectrum question for face-magic labeling of type (a, b, c) asks: For which $a, b, c \in \{0, 1\}$ does G admit a face-magic labeling of type (a, b, c)? To our knowledge, this problem has not yet been explored for any class of graphs. We completely answer the spectrum question for ladders and subdivided ladders, fans and subdivided fans, wheels and subdivided wheels, and a family of chained cycles.

Partitioning a set into subsets in which the sum of the elements in each subset is equal to the same fixed constant for every subset plays an important role in the aforementioned magic-type labelings. Let $n_1 + n_2 + \cdots + n_p = n$ be a partition of the number n, and $S = \{1, 2, \ldots, n\}$. If S can be partitioned into sets A_i for $i \in \{1, 2, \ldots, p\}$ such that $S = \bigcup A_i$, $|A_i| = n_i$, and $\sum_{a \in A_i} a = \mu$ for some fixed constant μ, then we call $\bigcup A_i$ a *constant sum partition* of S. For partitions of equal size, Miller et al. proved the following in [11], though it was done in the context of graph labeling.

Theorem 1.4. [11] *Let $n = pm$ and $S = \{1, 2, \ldots, n\}$. There exists a constant sum partition of S into p sets of size m if and only if m is even or both m and p are odd.*

A (p, q) graph is a simple graph with p vertices and q edges. For any integer n, we denote the set $S = \{1, 2, \ldots, n\}$ by $S = [1, n]$. For any integer k and set S, by $S + k$, we mean the set $S + k = \{s + k | s \in S\}$.

2 General results

The first two results in this section are corollaries of Theorem 1.4. The first deals with n disjoint copies of P_2 (a 1-factor), while the second deals with labeling only edges.

Observation 2.1. *The graph $G \cong nP_2$ admits an* $\mathrm{EMT}(\alpha, \beta)$ *if and only if β is even or both β and n are odd.*

Proof. Clearly G admits an $\mathrm{EMT}(\alpha, \beta)$ if and only if a constant sum partition of $[1, n(2\alpha + \beta)]$ into n sets of size $2\alpha + \beta$ exists. Therefore, the result follows from Theorem 1.4. □

Observation 2.2. *A (p,q) graph G admits an* EMT$(0,\beta)$ *labeling if and only if β is even or $\beta \geq 3$ and q are both odd.*

Proof. It is easy to see that G admits an EMT$(0,\beta)$ labeling if and only if there is a constant sum partition of $S = [1, q\beta]$ into β sets of size q, which exists, (by Theorem 1.4) if and only if β is even or β and q are both odd. □

Observation 2.3. *A (p,q) graph G admits an* SEMT$(\alpha,0)$ *if α is even or α and p are both odd.*

Proof. Partition the set $S = [1, p\alpha]$ into sets S_i of size α for $i \in [1,p]$ such that the sum of the elements in each set is equal to the same constant σ. Then for $i \in [1,p]$, give to each vertex the set of labels S_i. Clearly the weight of every edge is 2σ. □

To address the case missing from Observation 2.3, we know by Theorem 1.4 that when p is even and α is odd there does not exist a constant sum partition of the set $S = [1, p\alpha]$ into sets of size α. But if the graph G is bipartite with bipartition $V(G) = X \cup Y$, an EMT$(\alpha, 0)$ labeling would result from a partition of S into sets of size α such that $|X|$ of the sets have one constant sum and $|Y|$ of the sets have another constant sum. We show this can be done next.

Theorem 2.4. *Let G be a bipartite (p,q) graph with vertex bipartition $V(G) = X \cup Y$. If at least one of the conditions below are true, then G admits an* SEMT$(\alpha,0)$ *labeling for all $\alpha \geq 2$.*

- $\alpha \equiv 0 \pmod 2$
- $\alpha \equiv p \equiv 1 \pmod 2$
- $|X| \equiv |Y| \equiv 1 \pmod 2$
- $|X| = |Y|$

Proof. Let $G = (V,E)$ be a bipartite graph with vertex bipartition $V = X \cup Y$. If α is even or both α and p are odd, the proof follows from Theorem 1.4. So assume $\alpha \geq 3$ is odd and p is even. Then, $|X| \equiv |Y| \pmod 2$. If $|X|$ and $|Y|$ are both odd, then Theorem 1.4 tells us we can partition

$[1, |X|]$ and $[1, |Y|]$ into subsets of size α all having constant sum σ_X and σ_Y, respectively. Assign each vertex in X all α of the integers in each respective subset. Add $|X|$ to every integer in the partition of $[1, |Y|]$ and assign each vertex in Y all α of the integers in each translated subset. Since every edge of G has the form xy where $x \in X$ and $y \in Y$, we have $w(xy) = \sigma_X + \sigma_Y + \alpha|Y|$.

Finally, if $|X| = |Y|$, we can write $X = \{v_i : i = 1, 3, \dots, p-1\}$ and $Y = \{v_i : i = 2, 4, \dots, p\}$. Let $\alpha = 2t+1$ and define $f : V \to [1, 3p]$ as follows. For $i = 1, 3, \dots, p-1$, let

$$f(v_i) = \{\frac{i+1}{2}, \frac{3p+1+i}{2}, 3p-i\}$$

and

$$f(v_{i+1}) = \{\frac{p+i+1}{2}, \frac{2p+1+i}{2}, 3p+1-i\}.$$

We have $\sum_{u \in f(v_i)} u = \frac{9p+2}{2}$ and $\sum_{u' \in f(v_{i+1})} u' = \frac{9p+4}{2}$. If $t = 1$ we are done. If $t > 1$, the remaining numbers from the set $[3p+1, p\alpha]$ may be partitioned into pairs $\{a_1, a_2\}$ such that $a_1 + a_2 = (\alpha+3)p + 1 =: \sigma$. Add $t-1$ of these pairs to the set of labels for each vertex. We have,

$$w(e) = 9p + 3 + (t-1)\sigma$$

for every edge $e \in E$, so we have proved the theorem. □

Theorem 2.5. *Every graph admits an* SEMT(2, 2) *labeling.*

Proof. Let $G = (V, E)$ be a (p, q) graph and let $S_1 = [1, 2p]$ and $S_2 = [2p+1, 2(p+q)]$. Partition S_1 into p pairs $\{s_1, s_1'\}$ such that $s_1 + s_1' = 2p+1 =: \sigma_1$, and give one such pair to every vertex of G. Then partition S_2 into pairs $\{s_2, s_2'\}$ such that $s_2 + s_2' = 2(2p+q)+1 =: \sigma_2$, and give one such pair to every edge of G. It is easy to see that the weight of every edge $e \in E$ is $w(e) = 2\sigma_1 + \sigma_2$, so the labeling described is an SEMT(2, 2) labeling of G. □

Theorem 2.6. *If a graph G admits an* (S)EMT(α, β) *labeling, then G admits an* (S)EMT$(\alpha + 2c, \beta + 2d)$ *labeling for any integers c and d.*

Proof. Let $c, d \geq 0$ and $G = (V, E)$ be a (p, q) graph with (S)EMT(α, β) labeling f having associated magic constant μ. Observe that $f(V \cup E) = [1, p\alpha + q\beta]$. We will define a type $(\alpha + 2c, \beta + 2d)$ labeling g as follows.

Case 1. f is an SEMT(α, β) labeling.
We have $f(V) = [1, p\alpha]$. Let $S_1 = [p\alpha+1, p(\alpha+2c)]$ and $S_2 = [p\alpha+q\beta+2cp+1, p(\alpha+2c)+q(\beta+2d)]$. Partition S_1 into constant sum pairs $\{s, s'\}$ such that $s+s' = 2p(\alpha+c)+1 =: \sigma_1$. Let $\mathcal{P}_1 = \{\{s_{v,i}, s'_{v,i} : v \in V, i \in [1, c]\}$ be one of these partitions of S_1. Partition S_2 into constant sum pairs $\{t, t'\}$ such that $t+t' = 2p(\alpha+2c)+2q(b+d)+1 =: \sigma_2$. Let $\mathcal{P}_2 = \{\{t_{e,i}, t'_{e,i} : e \in E, i \in [1, d]\}$ be one of these partitions of S_2.

Then for every vertex $v \in V$, let

$$g(v) = f(v) \cup \{s_{v,i}, s'_{v,i} : i \in [1, c]\},$$

and for every edge $e \in E$, let

$$g(e) = f(e) + 2cp \cup \{t_{e,i}, t'_{e,i} : i \in [1, d]\}.$$

Therefore, the weight of each edge is $w(e) = \mu + 2cp\beta + \sigma_1 c + \sigma_2 d$, so g is an SEMT$(\alpha + 2c, \beta + 2d)$ labeling of G.

Case 2. f is not an SEMT(α, β) labeling.
Similar to the previous case, partition the set $S = [p\alpha + q\beta + 1, (\alpha + 2c)p + (\beta + 2d)q]$ of new labels into $c + d$ pairs $\{s, s'\}$ such that $s + s' = 2(\alpha + c)p + 2(\beta + d)q + 1 =: \sigma$. Then for every $v \in V$, let

$$g(v) = f(v) \cup \{s_i, s'_i : s_i, s'_i \in S, s_i + s'_i = \sigma, i \in [1, c]\}.$$

and for every edge $e \in E$, let

$$g(e) = f(e) \cup \{s_i, s'_i : s_i, s'_i \in S, s_i + s'_i = \sigma, i \in [c + 1, c + d]\}.$$

Therefore, the weight of each edge is $w(e) = \mu + \sigma(c + d)$, so g is an EMT$(\alpha + 2c, \beta + 2d)$ labeling of G.

$\square$

3 Paths

In this section, we will show that the path P_n is an SEMT(α, β) graph for any $\alpha, \beta \geq 1$. In addition, we will determine when the path P_n is either an SEMT$(\alpha, 0)$ graph or an SEMT$(0, \beta)$ graph. For the proofs in this section, assume $G = (V, E) \cong P_n$ with $V = \{v_i : 1 \leq i \leq n\}$ and $E = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. If $\alpha = 0$ or $\beta = 0$, the results of the previous section can be applied to yield the following two observations.

Observation 3.1. *The path P_n admits an* EMT$(0, \beta)$ *labeling if and only if β is even or $\beta \geq 3$ is odd and n is even.*

Proof. The proof follows directly from Observation 2.2. □

Observation 3.2. *The path P_n admits an* SEMT$(\alpha, 0)$ *for all $\alpha, n \geq 2$.*

Proof. If n is even, then P_n is a balanced bipartite graph so the proof follows directly from the fourth condition of Theorem 2.4. If n is odd, the proof follows from the first or second condition of the same theorem. □

It is easy to see that no graph admits an EMT$(0, 1)$ labeling and P_n admits an EMT$(1, 0)$ labeling if and only if $n = 2$. Having classified paths for $\alpha = 0$ or $\beta = 0$, we now assume $\alpha, \beta \geq 1$. Due to Theorem 2.6, if α and β are of different parities it suffices to provide a labeling for $(\alpha, \beta) = (1, 2)$ or $(2, 1)$.

Lemma 3.3. *The path P_n admits an* SEMT$(1, 2)$ *labeling for any $n \geq 2$.*

Proof. For $i \in [1, n]$, let

$$f(v_i) = i$$

and

$$f(v_i v_{i+1}) = \{2n - i, 3n - (i + 1)\}.$$

The weight of each edge is clearly $5n$ and $f(V(G)) = [1, n]$ so we have proved the lemma. □

Lemma 3.4. *The path P_n admits an* SEMT$(2, 1)$ *labeling for any $n \geq 2$.*

Proof. For $i \in [1, n-1]$, let

$$f(v_i v_{i+1}) = 3n - i.$$

If n is even, let

$$f(v_i) = \begin{cases} \{i, \frac{4n+1-i}{2}\}, & i = 1, 3, \ldots, n-1 \\ \{i, \frac{3n+2-i}{2}\}, & i = 2, 4, \ldots, n \end{cases}.$$

Otherwise, let

$$f(v_i) = \begin{cases} \{i, \frac{3n+2-i}{2}\}, & i = 1, 3, \ldots, n \\ \{i, \frac{4n+2-i}{2}\}, & i = 2, 4, \ldots, n-1 \end{cases}.$$

Let $e = v_i v_{i+1} \in E$. If n is even, then

$$\begin{array}{rcl} w(e) & = & 2i + 1 + \frac{7n+2-2i}{2} + 3n - i \\ & = & \frac{13n+4}{2}, \end{array}$$

and when n is odd, we have

$$\begin{array}{rcl} w(e) & = & 2i + 1 + \frac{7n+3-2i}{2} + 3n - i \\ & = & \frac{13n+5}{2}. \end{array}$$

In either case, $w(e)$ is independent of i and $f(V(G)) = [1, n]$ so we have proved the lemma. □

We have now generalized Theorem 1.1 to say the following.

Theorem 3.5. *The path P_n admits an* $\mathrm{SEMT}(\alpha, \beta)$ *for any* $\alpha, \beta \geq 1$.

Proof. If $\alpha \equiv \beta \pmod 2$, the proof follows from Theorems 1.1, 2.5, and 2.6. Otherwise, the proof follows from Lemmas 3.3 and 3.4, and Theorem 2.6. □

4 Cycles

We turn our attention to cycles and proceed in much the same way as the previous section to classify α and β such that the cycle C_n is (S)EMT(α, β). For the proofs in this section, assume $G = (V, E) \cong C_n$ where $V = \{v_i : 1 \leq i \leq n\}$ and $E = \{v_i v_{i+1} : 1 \leq i \leq n\}$, with arithmetic taken modulo n in the subscript. We begin with a necessary condition.

Theorem 4.1. *If the cycle C_n admits an* $\mathrm{SEMT}(\alpha, \beta)$ *for some $\alpha \geq 0$ and $\beta \geq 1$, then n is odd or β is even.*

Proof. The proof is a simple counting argument. Suppose such a labeling exists and let $S_1 = [1, \alpha n]$ and $S_2 = [\alpha n + 1, (\alpha + \beta)n]$. Because each vertex label is counted twice, each edge label only once, and all n edges have the same weight, we have

$$\begin{aligned} w(e) &= \frac{\sum_{s\in S_2} s + \sum_{t\in S_1} t}{n} \\ &= \frac{2\alpha((\alpha+\beta)n+1)+\beta(\beta n+1)}{2}. \end{aligned}$$

Therefore, $2|\beta$ or $2|(\beta n+1)$. Hence, β is even or n is odd. □

If $\alpha = 0$ or $\beta = 0$, we obtain the following two results.

Observation 4.2. *The cycle, C_n admits an* $\mathrm{EMT}(0,\beta)$ *labeling if and only if β is even or $\beta \geq 3$ and n are both odd.*

Proof. The proof follows directly from Observation 2.2. □

Observation 4.3. *The cycle, C_n admits an* $\mathrm{SEMT}(\alpha,0)$ *for all $\alpha, n \geq 2$.*

Proof. The proof follows directly from Theorem 2.4. □

From now on, we assume $\alpha, \beta \geq 1$.

Lemma 4.4. *The cycle C_n admits an* $\mathrm{SEMT}(1,2)$ *for $n \geq 3$.*

Proof. We prove the theorem by constructing a type $(1,2)$ labeling f as follows. For convenience, we let $v_{n+1} = v_1$. For $i \in [2,n]$, let

$$\begin{aligned} f(v_i) &= i-1 \\ f(v_iv_{i+1}) &= \{2n+2-i, 3n+1-i\}, \end{aligned}$$

and define $f(v_1) = n$ and $f(v_1v_2) = \{n+1, 3n\}$. If $e = v_iv_{i+1} \in E$, then

$$w(e) = 5n+2.$$

Since $f(V) = [1,n]$ and $w(e)$ is independent of i, we have completed the proof. □

Lemma 4.5. *The cycle C_n admits an* $\mathrm{SEMT}(2,1)$ *for any odd $n \geq 3$.*

Proof. We prove the theorem by constructing a type $(2,1)$ labeling f as follows. For $i \in [1,n]$, let

$$t_i = \begin{cases} \frac{3n+2-i}{2}, & \text{if } i \text{ is odd} \\ \frac{4n+2-i}{2}, & \text{if } i \text{ is even} \end{cases},$$

$$f(v_i) = \{i, t_i\},$$

and

$$f(v_i v_{i+1}) = 3n - i.$$

Define $f(v_n v_1) = 3n$. If $e = v_i v_{i+1} \in E$, then

$$w(e) = \frac{13n+5}{2},$$

so we have completed the proof. □

Lemma 4.6. *The cycle C_n admits an* EMT$(2,1)$ *for any even $n \geq 4$.*

Proof. We construct a type $(2,1)$ labeling f as follows.

Case 1. $n \equiv 0 \pmod 4$.
Let

$$f(v_i v_{i+1}) = \begin{cases} 2i & \text{for } i = 1, 2 \\ 4(i-1) & \text{for } i = 3, 4, \ldots, \frac{n}{2}+1 \\ 4(n-i)+6 & \text{for } i = \frac{n}{2}+2, \ldots, n \end{cases}.$$

Then label the vertices as follows. Let $f(v_1) = \{\frac{5n}{2}, 3n\}$ and

$$f(v_i) = \begin{cases} \{\frac{5n}{2}+2-i, 3n+2-i\} & \text{for } i = 3, 5, \ldots, \frac{n}{2}+1 \\ \{i+\frac{3n}{2}-1, i+2n-1\} & \text{for } i = \frac{n}{2}+3, \frac{n}{2}+5, \ldots, n-1 \\ \{\frac{n}{2}+3-i, 2n+1-i\} & \text{for } i = 2, 4, \ldots, \frac{n}{2}+2 \\ \{i-1, i-3+\frac{n}{2}\} & \text{for } i = \frac{n}{2}+4, \frac{n}{2}+6, \ldots, n \end{cases}.$$

For any edge $v_i v_{i+1} = e$, it is straight forward to check that $w(e) = 8n+2$.

Case 2. $n \equiv 2 \pmod 4$.
Let $f(v_1 v_2) = 2$, $f(v_{\frac{n}{2}+1} v_{\frac{n}{2}+2}) = 3n-4$, and

$$f(v_i v_{i+1}) = \begin{cases} 6i-8 & \text{for } i = 2, 3, \ldots, \frac{n}{2} \\ 6(n+1-i) & \text{for } i = \frac{n}{2}+2, \ldots, n \end{cases}.$$

Then label the vertices as follows. Let $f(v_1) = \{\frac{3n}{2} + 2, 3n - 2\}$, $f(v_2) = \{3, 3n - 1\}$, $f(v_3) = \{\frac{3n}{2} - 2, 3n\}$, and

$$f(v_i) = \begin{cases} \{3n + 14 - 6i, \frac{3n}{2} - 7 + 3i\} & \text{for } i = 5, 7, \ldots, \frac{n}{2} \\ \{6i - 3n - 4, \frac{9n}{2} + 1 - 3i\} & \text{for } i = \frac{n}{2} + 2, \ldots, n - 1 \\ \{3n + 15 - 6i, 3i - 7\} & \text{for } i = 4, 6, \ldots, \frac{n}{2} + 1 \\ \{6i - 3n - 3, 3n + 1 - 3i\} & \text{for } i = \frac{n}{2} + 3, \ldots, n \end{cases}.$$

It is again straightforward to check that for any edge $v_i v_{i+1} = e$, we have $w(e) = \frac{15n}{2} + 4$. Therefore, we have proved the claim. □

Figure 1 (a) shows an $\mathrm{EMT}(2, 1)$ labeling of C_8. We can now conclude the following about edge-magic type (α, β) labelings of cycles.

Theorem 4.7. *Suppose $\alpha \geq 1$, $\beta \geq 0$, and $n \geq 3$ are integers. Then the cycle C_n admits an $\mathrm{SEMT}(\alpha, \beta)$ labeling if and only if n is odd or β is even. Otherwise, C_n admits an $\mathrm{EMT}(\alpha, \beta)$ labeling.*

Proof. Theorem 4.1 provides the necessary conditions for the $\mathrm{SEMT}(\alpha, \beta)$ labeling. If n is odd, the proof follows from Theorem 1.3 and Lemmas 4.4 and 4.5 and Theorem 2.6. If β is even and α is odd, the proof follows from Lemma 4.4 and Theorem 2.6. If α and β are both even, the proof follows from Theorems 2.5 and 2.6. Otherwise, Lemma 4.4 and Theorems 1.2 and 2.6 complete the proof. □

5 Application to face-magic labelings

Figure 1 illustrates how elegantly the labeling introduced in the previous section translates to a face-magic type labeling of a related graph; the weight carried by an edge in Figure 1(a) corresponds to the weight carried by a face (minus the hub's label) in Figure 1(b). In this section, we use edge-magic type (α, β) labelings to find new results for face-magic labelings of type (a, b, c).

Let $G = (V, E)$ be a graph and H a subgraph of G such that every edge of G is contained in a subgraph isomorphic to H. A bijection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ is called *H-magic* if there exists a constant $k(f)$ such that $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = k(f)$ for every $H' = (V', E') \cong H$ [5]. Clearly, cycle-magic labelings and face-magic labelings of type $(1, 1, 0)$ are close relatives. The distinction between the two is the relationship

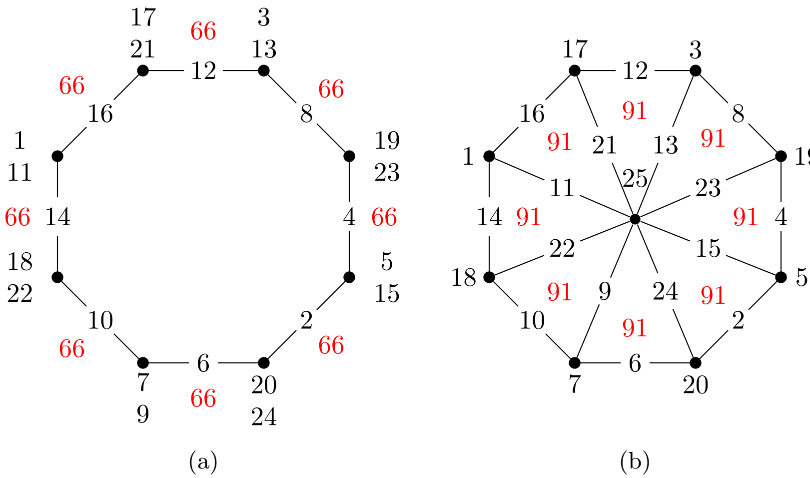

Figure 1: Weights (shown in red) of (a) an edge-magic type $(2, 1)$ labeling of C_8 and (b) a corresponding type $(1, 1, 0)$ face-magic labeling of the wheel graph W_8.

to the embedding of G; an H-magic labeling of G is independent of this embedding, while a type (a, b, c) face-magic labeling of G is not. Still, many cycle-magic labelings are indeed face-magic type $(1, 1, 0)$ labelings. For example, a C_3-magic labeling of the wheel W_n is a face-magic type $(1, 1, 0)$ labeling. However, it need not be the case that a C_4-magic labeling of an n-prism $(n \neq 4)$ is a face-magic labeling of type $(1, 1, 0)$ since an n-prism contains two n-sided faces in addition to the 4-sided faces, assuming the natural embedding in the plane.

In general, the process of subdividing a graph is the replacement of its edges with paths of a given length. In this section, we demonstrate how edge-magic type (α, β) labelings of paths and cycles can be used to completely answer the type (a, b, c) face-magic spectrum question for some families of subdivided graphs.

Rizvi et al. defined the following in [13]. An edge is *good* if it belongs to exactly one subgraph isomorphic to H. Let S be the collection of good edges obtained from taking $s \geq 1$ good edges from every subgraph isomorphic to H. The *uniform subdivided graph* is the graph obtained by subdividing every edge in S with $k \geq 1$ vertices. A *non-uniform subdivided graph* is obtained by subdividing every edge of $E \setminus S$. They proved the following.

Theorem 5.1. [13] *Let G be a C_n-supermagic graph. Then its uniform subdivided graph is C_{n+sk}-magic for $s, k \geq 1$.*

They provide some cycle-magic labelings for non-uniformly divided fans and triangular ladders and pose the open problem: *If a graph has a cycle-(super) magic labeling, determine whether its non-uniform subdivided graph has a cycle-(super) magic labeling.* Our results in this section answer this question for chains, ladders, and wheels.

Many of the labelings we will describe call for an arbitrary bijection between elements of two sets. For two sets A and B of the same cardinality, we will use a phrase such as, "let $f(A) = B$" to denote any such bijection $f : A \to B$. Also, for any set S, let $1S = S$ and $0S = \emptyset$.

5.1 Chains

Consider the graph consisting of k copies of C_n such that the i^{th} and $i+1^{\text{st}}$ copies of C_n are connected by a bridge for $i = 1, 2, \ldots, k-1$. Contracting every bridge results in a graph Ngurah et al. call a *kC_n-path* [12]. Let $v_1, v_2, \ldots, v_{k-1}$ be the cut-vertices of kC_n, and let d_i be the distance v_i to v_{i+1} for $1 \leq i \leq k-2$. We call $(d_1, d_2, \ldots, d_{k-2})$ the *string* of kC_n. Since we only consider face-magic labelings of type (a, b, c) on kC_n, the results of this section are independent of the string of kC_n. See Figure 2 for an example of a $4C_5$-path. Ngurah et al. proved the following.

Theorem 5.2. [12] *For any integers $k \geq 2$ and $n \geq 3$, the kC_n-path admits a super face-magic labeling of type* $(1, 1, 0)$.

We will use EMT(α, β) labelings of paths to answer the type (a, b, c) face-magic spectrum question for these graphs next. We include the type $(1, 1, 0)$ in our result for completeness and because our proof of that case is shorter than the proof in [12].

Theorem 5.3. *For any integers $a, b, c \in \{0, 1\}$, $k \geq 2$, and $n \geq 3$, the kC_n-path admits a face-magic labeling of type (a, b, c), except in the following cases.*

- $a = b = 0$.
- $a = c = 0$, $b = 1$*; n is odd and k is even.*
- $a = 0$, $b = c = 1$*; n and k are even.*

Proof. Since it is obvious that no graph admits a face-magic labeling of type $(0,0,1)$, we may assume that $(a,b) \neq (0,0)$. Let $G = (V,E,F)$ be a kC_n-path embedded in the plane in the natural way and denote the k C_n-components by B_i for $i \in [1,k]$. Let v_i be the shared vertex of B_i and B_{i+1} for $i \in [1,k-1]$, and let $v_0 \neq v_1$ and $v_k \neq v_{k-1}$ be any vertex in B_1 and B_k, respectively. Let F_∞ be the exterior face. We denote the path P_m by $x_0, x_1, \ldots, x_{m-1}$. For each triple (a,b,c), we describe a bijective labeling $f_{(a,b,c)} : aV \cup bE \cup cF \to [1, a|V| + b|E| + c|F|]$ as follows.

Case 1. Type $(1,0,0)$.
Let λ_1 be an EMT$(1, n-2)$ labeling of P_{k+1} with magic constant μ_1. Define $f_{(1,0,0)}(v_i) = \lambda(x_i)$ for $i \in [0,k]$, and let $f_{(1,0,0)}(V(B_i \setminus \{v_i \cup v_0\}) = \lambda_1(x_{i-1}x_i)$ for $i \in [1,k]$.

Case 2. Type $(0,1,0)$.
Let λ_2 be an EMT$(0,n)$ labeling of P_{k+1} with magic constant μ_2. Define $f_{(0,1,0)}(E(B_i)) = \lambda_2(x_{i-1}x_i)$ for $i \in [1,k]$.

Case 3. Type $(0,1,1)$.
Let λ_3 be an EMT$(0, n+1)$ labeling of P_{k+1} with magic constant μ_3. Let $f_{(0,1,1)}(E(B_i) \cup F(B_i)) = \lambda_3(x_{i-1}x_i)$ for $i \in [1,k]$ and $f_{(0,1,1)}(F_\infty) = |E| + |F|$.

Case 4. Type $(1,1,0)$.
Let λ_4 be an EMT$(1, 2n-2)$ labeling of P_{k+1} with magic constant μ_4. Let $f_{(1,1,0)}(v_i) = \lambda(x_i)$ for $i \in [0,k]$, and $f_{(1,1,0)}(E(B_i) \cup V(B_i \setminus \{v_i \cup v_0\}) = \lambda_4(x_{i-1}x_i)$ for $i \in [1,k]$.

Case 5. Type $(1,0,1)$.
Let λ_5 be an EMT$(1, n-1)$ labeling of P_{k+1} with magic constant μ_5. Let $f_{(1,0,1)}(v_i) = \lambda(x_i)$ for $i \in [0,k]$, and $f_{(1,0,1)}(F(B_i) \cup V(B_i \setminus \{v_i \cup v_0\})) = \lambda_5(x_{i-1}x_i)$ for $i \in [1,k]$. Then let $f_{(1,0,1)}(F_\infty) = |V| + |F|$.

Case 6. Type $(1,1,1)$.
Let λ_6 be an EMT$(1, 2n-1)$ labeling of P_{k+1} with magic constant μ_6. Let $f_{(1,1,1)}(v_i) = \lambda(x_i)$ for $i \in [0,k]$, and $f_{(1,1,1)}(E(B_i) \cup F(B_i) \cup V(B_i \setminus \{v_{i-1}, v_i\})) = \lambda_6(x_{i-1}x_i)$ for $i \in [1,k]$. Then let $f_{(1,1,1)}(F_\infty) = |V| + |E| + |F|$.

Since $\lambda_1, \lambda_4, \lambda_5$, and λ_6 exist by Theorem 3.5, and Observation 2.2 provides the necessary and sufficient conditions for the existence of λ_2 and λ_3, the weight of every n-sided face of G under $f_{(a,b,c)}$ is μ_i, and the weight of the external face is irrelevant (since $k \geq 2$), we have described a face-magic labeling of G in every case. $\square$

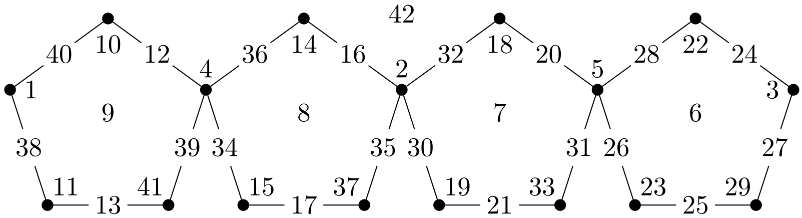

Figure 2: A type $(1,1,1)$ face-magic labeling of the $4C_5$-path

The natural next step is to chain the cycles together in a cycle. Consider the graph consisting of k copies of C_n such that the i^{th} and $i+1^{\text{st}}$ copies of C_n are connected by a bridge for $i = 1, 2, \ldots, k$, with i taken modulo k. Contracting every bridge results in a graph we will call a *kC_n-cycle.*

Theorem 5.4. *For any integers $a, b, c \in \{0, 1\}$, $n, k \geq 3$, the kC_n-cycle admits a face-magic labeling of type (a, b, c), except in the following cases.*

- *$a = b = 0$.*
- *$a = c = 0$, $b = 1$; n and k are both odd.*
- *$a = 0$, $b = c = 1$; n even and k is odd.*

Proof. Since $k \geq 2$, we may assume that $(a, b) \neq (0, 0)$. Let $G = (V, E, F)$ be a kC_n-cycle embedded in the plane in the natural way and denote the k C_n-components by B_i for $i \in [1, k]$. Let v_i be the shared vertex of B_i and B_{i+1} for $i \in [1, k]$, with arithmetic in the subscript taken modulo k. Denote the cycle C_m by $x_0, x_1, \ldots, x_{m-1}, x_0$. The rest of the proof follows in the same way as the proof of Theorem 5.3 with the only exception of replacing each EMT(α, β) labeling λ_i of the path P_{k+1} with an EMT(α, β) labeling of the cycle C_k. Since these labelings exist by Theorem 4.7, we omit further details. $\square$

5.2 Ladders and subdivided ladders

Let $G \cong L_n \cong P_2 \square P_n$ be the ladder graph with $V(G) = \{u_i, v_i : i \in [1, n]\}$ and $E(G) = S \cup R$ where $S = \{u_i u_{i+1}, v_i v_{i+1} : i \in [1, n-1]\}$ are the *sides* of the ladder and $R = \{u_i v_i : i \in [1, n]\}$ is the set of *rungs*. Embed G in the plane in accordance with its namesake and denote by $F_1, F_2, \ldots, F_{n-1}$, the $(n-1)$ 4-sided faces in the natural order and F_∞, the exterior $2n$-sided face.

Bača (type $(1,1,1)$) and Ngurah (type $(1,1,0)$) investigated face-magic labelings of type (a,b,c) for ladders in [3] and [12], respectively. Their results follow.

Theorem 5.5. [3] *The ladder $L_n \cong P_2 \square P_n$ admits super face-magic labelings of types $(1,0,0)$ and $(1,1,1)$ for all even $n \geq 2$.*

Theorem 5.6. [12] *The ladder $L_n \cong P_2 \square P_n$ admits a super face-magic labeling of type $(1,1,0)$ for all $n \geq 2$.*

The *subdivided ladder graph* $L_n(r,s)$ is the graph that results from subdividing every rung with $r \geq 0$ vertices and every side with $s \geq 0$ vertices. Due to Theorems 5.1 and 5.6, a face-magic labeling of type $(1,1,0)$ can be found for the graph that results from subdividing the exterior edges (all of the sides and the first and last rung) of L_n with $k \geq 1$ vertices for $n \geq 2$.

Our next result shows how one can use an EMT(α, β) labeling of P_n to obtain face-magic labelings of all types for every subdivided ladder. We emphasize that since we allow $r = s = 0$, the next theorem provides new results for ladders L_n as well.

Theorem 5.7. *Let $a, b, c \in \{0, 1\}$. The subdivided ladder graph $L_n(r,s)$ admits a face-magic labeling of type (a,b,c) for all $n \geq 2$, and $r, s \geq 0$, unless $a = b = 0$.*

Proof. Obviously no graph admits a type $(0,0,1)$ face-magic labeling, so assume a and b are not both 0. Let $G \cong L_n(r,s) = (V, E, F)$ be embedded in the plane in the natural way. If $n = 2$, the labeling is trivial, so we may assume $n \geq 3$. For $i \in [1, n]$, let $R_i \cong P_{r+2}$ denote the i^{th} rung and for $j \in [1, n-1]$, let $S_j \cong 2P_{s+2}$ denote the pair of j^{th} sides. For the purposes of the labelings that follow, a vertex v that belongs to both a rung R_i and a side, we make the convention that $\{v\} \in R_i$ only. So, $|V(R_i)| = r + 2$ and $|V(S_j)| = 2s$. Furthermore, $|V| = n(r+2) + 2s(n-1)$,

$|E| = n(r+1)+2(s+1)(n-1)$, and $F(G)$ contains $n-1$ $(2s+2r+4)$-sided faces, and one external $[2(r+1)+2(s+1)(n-1)]$-sided face. We proceed by describing a bijective labeling $f_{(a,b,c)} : aV \cup bE \cup cF \to [1, a|V|+b|E|+c|F|]$ for each type. We will denote the path P_n by $x_1, x_2, \ldots, x_n$.

Case 1. Type $(1,0,0)$.
Let λ_1 be an EMT$(r+2, 2s)$ labeling of P_n with magic constant μ_1. Let $f_{(1,0,0)}(V(R_i)) = \lambda_1(x_i)$ for $i \in [1,n]$, and $f_{(1,0,0)}(V(S_i)) = \lambda_1(x_i x_{i+1})$ for $i \in [1, n-1]$.

Case 2. Type $(0,1,0)$.
Let λ_2 be an EMT$(r+1, 2s+2)$ labeling of P_n with magic constant μ_2. Let $f_{(0,1,0)}(E(R_i)) = \lambda_2(x_i)$ for $i \in [1,n]$, and $f_{(0,1,0)}(E(S_j)) = \lambda_2(x_j x_{j+1})$ for $j \in [1, n-1]$.

Case 3. Type $(0,1,1)$.
Let λ_3 be an EMT$(r+1, 2s+3)$ labeling of P_n with magic constant μ_3. Let $f_{(0,1,1)}(E(R_i)) = \lambda_3(x_i)$ for $i \in [1,n]$. and $f_{(0,1,1)}(E(S_j) \cup F_j) = \lambda_3(x_j x_{j+1})$ for $j \in [1, n-1]$. Then give the external face the largest label by defining $f_{(0,1,1)}(F_\infty) = |E| + |F|$.

Case 4. Type $(1,0,1)$.
Let λ_4 be an EMT$(r+2, 2s+1)$ labeling of P_n with magic constant μ_4. Let $f_{(1,0,1)}(V(R_i)) = \lambda_4(x_i)$ and $f_{(1,0,1)}(V(S_j) \cup F_j) = \lambda_4(x_j x_{j+1})$ for $j \in [1, n-1]$. Then define $f_{(1,0,1)}(F_\infty) = |V| + |F|$.

Since for $i \in [1,4]$, λ_i exists by Theorem 3.5 (or Observation 2.3 in Case 1 when $s = 0$), the weight of interior face of G under $f_{(a,b,c)}$ is μ_i, and the weight of the external face is irrelevant (since $2(r+1) + 2(s+1)(n-1) > 2s + 2r + 4$), we have described a type (a,b,c) face-magic labeling of G in every case above. Since a translation of the labels preserves the equal weight property, the remaining cases, types $(1,1,0)$ and $(1,1,1)$, follow from Cases 1 and 2, and Cases 2 and 4, respectively. $\square$

Figure 3 shows a type $(1,0,0)$ face-magic labeling of $L_5(1,1)$.

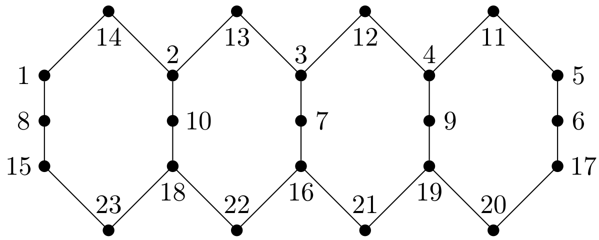

Figure 3: Type $(1, 0, 0)$ face-magic labeling of $L_5(1, 1)$

5.3 Fans and subdivided fans

Let $G \cong F_n$ the fan graph $P_n + K_1$ with $V(G) = \{v_i : i \in [1, n]\} \cup \{h\}$, where $\{h\}$ is called the *hub*, and $E(G) = S \cup R$ where $S = \{hv_i : i \in [1, n]\}$ are called *spoke* edges and $R = \{v_i v_{i+1} : i \in [1, n-1]\}$ are called *rim* edges.

Bača proved the following in 1987 [3].

Theorem 5.8. [3] *The fan F_n admits a face-magic labeling of type $(1, 1, 1)$ for $n \geq 2$.*

Twenty-five years later, Jeyanthi et al. showed fans are also type $(1, 1, 0)$ super face-magic [6].

Theorem 5.9. [6] *The fan F_n admits a super face-magic labeling of type $(1, 1, 0)$ for $n \geq 2$.*

The *subdivided fan graph $F_n(r, s)$* is the graph that results from subdividing every rim edge of F_n with $r \geq 0$ vertices and every spoke edge with $s \geq 0$ vertices. Of course, $F_n(0, 0) \cong F_n$. Rizvi et al. investigated type $(1, 1, 0)$ labelings for two special cases of subdivided fans [13]. They proved the following.

Theorem 5.10. [13] *Let $n \geq 3$. The subdivided fan graph $F_n(r, s)$ admits a face-magic labeling of type $(1, 1, 0)$ if $r = s \geq 1$ or $r = 0$.*

Since we allow $r = s = 0$, our next result completely answers the face-magic type (a, b, c) spectrum question for fans and subdivided fans.

Theorem 5.11. *Let $a, b, c \in \{0, 1\}$ and $r, s \geq 0$. The subdivided fan $F_n(r, s)$ admits a face-magic labeling of type (a, b, c) for any $n \geq 3$ unless $a = b = 0$.*

Proof. Obviously no graph admits a face-magic labeling of type $(0,0,1)$, we may assume $(a,b) \neq (0,0)$. Let $G=(V,E,F) \cong F_n(r,s)$ be embedded in the plane in the natural way. We have $|V| = sn + r(n-1) + n + 1$, $|E| = n(s+1) + (n-1)(r+1)$, and there are $n-1$ interior $(2s+r+3)$-sided faces and one exterior $(2(s+1)+(n-1)(r+1))$-sided face. Since $n \geq 3$, we may ignore the weight of the external face. Let $V_i^s \subseteq V \setminus \{h\}$, $E_i^s \subseteq E$ be the set of vertices, edges, respectively, associated with the spoke hv_i in $E(F_n)$. Similarly, let $V_i^r \subseteq V$, $E_i^r \subseteq E$ be the set of vertices, edges, respectively, associated with the rim v_iv_{i+1} in $E(F_n)$. For a vertex v on a spoke and on a rim, we include v in the set V_i^s only. For $i \in [1, n-1]$, denote by B_i the interior face corresponding to the face of F_n bounded by the edges hv_i, v_iv_{i+1}, and hv_{i+1}, and denote the exterior face B_∞. For each triple, (a,b,c), we will describe a bijective labeling $f_{(a,b,c)} : aV \cup bE \cup cF \to [1, a|V| + b|E| + c|F|]$ based on an EMT(α,β) labeling of the path $P_n \cong x_1, x_2, \ldots, x_n$.

Case 1. Type $(0,1,0)$.
Let λ_1 be an EMT$(s+1, r+1)$ labeling of P_n with magic constant μ_1. For $i \in [1,n]$, let $f_{(0,1,0)}(E_i^s) = \lambda_1(x_i)$ and for $i \in [1, n-1]$, let $f_{(0,1,0)}(E_i^r) = \lambda_1(x_ix_{i+1})$. Clearly the weight of every triangular face is μ_1.

Case 2. Type $(0,1,1)$.
Let λ_2 be an EMT$(s+1, r+2)$ labeling of P_n with magic constant μ_2. For $i \in [1,n]$, let $f_{(0,1,1)}(E_i^s) = \lambda_2(x_i)$ and for $i \in [1, n-1]$, let $f_{(0,1,1)}(E_i^r \cup B_i) = \lambda_2(x_ix_{i+1})$. Label the exterior face the largest label, $f_{(0,1,1)}(B_\infty) = |E| + |F|$. The weight of every interior face is $\mu_2 + |E| + |F|$.

Case 3. Type $(1,0,0)$.
Let λ_3 be an EMT$(s+1, r)$ labeling of P_n with magic constant μ_3. For $i \in [1,n]$, let $f_{(1,0,0)}(V_i^s) = \lambda_3(x_i)$ and for $i \in [1, n-1]$, let $f_{(1,0,0)}(V_i^r) = \lambda_3(x_ix_{i+1})$. Label the hub the largest label $f_{(1,0,0)}(h) = |V|$. The weight of every interior face is $\mu_3 + |V|$.

Case 4. Type $(1,0,1)$.
Let λ_4 be an EMT$(s+1, r+1)$ labeling of P_n with magic constant μ_4. For $i \in [1,n]$, let $f_{(1,0,1)}(V_i^s) = \lambda_4(x_i)$ and for $i \in [1, n-1]$, let $f_{(1,0,1)}(V_i^r \cup B_i) = \lambda_4(x_ix_{i+1})$. Label the hub $f_{(1,0,1)}(h) = |V| + |F| - 1$ and the exterior face $f_{(1,0,1)}(B_n) = |V| + |F|$. The weight of every interior face is $\mu_4 + 2(|V| + |F|) - 1$.

Since for $i \in [1,4]$, λ_i exists by Theorem 3.5 (or Observation 3.2 in Case 3 when $r = 0$), we have described a face-magic labeling of G in every case above. Since a translation of the labels preserves the equal weight property, the remaining cases, types $(1,1,0)$ and $(1,1,1)$, follow from Cases 1,3, and 4. $\square$

5.4 Wheels and subdivided wheels

Let $G \cong W_n$ the wheel graph $C_n + K_1$ with $V(G) = \{v_i : i \in [1,n]\} \cup \{h\}$, where $\{h\}$ is called the *hub*, and $E(G) = S \cup R$ where $S = \{hv_i : i \in [1,n]\}$ are called *spoke* edges and $R = \{v_iv_{i+1} : i \in [1,n]\}$ (with arithmetic in the subscript modulo n) are called *rim* edges.

In the first paper published on face-magic labelings of type (a,b,c), Lih [8] proved the following in 1983.

Theorem 5.12. [8] *The wheel W_n admits a face-supermagic labeling of type $(1,1,0)$ if $n \not\equiv 2 \pmod 4$.*

In 2007, Lladó and Moragas proved W_n admits a face-supermagic labeling of this type for $n \geq 5$ odd [9]. Six years later, Ali et al. obtained the following results [1].

Theorem 5.13. [1] *The wheel W_n admits a face-magic labeling of the following types.*

- $(1,1,1), n \geq 3$
- $(0,1,1), n \geq 3$
- $(0,1,0), n \geq 3$ *odd.*

Later that year, Roswitha et al. complemented the partial result of Lladó and Moragas by providing a face-supermagic labeling of type $(1,1,0)$ for W_n with n even [14]. The two results are combined in the next theorem.

Theorem 5.14. [9, 14] *The wheel W_n admits a face-supermagic labeling of type $(1,1,0)$ if $n \geq 4$.*

These results leave open the following cases; type $(1,0,0)$, type $(1,0,1)$, and type $(0,1,0)$ (partially open). We close these cases and more in Theorem 5.16. But first, we define an encompassing family of graphs and review the corresponding known results.

The *subdivided wheel graph* $W_n(r,s)$ is the graph that results from subdividing every rim edge of W_n with $r \geq 0$ vertices and every spoke edge with $s \geq 0$ vertices. Not only do wheels W_n belong to this family, but so too does the family of generalized Jahangir graphs. The *generalized Jahangir graph* $J_{n,d}$ is a graph on $nd+1$ vertices consisting of a cycle C_{nd} and one additional vertex that is adjacent to a set of n vertices on C_{nd} all of which are a distance d apart [14]. Notice that $J_{n,2}$ is the gear graph and $J_{n,d} \cong W_n(d-1,0)$.

Roswitha et al. proved the following in [14].

Theorem 5.15. [14] *The generalized Jahangir graph $J_{n,d}$ admits a face-magic type $(1,1,0)$ labeling if n is odd.*

Our next result answers the face-magic type (a,b,c) spectrum question for wheels and subdivided wheels. See Figure 4 for a type $(1,0,1)$ face-magic labeling of $W_6(1,3)$.

Theorem 5.16. *Let $a,b,c \in \{0,1\}$ and $r,s \geq 0$. The subdivided wheel $W_n(r,s)$ admits a face-magic labeling of type (a,b,c) for any $n \geq 3$ unless $a=b=0$.*

Proof. The proof is essentially the same as the proof of Theorem 5.11 after replacing every $\mathrm{EMT}(\alpha,\beta)$ labeling of P_n in that proof with an $\mathrm{EMT}(\alpha,\beta)$ labeling of C_n. Since these labelings exist by Theorem 4.7, we omit further details. $\square$

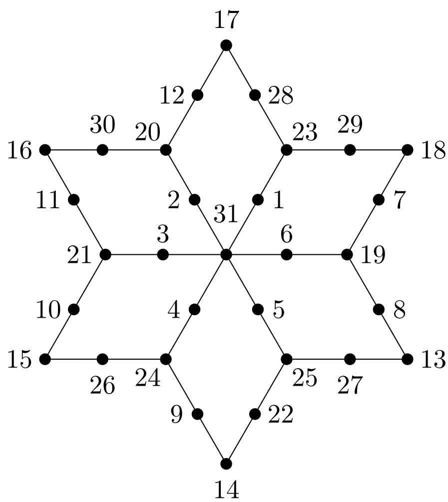

Figure 4: A type $(1,0,0)$ face-magic labeling of $W_6(1,3)$.

6 Conclusion

We have introduced a generalization of edge-magic total labelings and constructed such labelings for paths and cycles. We applied these results to answer the spectrum question for face-magic labelings of type (a, b, c) for some infinite families of graphs. An answer to the problem below would yield face-magic labelings of type (a, b, c) for more families of graphs using similar techniques as ours.

Open Problem. For what integers α, β, k, n does an EMT(α, β) labeling of kP_n or kC_n exist?

References

[1] K. Ali, M. Hussain, A. Ahmad and M. Miller, Magic Labelings of type (a, b, c) of families of wheels, *Math. Computer Sci.*, **7** (2013), 315–319.

[2] S. Avadayappan, R. Vasuki and P. Jeyanthi, Magic strength of a graph, *Indian J. Pure Appl. Math.*, **31** (2000), 873–883.

[3] M. Bača, On magic and consecutive labelings for the special classes of plane graphs, *Util. Math.*, **32** (1987), 59–65.

[4] H. Enomoto, A.S. Lladó, T. Nakamigawa and G.S. Ringel, Super edge-magic graphs, *SUT J. Math.*, **34** (1998), 105–109.

[5] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, (2020), **23**, DS6.

[6] P. Jeyanthi, P. Selvagopal and S.S. Sundaram, Some C_3-super magic graphs, *Util. Math.*, **89** (2012), 357–366.

[7] A.Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451–461.

[8] K. Lih, On magic and consecutive labelings of plane graphs, *Util. Math.*, **24** (1983), 165–197.

[9] A. Lladó, and J. Moragas, Cycle-magic graphs, *Discrete Math.*, **307** (2007), 2925–2933.

[10] A.M. Marr and W.D. Wallis, *Magic Graphs*, Birkhäuser, 2001.

[11] M. Miller, M., C. Rodger, and R. Simanjuntak, Distance magic labelings of graphs, *Austral. J. Combin.*, **28** (2003), 305–315.

[12] A.A.G. Ngurah, A.N.M. Salman and L. Susilowati, H-supermagic labeling of graphs, *Discrete Math.*, **310** (2010), 1293–1300.

[13] S.T.R. Rizvi, M. Khalid, K. Ali, M. Miller and J. Ryan, On cycle-supermagicness of subdivided graphs, *Bull. Aust. Math. Soc.*, **92(1)** (2015), 11–18.

[14] M. Roswitha, E.T. Baskoro, T.K. Maryati, N.A. Kurdhi, and I. Susanti, I. Further results on cycle-supermagic labeling, *AKCE Int. J. Graphs Comb.*, **10(2)** (2013) 211–220.

[15] W.D. Wallis, E.T. Baskoro and M. Miller, and Slamin, Edge-magic total labelings, *Australas. J. Combin.*, **22** (2000), 177–190.