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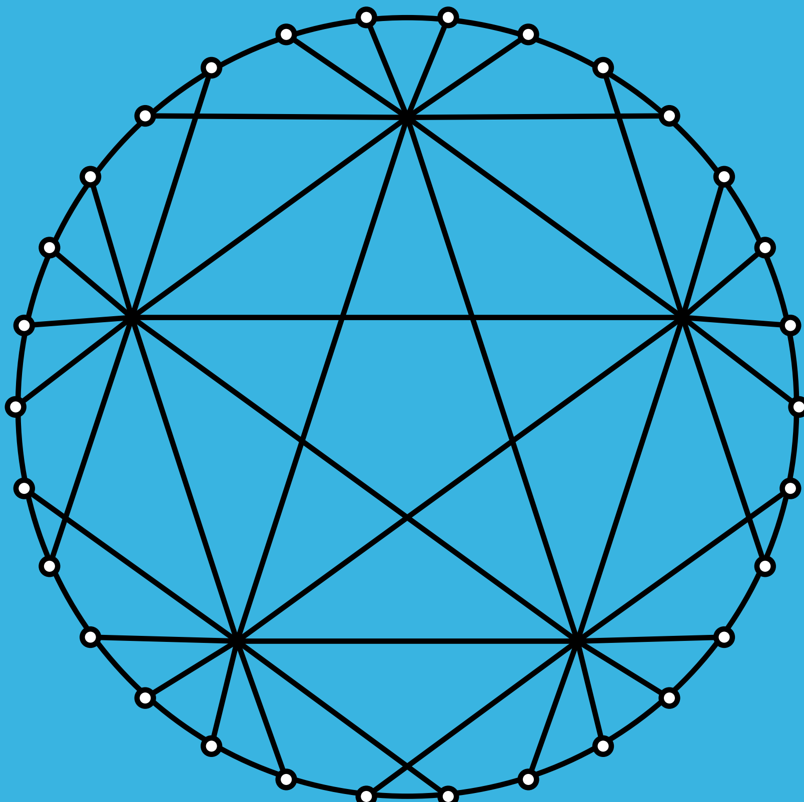

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Missing blocks of two designs in Hanani's paper

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Abstract. In a classical paper "On Some Tactical Configurations", Hanani proved that the necessary conditions are sufficient for the existence of 3-$(v, 4, \lambda)$ design for every λ. There he defined and constructed several configurations to help solve the general problem. One such design of a configuration is $Q''_6[4, 1, 18]$. Unfortunately: He gave only 108 blocks out of the required 189 blocks. Similarly he constructed a $Q''_3[4, 4, 15]$ and did not list 90 blocks. Main aim of the present note is to provide the missing 81 and 90 blocks in respective designs and to generalize a construction to obtain $Q''_{3t}[4, 2\lambda, 15t]$ for all positive integers λ, t.

1 Preliminaries

Definition 1.1. A group divisible design GDD$(n, m, k; \lambda_1, \lambda_2)$ is a collection of k-subsets, called blocks, of an nm-set X, where the elements of X are partitioned into m subsets (called groups) of size n each; pairs of distinct elements within the same group are called first associates and appear

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together in λ_1 blocks while any two elements not in the same group are called second associates and appear together in λ_2 blocks.

Another common notation for a GDD$(n, m, k; \lambda_1{=}0, \lambda_2)$ is (k, λ_2)-GDD(n^m).

Definition 1.2. A t-(v, k, λ) design, or a t-design, is a pair (X, B) where X is a v-set of points and B is a collection of k-subsets (blocks) of X with the property that every t-subset of X is contained in exactly λ blocks. The parameter λ is called the index of the design.

Hanani ([1], pp 706–707) proved that the necessary conditions are sufficient for the existence of 3-$(v, 4, \lambda)$ design, i.e., 3-$(v, 4, \lambda)$ design exists if and only if :

$$\begin{aligned}
&\lambda \equiv 1, 5, 7, \text{ or } 11 \pmod{12} \text{ and } v \equiv 2 \text{ or } 4 \pmod{6};\\
&\lambda \equiv 2 \text{ or } 10 \pmod{12} \text{ and } v \equiv 1, 2, 4, 5, 8, \text{ or } 10 \pmod{12};\\
&\lambda \equiv 3 \text{ or } 9 \pmod{12} \text{ and } v \equiv 0 \pmod{2};\\
&\lambda \equiv 4 \text{ or } 8 \pmod{12} \text{ and } v \equiv 1 \text{ or } 2 \pmod{3};\\
&\lambda \equiv 6 \pmod{12} \text{ and } v \equiv 0, 1, \text{ or } 2 \pmod{4};\\
&\lambda \equiv 0 \pmod{12}.
\end{aligned}$$

Sarvate and Bezire defined a 3-GDD$(n, 2, k; \lambda_1, \lambda_2)$ in [3] and obtained some necessary conditions for the case $k = 4$.

Definition 1.3. A 3-GDD$(n, 2, k; \lambda_1, \lambda_2)$ is a set X of $2n$ elements partitioned into two parts of size n called groups together with a collection of k-subsets of X called blocks, such that

(i) every 3-subset of each group occur in λ_1 blocks and

(ii) every 3-subset where two elements are from one group and one element from the other group occurs in λ_2 blocks.

They also gave the following fundamental construction.

Theorem 1.1. *A 3-GDD$(n, 2, 4; 0, 1)$ exists for even n and a 3-GDD$(n, 2, 4; 0, 2)$ exists for all positive integers n.*

Definition 1.3 is generalized below:

Definition 1.4. A 3-GDD$(n, m, k; \lambda_1, \lambda_2)$ is a set X of mn elements partitioned into m parts of size n called groups together with a collection of k-subsets of X called blocks, such that

(i) every 3-subset of configuration $(3, 0)$, i.e. where all 3 elements are from the same group occur in λ_1 blocks,

(ii) every 3-subset of configuration $(2, 1)$ where two elements are from one group and one element from the other group, or of configuration $(1, 1, 1)$, i.e. three elements from different groups, occurs in λ_2 blocks.

In addition, one may relax condition (ii) from the definition of 3-GDD and require that every 3-subset where each element is from a different group occurs in λ_3 blocks, where λ_3 may not be equal to λ_2 and we will denote such design by a 3-PBIBD$(n, m, k; \lambda_1, \lambda_2, \lambda_3)$ or a 3-GDD$(n, m, k; \lambda_1, \lambda_2, \lambda_3)$.

In fact, Hanani has used the concept of 3-GDDs with $\lambda_1 = 0$ to construct 3-$(v, 4, \lambda)$ designs. He used the notation

$$P_n^{''}[k, \lambda, nt] \text{ to represent 3-PBIBD}(n, t, k; 0, 0, \lambda)$$

and the notation

$$Q_n^{''}[k, \lambda, nt] \text{ to represent 3-PBIBD}(n, t, k; 0, \lambda, \lambda)$$

i.e. 3-GDD$(n, t, k; 0, \lambda)$. His following result and existence of designs such as $Q_6^{''}(4, 1)$ (i.e. 3-GDD$(6, 3, 4; 0, 1)$) was used in a note by the present authors [4] to construct some 3-GDDs with three groups and block size four.

Theorem 1.2. *([1], Proposition 5) If $n'|n$ and a 3-GDD$(n', t, 4; 0, 0, \lambda)$ exists then a 3-GDD$(n, t, 4; 0, 0, \lambda)$ exists.*

2 Hanani's design $Q_6''[4, 1, 18]$

The points and the blocks of a $Q_6''[4, 1, 18]$, (i.e. a 3-GDD$(6, 3, 4; 0, 1)$), given by Hanani in [1] on page 718 Equation (40), are

Points: (i, j), $i = 0, 1, \ldots, 5$, $j = 0, 1, 2$;

Blocks: { (a_0, j), (a_0+1, j), $(a_1, j+1)$, $(a_2, j+2)$ }, where $\sum_{h=0}^{2} a_h \equiv 2j \pmod 6$.

Here the groups are, say G_j=$\{(i, j),\ i = 0, 1, \ldots, 5\}$, $j = 0, 1, 2$.

These blocks account for 108 of the required 189 blocks. How can we construct the remaining 81 blocks? First we notice that the given blocks are all of the form $(2, 1, 1)$, meaning there are two elements (points) from one of the groups and a single point from each of the remaining two groups. All triples of the form $(1, 1, 1)$ occur only in these blocks. For the reader's convenience, we would like to demonstrate that all such triples are accounted in these blocks. First, we note that if $((a, 0), (b, 1), (c, 2))$ is a triple of the type $(1, 1, 1)$; consider the distinct sums $a + b + c - 1$ and $a + b + c$, as these are two consecutive integers, at least one of these sums is $0, 2$ or $4 \pmod 6$. There are three cases:

1. If $a+b+c = 0$ then the triple is in $\{(a, 0), (a+1, 0), (b, 1), (c, 2)\}$ and if $a + b + c = 1$ then it is in $(\{a - 1, 0), (a, 0), (b, 1), (c, 2)\}$.
2. If $a+b+c = 2$ then the triple is in $\{(b, 1), (b+1, 1), (c, 2), (a, 0)\}$ and if $a + b + c = 3$ then it is in $\{(b - 1, 1), (b, 1), (c, 2), (a, 0)\}$.
3. If $a+b+c = 4$ then the triple is in $\{(c, 2), (c+1, 2), (a, 0), (b, 1)\}$ and if $a + b + c = 5$ then it is in $\{(c - 1, 2), (c, 2), (a, 0), (b, 1)\}$.

Now as the total number of such blocks is 108 and the number of the triple of the type $(1, 1, 1)$ is $6^3 = 216$, every triple must occur exactly once. Notice that each block contains one edge of a K_6 labeled with the elements of G_j, which forms a 2-factor, namely the cycle

$$((0, j), (1, j), (2, j), (3, j), (4, j), (5, j), (0, j)),$$

for each $j = 0, 1$ and 2.

The remaining edges of the K_6, can be decomposed into 3 one-factors: for example, ignoring the second coordinates, we have

$$F_{1j} = \{((0, j), (2, j)), ((1, j), (4, j)), ((3, j), (5, j))\},$$
$$F_{2j} = \{((0, j), (3, j)), ((1, j), (5, j)), ((2, j), (4, j))\}$$

and

$$F_{3j} = \{((0, j), (4, j)), ((1, j), (3, j)), ((2, j), (5, j))\}.$$

Now we form the missing 81 blocks of the design as follows: form blocks of size 4 by taking union of each edge of F_{x1} with each edge of F_{x2} and F_{x3} and by taking the union of each edge of F_{x2} with each edge of F_{x3} for $x = 1, 2, 3$. In the process notice that each edge of these one factors is in six blocks.

3 Hanani's design $Q_3''[4, 4, 15]$

The points and the blocks of a $Q_3''[4, 4, 15]$, (i.e. a 3-GDD$(3, 5, 4; 0, 4)$), given by Hanani in [1] on page 712 Equation (17), are:

Points: (i, j), $(i = 0, 1, 2;\ j = 0, 1, 2, 3, 4)$;

Blocks:

$$\{(i, j), (i + 1, j), (i + \gamma, j - \alpha), (i + \gamma, j + \alpha)\}, (\alpha = 1, 2; \gamma = 0, 1, 2);$$
$$\{(i, j - \alpha), (i, j + \alpha), (i + 1, j - 2\alpha), (i + 1, j + 2\alpha)\}, (\alpha = 1, 2);$$
$$\{(i, j - \alpha), (i, j + \alpha), (i - \epsilon, j - 2\alpha), (i + \epsilon, j + 2\alpha)\}, (\alpha = 1, 2; \epsilon = \pm 1);$$

all these blocks are taken twice.

Here the groups are, say G_j=$\{(i, j),\ i = 0, 1, 2\}$, $j = 0, 1, 2, 3, 4$.

These blocks account for 360 of the required 450 blocks. How can we construct the missing 90 blocks?

We notice that the given blocks are all of the type $(2, 1, 1)$ or $(1, 1, 1, 1)$. Also, only the triples of the type $(2, 1)$ occur twice in the given blocks instead of four times as required. All other triples occur four times as required. We apply the fundamental construction 1.1 to get the missing blocks of type $(2, 2)$ on each pair of groups: $(G_i,\ G_j)$, $i < j$, $1 \leq i \neq j \leq 5$.

3.1 Generalization

Theorem 3.1. *A 3-GDD$(n = 3t, 5, 4; 0, 2\lambda)$ exists for all positive integers λ, t.*

Proof. We will construct a 3-GDD$(3t, 5, 4; 0, 2)$, λ copies of it will give a 3-GDD$(n = 3t, 5, 4; 0, 2\lambda)$. Given m groups of size n, consider the set of all triple of the type $(1,1,1)$. If a $(3,1)$-GDD(n^m) exists, a question can be asked: Can we partition these $\binom{m}{3}n^3$ triples into $(m-2)n$ $(3,1)$-GDD(n^m)? If such a partition exists then it is called a $(3,1)$-LGDD(n^m).

It is known, (see Niu, Cao and Javadi[2] and the references therein), that there exists a $(3,1)$-LGDD(n^m) if and only if $n(m-1) \equiv 0 \pmod 2$, $n^2m(m-1) \equiv 0 \pmod 6$, $m \geq 3$ and $(n,m) \neq (1,7)$.

Hence a $(3,1)$-LGDD$((3t)^3)$ exists with $3t$ disjoint GDDs. Now we will construct a 3-GDD$(3t, 5, 4; 0, 2)$ as follows.

Let the five groups of size $3t$ be G_i, $i = 1, 2, \ldots, 5$. Consider each set of four groups in turn as follows. Without loss of generality, let the set be $\{G_1, G_2, G_3, G_4\}$ and let $G_4 = \{x_1, x_2, \ldots, x_{3t}\}$. Construct the large set $(3,1)$-LGDD$(3t)^3$ with $3t$ disjoint GDDs with the set of blocks B_i, $i = 1, 2, \ldots, 3t$. Recall that the blocks in B_i are triples of the type $(1,1,1)$. Construct a subset of the blocks of the required design by taking union of each triple of B_i with the element x_i of G_4. Repeat this procedure with the remaining sets of the four groups namely $\{G_1, G_2, G_3, G_5\}$, $\{G_1, G_2, G_4, G_5\}$, $\{G_1, G_3, G_4, G_5\}$, and $\{G_2, G_3, G_4, G_5\}$.

Notice that these blocks are of the type $(1,1,1,1)$ and cover all $(1,1,1)$ type triples exactly twice. Hence the blocks constructed until now give a 3-PBIBD$(3t, 5, 4; 0, 0, \lambda = 2)$.

Now we use the fundamental construction 1.1, as used in the example above, to get a 3-PBIBD$(3t, 5, 4; 0, \lambda = 2, 0)$ and hence together we have the required 3-GDD$(3t, 5, 4; 0, 2)$. $\square$

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