A note on the Buratti-Horak-Rosa conjecture about hamiltonian paths in complete graphs

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Abstract. The conjecture posed by Buratti, Horak and Rosa states that a (multiset) list $L$ of $v - 1$ positive integers not exceeding $\lfloor v/2 \rfloor$ is the list of edge-lengths of a suitable Hamiltonian path of the complete graph with vertex-set $\{0, 1, \ldots, v - 1\}$ if and only if for every divisor $d$ of $v$, the number of multiples of $d$ appearing in $L$ is at most $v - d$. A list $L$ is called realizable if there exists such Hamiltonian path $P$ of the complete graph with $|L| + 1$ vertices whose edge-lengths is the given list $L$. If the initial and the final vertices in $P$ are 0 and $v - 1$, respectively, then $P$ is called perfect.

In this note, we show some properties of some perfect linear realizations. Also, we present a new operation over well-known linear realizations. For example, we give a linear realizations of the lists $\{1^a, 2^b, 4^c\}$, where $a, c \geq 1$ and $b \geq 3$ integers, $\{1^a, 2^b, 4^2c, 8^d\}$, for all $a, d \geq 1$, $b \geq 3$ and $c \geq 2$ integers, and $\{1^a, 2^b, 4^c, 8^d\}$, for all $a, d \geq 1$, $b \geq 3$ and $c \geq 8$ integers.

1 Introduction

Throughout the paper, $K_p$ will denote the complete graph on $p$ vertices, labeled by the integers of the set $\{0, 1, \ldots, p - 1\}$. For the basic terminology on graphs we refer to [1] and for basic facts about the Buratti-Horak-Rosa
conjecture we refer to [10]. The length of the edge $xy$, where $x, y \in V(K_p)$ is given by
\[
\ell(x, y) = \min\{|y - x|, p - |y - x|\}.
\]
Given a path $P = (x_0, x_1, \ldots, x_k)$, the list of edge-lengths of $P$ is the list $\ell(P)$ of the lengths (taken with their respective multiplicities) of all the edges of $P$. Hence, if a list $L$ consists of $a_1$ 1s, $a_2$ 2s, \ldots, $a_t$ ts, then we write $L = \{1^{a_1}, 2^{a_2}, \ldots, t^{a_t}\}$ and $|L| = \sum_{i=1}^{t} a_i$. The set $U_L = \{i : a_i > 0\} \subseteq L$ is called the underlying set of $L$.

The following conjecture was proposed in a private communication by Buratti to Rosa in 2007:

**Conjecture 1.1 (M. Buratti).** For any prime $p = 2n + 1$ and any list $L$ of $2n$ positive integers not exceeding $n$, there exists a Hamiltonian path $P$ of $K_p$ with $\ell(P) = L$.

Talking with Professor Buratti, the origin of this problem comes from the study of dihedral Hamiltonian cycle decompositions of the cocktail party graph (see comments before Corollary 3.19 in [2]).

Buratti’s conjecture is almost trivially true in the case when $|U_L| = 1$. On the other hand, the case of exactly two distinct edge-lengths has been solved independently by Dinitz and Janiszewski [4] and Horak and Rosa [5]. Using a computer, Meszka has verified the validity of Buratti’s conjecture for all primes $\leq 23$. Monopoli [6] showed that the conjecture is true when all the elements of the list $L$ appear exactly twice.

In [5] Horak and Rosa proposed a generalization of Buratti’s conjecture, which has been restated in an easier way in [9] as follows:

**Conjecture 1.2 (P. Horak and A. Rosa).** Let $L$ be a list of $v - 1$ positive integers not exceeding $\lfloor v/2 \rfloor$. Then there exists a Hamiltonian path $P$ of $K_v$ such that $\ell(P) = L$ if and only if the following condition holds:

\[
\text{for any divisor } d \text{ of } v, \text{ the number of multiples of } d \text{ appearing in } L \text{ does not exceed } v - d.
\]

The case of exactly three distinct edge-lengths has been solved when the underlying set is $U_L = \{1, 2, 3\}$ in [3], when $U_L$ is one of the sets
\[
\{1, 2, 5\}, \{1, 3, 5\}, \{2, 3, 5\}
\]
in [9], and when $U_L = \{1, 3, 4\}$ or $U_L = \{2, 3, 4\}$ in [8]. In [10] the authors give a complete solution when $U_L = \{1, 2, t\}$, where $t \in \{4, 6, 8\}$, and when $L = \{1^a, 2^b, t^c\}$ with $t \geq 4$ an even integer and $a + b \geq t - 1$. The case with four distinct edge-lengths for which the conjecture has been shown to be true is when $U_L = \{1, 2, 3, 4\}$ or $U_L = \{1, 2, 3, 5\}$, see [6] and [10]. Recently, Ollis et al. [8] proved some partial results in which $U_L = \{x, y, x + y\}$, $U_L = \{1, 2, 4, \ldots, 2x\}$ and $U_L = \{1, 2, 4, \ldots, 2x, 2x + 1\}$; many other lists were considered, see [8].

A cyclic realization of a list $L$ with $v - 1$ elements each from the set $\{1, 2, \ldots, \lfloor v/2 \rfloor\}$ is a Hamiltonian path $P$ of $K_v$ such that the multiset of edge-lengths of $P$ equals $L$. Hence, it is clear that the Conjecture 1.2 can be formulated as follow: every such a list $L$ has a cyclic realization if and only if condition (1,1) is satisfied.

**Example 1.** The path $P = (0, 1, 2, 3, 6, 4, 5, 7)$ is a cyclic realization of the list $L = \{1^4, 2^2, 3\}$.

A linear realization of a list $L$ with $v - 1$ positive integers not exceeding $v - 1$ is a Hamiltonian path $P = (x_0, x_1, \ldots, x_{v-1})$ of $K_v$ such that $L = \{|x_i - x_{i+1}| : i = 0, \ldots, v - 2\}$. The linear realization is standard if $x_0 = 0$ (see [8]). In this note we assume that any realization $P$ of a given list $L$ is standard. On the other hand, if $x_{v-1} = v - 1$, the (standard) linear realization is called perfect (see [3]).

**Example 2.** The path $P = (0, 2, 4, 6, 5, 3, 1, 7)$ is a perfect linear realization of the list $L = \{1^1, 2^5, 6\}$.

**Remark 1.** From the definitions presented before, it is not hard to see that any linear realization of a list $L$ can be viewed as a cyclic realization of a list $\hat{L}$ (not necessarily of the same list); however if all the elements in the list are less than or equal to $\lfloor \frac{|L|+1}{2} \rfloor$, then every linear realization of $L$ is also a cyclic realization of the same list $L$. For example, the path $P = (0, 5, 7, 8, 6, 4, 3, 1, 2)$ is a linear realization of the list $L = \{1^3, 2^4, 5\}$ and a cyclic realization of the list $\hat{L} = \{1^3, 2^4, 4\}$.

In this note, we show some properties of some perfect linear realizations. Also, we present a new operation over well-known linear realizations and we give several examples.
2 Some perfect linear realizations

Let \( P = (x_0, x_1, \ldots, x_{v-1}) \) and \( P' = (y_0, y_1, \ldots, y_{w-1}) \) be two paths (in general) such that \( V(P) \cap V(P') = \emptyset \). If \( x_{v-1} \) and \( y_0 \) are adjacent, then we can generate the path:

\[ P + P' = (x_0, x_1, \ldots, x_{v-1}, y_0, y_1, \ldots, y_{w-1}) \]

The path \( P + P' \) is also well-defined if \( x_{v-1} = y_0 \), in this case

\[ P + P' = (x_0, x_1, \ldots, x_{v-1}, y_1, \ldots, y_{w-1}) \]

**Theorem 2.1 ([3]).** Let \( P \) be a perfect linear realization of a list \( L \) and \( P' \) be a linear realization of the list \( L' \). Then there exists a linear realization \( P'' \) of the list \( L \cup L' \). Furthermore, if \( P' \) is also perfect, then \( P'' \) is perfect.

**Remark 2.** Let \( P = (x_0 = 0, x_1, \ldots, x_{v-1} = v - 1) \) be a perfect linear realization of a list \( L \). Applying the previous theorem to the perfect linear realization \((0, 1, \ldots, A) \) of \( \{1^A\} \), \( P' = P + (v - 1, v, \ldots, v - 1 + A) \) is a perfect linear realizations of \( L \cup \{1^A\} \), for all \( A \geq 0 \) integer, see [3].

Let \( P = (x_0, x_1, \ldots, x_{v-1}) \) be a path. For every \( k \in \mathbb{Z} \) integer, let \( \pi_k : \mathbb{Z} \to \mathbb{Z} \) given by \( \pi_k(x) = x + k \). Hence, if \( P = (x_0 = 0, x_1, \ldots, x_{v-1}) \) is a linear realization of a list \( L \), then \( P' = (0, 1, \ldots, A) + \pi_A(P) \) is a linear realization of the list \( L \cup \{1^A\} \).

Let \( P = (x_0, x_1, \ldots, x_{v-1}) \) be a path. For each \( j \in \{1, 2, \ldots, v - 1\} \), the path \( P \) is called \( j \)-partitionable if \( P = P_j + P_j^c \), where

\[ V(P_j) = \{x_0, x_1, \ldots, x_j\} = \{0, 1, \ldots, j\} \]

and \( x_j = j \). A path \( P \) is called partitionable if \( P \) is \( j \)-partitionable for some \( j \in \{1, 2, \ldots, v - 1\} \).

**Example 3.** The path \( P = (0, 1, 2, 5, 3, 4, 6) \) is \( j \)-partitionable for \( j \in \{1, 2, 6\} \). On the other hand, the path \( P' = (0, 1, 2, 5, 3, 4, 6, 7, 8) \) is \( j \)-partitionable for \( j \in \{1, 2, 6, 7, 8\} \). In particular, both paths are perfect.

Let \( P \) be a \( j \)-partitionable, for some \( j > 0 \). Then \( P \) is weakly \( j \)-partitionable if \( P \) is also \((j + 1)\)-partitionable; otherwise the path is called strong.

**Lemma 2.2 ([10]).** Let \( P = (x_0, x_1, \ldots, x_{v-1}) \) be a linear realization of a list \( L \). If there exists \( i \in \{0, 1, \ldots, v-2\} \) such that \( \{x_i, x_{i+1}\} = \{v-2, v-1\} \), then \( P = (x_0, \ldots, x_i, v, x_{i+1}, \ldots, x_{v-1}) \) is a linear realization of \( L \cup \{2\} \).
Corollary 2.3 ([10]). Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list $L$. If there exists $i \in \{0, 1, \ldots, v-2\}$ such that $\{x_i, x_{i+1}\} = \{v-2, v-1\}$, then the list $L' = L \cup \{2^b\}$ admits a linear realization, for any positive integer $b$.

**Lemma 2.4.** If a list $L$ admits a weakly $j$-partitionable linear realization, for some $j \in \{1, \ldots, |L| - 2\}$, then the list $L \setminus \{1\}$ admits a linear realization.

**Proof.** Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a weakly $j$-partitionable linear realization of a list $L$, for some $j \in \{1, \ldots, |L| - 2\}$. Since the path is weakly $j$-partitionable, then $j$ and $j+1$ are adjacent in $P$ and $1 \in L$. Therefore, the path $P' = (x_0, \ldots, x_j, \pi_{-1}(x_{j+2}), \ldots, \pi_{-1}(x_{v-1}))$ is a linear realization of $L \setminus \{1\}$. \hfill \Box

**Proposition 2.5.** If a list $L$ admits a perfect weakly $i$-partitionable linear realization, for all $i \in \{i_1, \ldots, i_k\}$, then $L = L_{i_1} \cup \cdots \cup L_{i_k} \cup L_{v-1}$ where $L_i$ admits a perfect strong linear realization for all $i \in \{i_1, \ldots, i_k\} \cup \{v-1\}$.

**Proof.** Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a perfect weakly $i$-partitionable linear realization of a list $L$, where $i \in \{i_1, \ldots, i_k\}$ and $i_1 < i_2 < \cdots < i_k$. Hence $P = (x_0, \ldots, x_{i_1}) + (x_{i_1+1}, \ldots, x_{i_2}) + \cdots + (x_{i_k+1}, \ldots, x_{v-1})$.

Setting $i_0 = 0$, $i_{k+1} = v - 1$ and $P_{i_j} = (x_{i_j-1+1}, \ldots, x_{i_j})$, for all $j \in \{1, \ldots, k\}$, then $P = P_{i_1} + P_{i_2} + \cdots + P_{i_{k+1}}$. Since $P$ is perfect and partitionable, $P_{i_1}, \pi_{-(x_{i_1}+1)}(P_{i_2}), \ldots, \pi_{-(x_{i_{k+1}+1})}(P_{i_{k+1}})$ are perfect strong linear realizations of $L_{i_1}, L_{i_2}, \ldots, L_{i_{k+1}}$, respectively, where $L_{i_j} \subseteq L$, for all $j \in \{1, \ldots, k+1\}$ and $L = L_{i_1} \cup \cdots \cup L_{i_{k+1}}$ (by Theorem 2.1). \hfill \Box

**Lemma 2.6 ([10]).** If a list $L = \{1^{a_1}, 2^{a_2}, \ldots, t^{a_t}\}$ admits a linear realization, then $a_i + i - 1 \leq |L|$ for all $i = 1, \ldots, t$.

**Proposition 2.7.** If a list $L = \{1^a, 2^b, t^c\}$ admits a perfect linear realization, then $b + (t - 1)c$ is even.

**Proof.** The proof is obtained straightforwardly of proof given by Proposition 3.1 in [3]. \hfill \Box
In particular of Lemma 2.6, if a list $L = \{1^a, 2^b, t^c\}$ admits a linear realization, then $a + b \geq t - 1$.

**Remark 3.** If $P = (x_0, x_1, \ldots, x_t)$ is a perfect linear realization of $L_t = \{1^a, 2^b, t\}$ with $a + b = t$, then either $x_1 = t$ or $x_{t-1} = 1$.

**Proposition 2.8.** There exist a perfect linear realization of the list $L_t = \{1, 2^{t-1}, t\}$, for all $t \geq 3$ integer.

**Proof.** It is very easy to see that the following paths are perfect linear realizations of $L$.

(a) $P_t = (0, 2, 4, \ldots, t, t - 1, t - 3, \ldots, 1, t + 1)$ if $t \geq 4$ is even.

(b) $P_t = (0, 2, 4, \ldots, t - 1, t, t - 2, \ldots, 1, t + 1)$ if $t \geq 3$ is odd.

(c) $\hat{P}_t = (0, t, t - 2, \ldots, 2, 1, 3, \ldots, t - 1, t + 1)$ if $t \geq 4$ is even.

(d) $\hat{P}_t = (0, t, t - 2, \ldots, 1, 2, 4, \ldots, t - 1, t + 1)$ if $t \geq 3$ is odd.

**Example 4.** The paths $P_4 = (0, 2, 4, 3, 1, 5)$ and $\hat{P}_4 = (0, 4, 2, 1, 3, 5)$ are perfect linear realizations of the list $L_4 = \{1, 2^3, 4\}$.

**Theorem 2.9.** Let $a + b = t \geq 3$ with $a, b \geq 1$ integers. The list $L_t = \{1^a, 2^b, t\}$ admits a perfect linear realization if and only if $(a, b) = (1, t - 1)$, in which the paths $P_t$ and $\hat{P}_t$ are the unique perfect linear realization of the list $L_t$.

**Proof.** Suppose that $t \geq 4$ is an even integer (the proof for $t \geq 3$ odd is completely analogous). Let $P = (x_0, x_1, \ldots, x_{t+1})$ be a perfect linear realization of $L_t$. By Remark 3 either $x_t = 1$ or $x_1 = t$. Without loss of generality assume that $x_t = 1$, which implies that $x_1 = 2$, which implies that $x_{t-1} = 3$, which implies that $x_2 = 4$, which implies that $x_{t-2} = 5$, and so on until $x_{t-2} = t - 3$ and $x_2 = t$. Which implies that $x_{t+1} = t - 1$. Hence, we have that $P = P_t$. The proof to the case $x_1 = t$ is analogous to the proof presented before.

\[\square\]
3 Even-odd applications over paths

If \( P = (x_0, x_1, \ldots, x_{v-1}) \) is a standard linear realization of a list \( L \), then this path is called \((x_1, x_{v-1})\)-realization of \( L \). Let \( P^* \) be the sub-path of \( P \) without initial vertex, that is \( P^* = P \setminus \{x_0\} \). Hence, \( P^* \) is a (non-standard) linear realization of the list \( L \setminus \{x_1\} \). The reverse of \( P \), \( \text{rev}(P) = (x_{v-1}, x_{v-2}, \ldots, x_0) \), is also a linear realization of \( L \), see [8]. The even-application of \( P \), \( E(P) \), is defined by the path
\[
E(P) = (2x_0, 2x_1, \ldots, 2x_{v-1}).
\]
This application satisfies that \( \ell(E(P)) = 2L \). Finally, the odd-application of \( P \), \( O(P) \), is defined by the path:
\[
O(P) = (2x_1 - 1, 2x_2 - 1, \ldots, 2x_{v-1} - 1)
\]
and the odd reverse-application of \( P \), \( OR(P) \), is defined as the path
\[
OR(P) = (2x_{v-1} - 1, 2x_{v-2} - 1, \ldots, 2x_1 - 1).
\]
These applications satisfy \( \ell(O(P)) = \ell(OR(P)) = 2L \setminus \{2x_1\} \).

We define two operations over a linear realization \( P \) of a list \( L \), called even-odd extension, \( EO(P) \), and even-odd reverse extension of \( P \), \( EOR(P) \), as follow:
\[
EO(P) = E(P) + O(P) \text{ and } EOR(P) = E(P) + OR(P).
\]
The even-odd extension of \( P \) is a linear realization of the list
\[
(2L \cup 2L \setminus \{2x_1\}) \cup \{|2(x_{v-1} - x_1) + 1|\}.
\]
On the other hand, the even-odd reverse extension of \( P \) is a linear realization of the list
\[
(2L \cup 2L \setminus \{2x_1\}) \cup \{1\}.
\]
To the next, we are going to construct some linear realization from well-known linear realizations.

**Example 5.** As we have already seen, \( P = (0, 1, \ldots, k) \) is a (perfect) linear realization of the list \( \{1^k\} \). On the other hand, \( E(P) = (0, 2, 4, \ldots, 2k) \) and
Let $O(P) = (1, 3, \ldots, 2k - 1)$. Hence, $\ell(E(P)) = \{2^k\}$ and $\ell(O(P)) = \{2^{k-1}\}$. It follows that the even-odd reverse extension of $P$:

$$EOR(P) = (0, 2, 4, \ldots, 2k, 2k - 1, 2k - 3, \ldots, 3, 1)$$

is a linear realization of the list $\{1, 2^{2k-1}\}$. Notice that the new path is a $(2, 1)$-realization.

**Example 6.** Let $k \geq 1$ be an integer, and take

$$P_k = (0, 2, \ldots, 2k, 2k - 1, 2k - 3, \ldots, 1),$$

$$P'_k = (0, 2, \ldots, 2k, 2(k + 1), 2k - 1, \ldots, 1).$$

It is easy to see that $P_k$ is a $(2, 1)$-realization of $\{1, 2^{2k-1}\}$ (see Example 5) and $P'_k$ is $(2, 1)$-realization of $\{1, 2^{2k}\}$. Hence, the even-application of $P_k$ and $P'_k$ are

$$E(P_k) = (0, 4, \ldots, 4k, 4k - 2, 4k - 6, \ldots, 2),$$

$$E(P'_k) = (0, 4, \ldots, 4k, 4k + 2, 4k - 2, \ldots, 2),$$

satisfying $\ell(E(P_k)) = \{2, 4^{2k-1}\}$ and $\ell(E(P'_k)) = \{2, 4^{2k}\}$, respectively. The odd-application of $P_k$ and $P'_k$ are

$$O(P_k) = (3, 7, \ldots, 4k - 1, 4k - 3, 4k - 7, \ldots, 1),$$

$$O(P'_k) = (3, 7, \ldots, 4k - 1, 4k + 1, 4k - 3, \ldots, 1),$$

satisfying $\ell(O(P_k)) = \{2, 4^{2k-2}\}$ and $\ell(O(P'_k)) = \{2, 4^{2k-1}\}$, respectively.

Hence, the even-odd extension of $P_k$ and $P'_k$, $EO(P_k)$ and $EO(P'_k)$, are $(4, 1)$-realization of the lists $\{1, 2^2, 4^{4k-3}\}$ and $\{1, 2^2, 4^{4k-1}\}$, respectively. Also, the even-odd reverse extension of $P_k$ and $P'_k$, $EOR(P_k)$ and $EOR(P'_k)$, are $(4, 3)$-realization of the same lists.

**Lemma 3.1** ([10], Lemma 9). Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a standard linear realization of a list $L$. If $x_{v-1} = 1$, then the list $L' = L \cup \{2^b\}$ is linear realizable, for any $b \geq 2$ integer.

By Remark 2, Example 6 and Lemma 3.1, we have the following result, which is a particular case of Proposition 20 in [10]:

**Corollary 3.2.** There are linear realizations of the lists $\{1^a, 2^2, 4^{2c-1}\}$ and $\{1^a, 2^b, 4^{2c-1}\}$, for all positive integers $a, b, c$ such that $b \geq 4$.

**Theorem 3.3** ([3]). If $a \geq 2$ and $b \geq 0$ are integers, then the list $\{1^a, 3^b\}$ admits a linear realization. Also, this realization can be assumed to be perfect when $b \not\equiv 1 \pmod{3}$.
Corollary 3.4. Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list $L$, where the vertices $v-1, v-2$ are adjacent. If $EO(P)$ is a linear realization of $L_{EO}$ and $EOR(P)$ is a linear realization of $L_{EOR}$, then the lists $L_{EO} \cup \{4\}$ and $L_{EOR} \cup \{4\}$ are linear realizable.

Proof. The proof is completely analogous to the proof of Lemma 9 of [10]. Since the vertices $v-1, v-2$ are adjacent in $P$, the vertices $2v-3, 2v-5$ are adjacent in $O(P)$ (and in $OR(P)$), and the vertices $2v-2, 2v-4$ are adjacent in $E(P)$. Hence, the new vertex $2v-1$ can be added between $2v-3, 2v-5$. So, there is a linear realization of $L_{EO} \cup \{4\}$ or $L_{EOR} \cup \{4\}$. Else, if we also add the vertex $2v$ between $2v-2$ ans $2v-4$, we obtain a linear realization of $L_{EO} \cup \{4^2\}$ and of $L_{EOR} \cup \{4^2\}$. □

Corollary 3.5. Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list $L$, where the vertices $v-1, v-2$ are adjacent. If $EO(P)$ is a linear realization of $L_{EO}$ and $EOR(P)$ is a linear realization of $L_{EOR}$, then the lists $L_{EO} \cup \{4^b\}$ and $L_{EOR} \cup \{4^b\}$ are linear realizable, for any positive integer $b$.

By Remark 2, Example 6, Corollary 3.5 and Lemma 3.1, we have the following:

Corollary 3.6. There are linear realizations of the lists $\{1^a, 2^2, 4^c\}$ and $\{1^a, 2^b, 4^c\}$, for all positive integers $a, b, c$ such that $b \geq 4$.

Corollary 3.7. Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a standard linear realization of a list $L$, where $x_{v-1} = 1$. There exists a linear realization of $2L \cup 2L \cup \{1, 4^{2b-1}\}$.

Proof. Following the proof of Lemma 9 of [10], there exists a $(2, 1)$-realization $P'$ of $L \cup \{2^b\}$. Then $EO(P')$ and $EOR(P')$ are linear realizations of $2L \cup 2L \cup \{1, 4^{2b-1}\}$. □

Proposition 3.8. There exists a standard linear realization of the list $\{1^a, 2^b, 4^c\}$, for all positive integers $a, b, c$ where $b \geq 3$.

Proof. Let $k \geq 2$ be an integer. Consider the path $P_k$ of Example 5, obtained by applying the even-odd reverse extension of the perfect linear realization $I_k = \{0, 1, 2, \ldots, k\}$ of the list $\{1^k\}$: $P_k = EOR(I_k)$. Then, we can write $P_k = P_{k,0}^E + rev(P_{k,0}^O)$, where

$$P_{k,0}^E = E(I_k) = (0, 2, \ldots, 2k) \text{ and } P_{k,0}^O = O(I_k) = (1, 3, \ldots, 2k-1).$$
So, $\ell(P_{k,0}^E) = \{2^k\}$ and $\ell(P_{k,0}^O) = \{2^{k-1}\}$. Now, let $t$ be a positive integer. For all $j = 1, \ldots, t$, we construct a path $P_{k,j}^E$ by adding the vertex $2k + 2j$ between the consecutive vertices $2k + 2(j - 2), 2k + 2(j - 1)$ of the path $P_{k,j-1}^E$. Then, $\ell(P_{k,j}^E) = \{2^k, 4^j\}$. Similarly, for all $j = 1, \ldots, t$, we construct a path $P_{k,j}^O$ by adding the vertex $2k + 2j - 1$ between the consecutive vertices $2k + 2j - 5, 2k + 2j - 3$ of the path $P_{k,j-1}^O$. In this case, $\ell(P_{k,j}^O) = \{2^{k-1}, 4^j\}$.

Hence, the path $P_{k,t}^E = P_{k,t}^E + \text{rev}(P_{k,t}^O)$ is a $(2, 1)$-realization of the list $\{1, 2^{2k-1}, 4^t\}$. Now, the path $P_{k,t}^E + \text{rev}(P_{k,t}^O) = (0, 2, 4, \ldots, 2k, 2k - 1, 2k + 1, 2k - 3, 2k - 5, \ldots, 1)$ is a $(2, 1)$-realization of the list $\{1, 2^{2k-1}, 4\}$. Finally, for any positive integer $t$, the path $P_{k,t}^E + P_{k,t+1}^O$ is a $(2, 1)$-realization of the list $\{1, 2^{2k-1}, 4^{2t+1}\}$.

Hence, there is a $(2, 1)$-realization of $\{1, 2^{2x+1}, 4^y\}$ for all positive integers $x, y$. Finally, by Remark 2, Lemma 3.1, Corollary 3.6 and Corollary 3.7, there is a linear realization of $\{1^a, 2^b, 4^c\}$, for all positive integers $a, b, c$ where $b \geq 3$.

**Example 7.** For instance, taking $t = 2$ and $k = 3$, we have

\[
P_{3,0}^E = (0, 2, 4, 6), \quad P_{3,1}^E = (0, 2, 4, 8, 6), \quad P_{3,2}^E = (0, 2, 4, 8, 10, 6),
\]

\[
P_{3,0}^O = (1, 3, 5), \quad P_{3,1}^O = (1, 3, 7, 5), \quad P_{3,2}^O = (1, 3, 7, 9, 5).
\]

Hence, $P_{3,2}^E + \text{rev}(P_{3,1}^O) = (0, 2, 4, 8, 10, 6, 5, 9, 7, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^5, 4^4\}$, $P_{3,0}^E + \text{rev}(P_{3,1}^O) = (0, 2, 4, 6, 5, 7, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^5, 4\}$, and $P_{3,2}^E = (0, 2, 4, 8, 10, 6, 5, 9, 11, 7, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^5, 4^5\}$.

Furthermore, $P_{3,2}^E + \text{rev}(P_{4,1}^O) = (0, 2, 4, 8, 10, 6, 7, 11, 9, 5, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^6, 4^4\}$, $P_{3,2}^E + \text{rev}(P_{4,2}^O) = (0, 2, 4, 8, 10, 6, 7, 11, 9, 5, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^6, 4\}$, and $P_{3,3}^E + P_{4,2}^O = (0, 2, 4, 8, 12, 10, 6, 7, 11, 9, 5, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^6, 4^5\}$.

**Proposition 3.9.** There exists a standard linear realization of the list $\{1^a, 2^b, 4^c, 8^d\}$, for all positive integers $a, b, c, d$ where $b \geq 3$ and $c \geq 2$. Moreover, there exists a standard linear realization of the list $\{1^a, 2^b, 4^c, 8^d\}$, for all positive integers $a, b, c, d$ such that $a \geq 2$, $b \geq 3$ and $c \geq 4$.

**Proof.** Let $Q = P_{2,4} = (0, 2, 6, 8, 4, 3, 7, 5, 1)$ be a $(2, 1)$-linear realization of $\{1, 2^3, 4^4\}$ (see Proposition 3.8). Let $Q^2 = (2, 6)$, $Q^0 = (4, 8)$, $Q^1 = (3, 7)$ and $Q^3 = (1, 5)$. So,

\[
Q = (0) + Q^2 + \text{rev}(Q^0) + Q^1 + \text{rev}(Q^2).
\]
Let \( l \geq 3 \) be an integer and \( i \in \{0,1,2,3\} \), we construct the path \( Q_{l+1,i} \) by adding the vertex \( 4l - i \) to the path \( Q_{l,i} \), where \( Q_3 = Q^i \) for \( i = 0,1,2,3 \), as follow:

- If \( i = 3 \), we add the vertex \( 4l - 3 \) between the vertices \( 4(l - 1) - 3 \) and \( 4(l - 2) - 3 \) to the path \( Q^3_l \). Hence,
  \[
  Q_{l+1,3} = (0) + Q^2_l + rev(Q^0_l) + Q^1_l + rev(Q^3_{l+1}).
  \]
- If \( i = 2 \), then \( Q^2_{l+1} = Q^3_{l+1} + 1 \) (since \( Q^2 = Q^3 + 1 \)), we are adding the vertex \( 4l - 2 \) between the vertices \( 4(l - 1) - 2 \) and \( 4(l - 2) - 2 \) of the path \( Q^2_l \). Hence,
  \[
  Q_{l+1,2} = (0) + Q^2_{l+1} + rev(Q^0_l) + Q^1_l + rev(Q^3_{l+1}).
  \]
- If \( i = 1 \), we add the vertex \( 4l - 1 \) between the vertices \( 4(l - 1) - 1 \) and \( 4(l - 2) - 1 \) to the path \( Q^1_l \). Hence,
  \[
  Q_{l+1,1} = (0) + Q^2_{l+1} + rev(Q^0_l) + Q^1_{l+1} + rev(Q^3_{l+1}).
  \]
- If \( i = 0 \), then \( Q^0_{l+1} = Q^1_{l+1} + 1 \) (since \( Q^0 = Q^0 + 1 \)), we are adding the vertex \( 4l \) between the vertices \( 4(l - 1) \) and \( 4(l - 2) \) of the path \( Q^0_l \). Hence,
  \[
  Q_{l+1,0} = (0) + Q^2_{l+1} + rev(Q^0_{l+1}) + Q^1_{l+1} + rev(Q^3_{l+1}).
  \]

So, \( \ell(Q^i_{l+1}) = \{4,8^i\} \), for \( i = 0,1,2,3 \). Therefore, the path \( Q_{l+1,i} \) is a \((2,1)\)-realization of \( \{1,2^3,4^4,8^{4l-8-i}\} \), proving that there exists a \((2,1)\)-realization of \( \{1,2^3,4^4,8^i\} \), for all positive integer \( l \). Proceeding as the same way as before taking \( Q = P_{2,2k} \) (see Proposition 3.8), for \( k \geq 1 \) integer, we can prove that there is a \((2,1)\)-realization of the list \( \{1,2^3,4^{4k},8^s\} \), for all positive integers \( k,s \).

Now, let \( \hat{Q} = (0,2,6,10,8,4,5,9,11,7,3,1) \) be a \((2,1)\)-linear realization of \( \{1,2^4,4^6\} \). If \( \hat{Q}^2 = (2,6), \hat{Q}^0 = (4,8), \hat{Q}^1 = (5,9) \) and \( \hat{Q}^3 = (3,7) \), we have
  \[
  \hat{Q} = (0) + \hat{Q}^2 + (10) + rev(\hat{Q}^0) + \hat{Q}^1 + (11) + rev(\hat{Q}^3) + (1).
  \]
As the same way as before, we can construct a \((2,1)\)-linear realization of the list \( \{1,2^4,4^6,8^s\} \), for \( s \geq 1 \) integer. Moreover, if we take \( \hat{Q} = P'_{2,2k+1} \) (see Proposition 3.8), for \( k \geq 1 \) integer, we can prove that there is a \((2,1)\)-realization of the list \( \{1,2^4,4^{4k+2},8^s\} \), for all positive integers \( k,s \).
On the other hand, let \( Q = (0, 2, 6, 8, 4, 5, 9, 7, 3, 1) \) be a \((2, 1)\)-linear realization of \( \{1, 2^4, 4^4\} \). Let \( Q^2 = (2, 6), Q^0 = (4, 8), Q^1 = (5, 9) \) and \( Q^3 = (3, 7) \), we have
\[
Q = (0) + Q^2 + \text{rev}(Q^0) + Q^1 + \text{rev}(Q^3) + (1).
\]
Let \( l \geq 3 \) be an integer and \( i \in \{0, 1, 2, 3\} \), we construct the path \( Q_{l+1,i} \) by adding the vertex \((4l - 3) + i\) to the path \( Q_{l,i} \), where \( Q^3_i = Q^i \) for \( i = 0, 1, 2, 3 \), as follow:

- If \( i = 0 \), we add the vertex \((4l - 2)\) between the vertices \(4(l - 2)\) and \(4(l - 3) - 2\) to the path \( Q^2_l \). Hence,
  \[
  Q_{l+1,0} = (0) + Q^2_{l+1} + \text{rev}(Q^0_l) + Q^1_l + \text{rev}(Q^3_l).
  \]
- If \( i = 1 \), then \( Q^3_{l+1} = Q^2_{l+1} + 1 \) (since \( Q^2 = Q^3 + 1 \)). Hence,
  \[
  Q_{l+1,1} = (0) + Q^2_{l+1} + \text{rev}(Q^0_l) + Q^1_l + \text{rev}(Q^3_{l+1}).
  \]
- If \( i = 2 \), we add the vertex \(4l - 1\) between the vertices \(4(l - 1) - 1\) and \(4(l - 2) - 1\) to the path \( Q^1_l \). Hence,
  \[
  Q_{l+1,2} = (0) + Q^2_{l+1} + \text{rev}(Q^0_l) + Q^1_{l+1} + \text{rev}(Q^3_{l+1}).
  \]
- If \( i = 3 \), then \( Q^0_{l+1} = Q^1_{l+1} + 1 \) (since \( Q^0 = Q^0 + 1 \)), we are adding the vertex \(4l\) between the vertices \(4(l - 1)\) and \(4(l - 2)\) of the path \( Q^0_l \). Hence,
  \[
  Q_{l+1,3} = (0) + Q^2_{l+1} + \text{rev}(Q^0_{l+1}) + Q^1_{l+1} + \text{rev}(Q^3_{l+1}).
  \]

So, we can construct a \((2, 1)\)-linear realization of the list \( \{1, 2^4, 4^4, 8^t\} \), for \( t \geq 1 \) integer. Moreover, if we take \( Q = P_{2,2k} \) (see Proposition 3.8), for \( k \geq 1 \) integer, we can prove that there is a \((2, 1)\)-realization of the list \( \{1, 2^4, 4^{4k}, 8^s\} \), for all positive integers \( k, s \). Finally, taking the path \( Q = P_{2,2k+1} \) (see Proposition 3.8) and all ideas presented before, we can construct a \((2, 1)\)-linear realization of the list \( \{1^3, 2^4, 4^{4c}, 8^d\} \), for all positive integers \( k, s \). By Remark 2 and Lemma 3.1, there is a linear realization of \( \{1^a, 2^b, 4^c, 8^d\} \), for all positive integers \( a, b, c, d \) such that \( b \geq 3 \) and \( c \geq 2 \). Moreover, By Remark 2, Corollary 3.7 and Lemma 3.1 there exists a linear realization of \( \{1^a, 2^b, 4^c, 8^d\} \), for all positive integers \( a, b, c, d \) such that \( a \geq 2, b \geq 3 \) and \( c \geq 4 \).

**Proposition 3.10.** There are linear realizations of the lists
\[
\{1^a, 2^4, 4^c, 6^{6d+1}\}, \{1^a, 2^5, 4^c, 6^{6d-2}\} \text{ and } \{1^a, 2^b, 4^c, 6^{6d-2}\},
\]
for all positive integer \( a, b, c, d \) such that \( b \geq 7 \).
Proof. Let $k \geq 1$ be an integer. The path
\[ Q_k = (0, 3, \ldots, 3k + 3, 3k + 2, 3k - 1, \ldots, 2, 1, 4, \ldots, 3k + 1), \]
is a realization of the list $\{1^2, 3^{3k+1}\}$. Then, the even-odd reverse extension of $Q_k$, $EOR(Q_k)$, is a linear realization of the list $\{1, 2^4, 6^{6k+1}\}$. By Remark 2 and Corollary 3.5, there exists a linear realization of $\{1^a, 2^4, 4^c, 6^{6k+1}\}$ for all positive integer $a, c$.

On the other hand, the path
\[ \hat{Q}_k = (0, 1, 4, \ldots, 3k + 1, 3k + 2, 3k - 1, \ldots, 2, 3, 6, \ldots, 3k), \]
is a linear realization of the list $\{1^3, 3^{3k-1}\}$. Then, the even-odd reverse extension of $\hat{Q}_k$, $EOR(\hat{Q}_k)$, is a $(2, 1)$-linear realization of the list $\{1, 2^5, 6^{6k-2}\}$. Using Remark 2, Corollary 3.5 and Lemma 3.1, there are linear realizations of the lists $\{1^a, 2^5, 4^4, 6^{6k-2}\}$ and $\{1^a, 2^b, 4^c, 6^{6k-2}\}$, for all positive integer $a, b, c$ such that $b \geq 7$.

Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list $L$, and let $P' = (y_0, y_1, \ldots, y_{v-1})$ be a standard linear realization of the list $L'$, such that $|L| = |L'|$. The even-odd extension of $P$ and $P'$, denoted by $EO(P, P')$, is defined as follow:
\[
EO(P, P') = E(P) + O(P')
= (2x_0, 2x_1, \ldots, 2x_{v-1}, 2y_1 - 1, 2y_2 - 1, \ldots, 2y_{v-1} - 1);
\]
the even-odd reverse extension of $P$ and $P'$, denoted by $EOR(P, P')$, is defined as follow:
\[
EOR(P, P') = E(P) + OR(P')
= (2x_0, 2x_1, \ldots, 2x_{v-1}, 2y_{v-1} - 1, 2y_{v-2} - 1, \ldots, 2y_1 - 1).
\]
The even-odd extension of $P$ and $P'$ is a linear realization of the list $(2L \cup 2L' \setminus \{2y_1\}) \cup \{|2(x_{v-1} - y_1) + 1|\}$, while the even-odd reverse extension of $P$ and $P'$ is a linear realization of the list $(2L \cup 2L' \setminus \{2y_1\}) \cup \{|2(x_{v-1} - y_{v-1}) + 1|\}$. In particular, if $P' = P$, then $EO(P, P) = EO(P)$ and $EOR(P, P) = EOR(P)$.

To the next, we are going to construct some linear realization from well-known linear realizations.

**Example 8.** Let $a \geq 2$ and $b \geq 1$ integers. Let $P = (x_0, x_1, \ldots, x_{a+b})$ be a linear realization of the list $\{1^a, 3^b\}$, and let $Q = (0, 1, 2, \ldots, a + b)$ be a
linear realization of the list \( \{1^{a+b}\} \). If \( P \) is a perfect linear realization, then the even-odd reverse extension of \( P \) and \( Q \), \( \text{EOR}(P,Q) \), is a (standard) linear realization of the list \( \{1,2^{2a+b-1},6^b\} \). Also, if \( x_{a+b} = 1 \), then the even-odd extension of \( P \) and \( Q \), \( \text{EO}(P,Q) \), is a linear realization of the same list.

**Example 9.** Let \( k = 2s \) and \( h = 3s + 1 \), where \( s \geq 1 \) is an integer. Let \( P_h = (0,1,\ldots,h) \) be a linear realization of the list \( \{1^h\} \). By Example 5, the even-odd reverse extension of \( P_h \), \( \text{EOR}(P_h) \), is a \((2,1)\)-realization of the list \( \{1,2^{2h-1}\} \). If \( \hat{P}_h = \text{EOR}(P_h) \), then the even-odd extension of \( \hat{P}_h \) and \( \hat{Q}_k \) (see Proposition 3.10), \( \text{EO}(\hat{P}_h,\hat{Q}_k) \), is a \((2,1)\)-realization of the list \( \{1,2^2,4^{2h-1},6^{3k}\} \); that is, the even-odd extension of \( \hat{P}_{3s+1} \) and \( \hat{Q}_{2s} \) is a \((2,1)\)-realization of the list \( \{1,2^2,4^{6s+1},6^{6s}\} \). By Remark 2, Corollary 3.5 and Lemma 3.1 there are linear realizations of \( \{1^a,2^2,4^c,6^d\} \) and \( \{1^a,2^k,4^c,6^d\} \) for all positive integers \( a, b, c, d \) such that \( b \geq 4 \) and \( c \geq 7 \).

### 4 \( k \)-extension of linear realizations

In this section, we are going to generalize the even-odd extension given in Section 3 for well-known linear realizations.

Let \( P = (x_0 = 0, x_1, \ldots, x_{v-1}) \) be a (standard) linear realization of a list \( L \). For each \( i \in \{1,2,\ldots,k-1\} \), the \( i \)-application of \( P \) is defined by the path

\[
P_{k,i} = (kx_1 - i, kx_2 - i, \ldots, kx_{v-1} - i).
\]

So, \( \ell(P_{k,i}) = kL \setminus \{kx_i\} \). We define the \( k \)-extension of \( P \), denoted by \( E_k(P) \), as follow:

\[
E_k(P) = P_{k,0} + P_{k,1} + \cdots + P_{k,k-1},
\]

where \( P_{k,0} = kP = (0, kx_1, \ldots, kx_{v-1}) \). Notice that

\[
|kx_1 - (i + 1) - (kx_{v-1} - i)| = |k(x_1 - x_{v-1}) - 1|.
\]

Hence, the \( k \)-extension of \( P \) is a linear realization of the list

\[
(kL \cup kL \setminus \{kx_1\} \cup kL \setminus \{kx_1\} \cup \cdots \cup kL \setminus \{kx_1\}) \cup \{k(x_1 - x_{v-1}) - 1\}^{k-1}.
\]

**Corollary 4.1.** Let \( P = (x_0, x_1, \ldots, x_{v-1}) \) be a standard linear realization of \( L \), where the vertices \( v-1 \) and \( v-2 \) are adjacent. If \( E_k(P) \) is the linear realization of \( L_k \), then the list \( L_k \cup \{(2k)^b\} \) admits a linear realization for any positive integer \( b \).
Proof. Note that the vertices \(k(v−1)−i, k(v−2)−i\) are adjacent in \(E_k(P)\) for all \(i = 0, \ldots, k−1\). Then, one can proceed as the proof of Lemma 7 of [10].

Example 10. Let \(P = (0, 1, \ldots, s)\) be a linear realization of the list \(\{1^s\}\). For each \(i = \{1, 2\}\) (in this case \(k = 3\)), the \(i\)-application of \(P\) is given by the path

\[
P_{3,i} = (3 \cdot 1 − i, 3 \cdot 2 − i, \ldots, 3 \cdot s − i),
\]

which satisfies that \(\ell(P_{3,i}) = \{3^{s−1}\}\). Hence, the 3-extension of \(P\):

\[
E_3(P) = (0, 3, 6, \ldots, 3s, 2, 5, \ldots, 3s−1, 1, 4, \ldots, 3s−2)
\]

is a linear realization of the list \(\{3^{3s−2}, (3s−2)^2\}\). By Remark 2 and Corollary 4.1 there exists a linear realization of the list \(\{1^a, 3^{3s−2}, 6^b, (3s−2)^2\}\) for all positive integer \(a, b, s\).

Proposition 4.2. There are linear realizations of the lists \(\{1^a, 2^b, 3^3, 6^c\}\) and \(\{1^a, 2^b, 3^3, 6^c\}\), for all positive integers \(a, b, c\) such that \(b ≥ 4\).

Proof. Let \(s ≥ 1\) be an integer. Consider the linear realizations \(P_s\) and \(P'_s\) of the lists \(\{1, 2^{2s−1}\}\) and \(\{1, 2^{2s}\}\), respectively, given in Example 6:

\[
P_s = (0, 2, \ldots, 2s, 2s−1, 2s−3, \ldots, 1),
\]

\[
P'_s = (0, 2, \ldots, 2s, 2s+1, 2s−1, \ldots, 1),
\]

For each \(i = \{1, 2\}\) (in this case \(k = 3\)), the \(i\)-realization of \(P_s\), and \(P'_s\) are:

\[
P_{s,i} = (3 \cdot 2 − i, \ldots, 3 \cdot 2s − i, 3 \cdot (2s−1) − i, 3 \cdot (2s−3) − i, \ldots, 3 \cdot 1 − i),
\]

\[
P'_ {s,i} = (3 \cdot 2 − i, \ldots, 3 \cdot 2s − i, 3 \cdot (2s+1) − i, 3 \cdot (2s−1) − i, \ldots, 3 \cdot 1 − i),
\]

respectively. So, \(\ell(P_{s,i}) = \{3, 6^{2s−2}\}\) and \(\ell(P'_{s,i}) = \{3, 6^{2s−1}\}\). Therefore, the 3-extensions of \(P_s, E_3(P_s),\) and \(P'_s, E_3(P'_s),\) are \((6, 1)\)-realization of the lists \(\{2^2, 3^3, 6^{6s−5}\}\) and \(\{2^2, 3^3, 6^{6s−2}\}\), respectively. By Remark 2, Corollary 4.1 and Lemma 3.1, there are linear realizations of the lists \(\{1^a, 2^b, 3^3, 6^c\}\) and \(\{1^a, 2^b, 3^3, 6^c\}\), for all positive integers \(a, b, c\) such that \(b ≥ 4\).

For each \(i ∈ \{0, 1, 2, \ldots, k−1\}\) let \(Q_i = (x_{i,0} = 0, x_{i,1}, \ldots, x_{i,v−1})\) be \(k\) (standard) linear realizations of the list \(L_i\), such that \(|L_i| = |L_j|\), for every \(0 ≤ i < j ≤ k−1\). A \(k\)-extension of \(Q_0, Q_1, \ldots, Q_{k−1}\), denoted by \(E_k(Q^{T_0}_{0}, Q^{T_1}_{1}, \ldots, Q^{T_{k−1}}_{k−1})\), is defined as follow:

\[
E_k(Q^{T_0}_{0}, Q^{T_1}_{1}, \ldots, Q^{T_{k−1}}_{k−1}) = Q^{T_0}_{k,0} + Q^{T_1}_{k,1} + \cdots + Q^{T_{k−1}}_{k,k−1}.
\]
where either $Q_{k,i}^T = Q_{k,i}$ if $Q_{k,i}^T = Q_i$ or $Q_{k,i}^T = \text{rev}(Q_{k,i})$ if $Q_{k,i}^T = Q_i^{rev}$, for all $i = 0, 1, \ldots, k-1$, and where $Q_{k,i} = (kx_{i,1} - i, kx_{i,2} - i, \ldots, kx_{i,v-1} - i)$, for all $i = 1, \ldots, k-1$, and $Q_{k,0} = (0, kx_{0,1}, \ldots, kx_{0,v-1})$. So, $\ell(Q_{k,i}) = kL_i \setminus \{kx_{i,1}\}$, for all $i = 1, \ldots, k-1$, and $\ell(Q_{k,0}) = kL_0$.

A $k$-extension of $Q_0, Q_1, \ldots, Q_{k-1}$, $E_k(Q_0^T, Q_1^T, \ldots, Q_{k-1}^T)$, is a linear realization of the list

\[
\{1^a, 2^b, (k(t-1)+1)^c, (2k)^d\}
\]

for all integers $a, b, c, t$ such that $a \geq k-1$ and $t, b \geq 2$.

**Proposition 4.3.** For all $k \geq 2$ an even integer and $s$ a positive integer, there exists a linear realization of the list

\[
\{1^{k-1}, k^k, (2k)^k, \ldots, ((s-1)k)^k, (sk)\}.
\]

**Proof.** Let $C = C_s = (x_0, x_1, \ldots, x_s)$ with $x_{2i} = i$ and $x_{2i+1} = s - i$, for all $i \in \{0, 1, \ldots, [s/2]\}$, be the well-known Walecki linear realization of the list $\{1, 2, \ldots, s\}$, see [8] (page 3). The following $k$-extension of $C$:

\[
E_k(C, C^{rev}, \ldots, C, C^{rev}) = C_{k,0} + \text{rev}(C_{k,1}) + \cdots + C_{k,k-2} + \text{rev}(C_{k,k-1}),
\]

is a linear realization of the list $\{1^{k-1}, k^k, (2k)^k, \ldots, ((s-1)k)^k, (sk)\}$. \qed

**Corollary 4.4.** For $i = 0, 1, \ldots, k$, let $Q_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,v-1})$ be a standard linear realization of $L_i$, where the vertices $v-1$ and $v-2$ are adjacent for all $i$. If $E_k(Q_0^T, Q_1^T, \ldots, Q_{k-1}^T)$ is a (standard) linear realization of $L$, then the list $L \cup \{(2k)^b\}$ is linear realizable, for any positive integer $b$.

**Proof.** See proof of Corollary 4.1. \qed

**Proposition 4.5.** Let $k \geq 2$ be an integer, then there exists a linear realization of the list

\[
\{1^a, 2^b, k^{(t-1)+1}, (2k)^c\}
\]

for all integers $a, b, c, t$ such that $a \geq k-1$ and $t, b \geq 2$. 

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Proof. Let $I = I_t = \{0, 1, \ldots, t\}$ be the linear realization of the list $\{1^t\}$, where $t \geq 2$ is a positive integer. The following $k$-extension of $I$:

$$E_k(I, I^{rev}, \ldots, I, I^{rev}) = I_{k,0} + rev(I_{k,1}) + \cdots + I_{k,k-2} + rev(I_{k,k-1}),$$

is a $(k, 1)$-realization of $\{1^{k-1}, k^{k(t-1)+1}\}$. By Remark 2, Corollary 4.4 and Lemma 3.1, there exists a linear realization of the list

$$\{1^a, 2^b, k^{k(t-1)+1}, (2k)^c\},$$

for all integers $a, b, c$ such that $a \geq k - 1$ and $b \geq 2$. □

**Proposition 4.6.** There exists a linear realization of the lists $\{1^a, 4^4, 8^c\}$, for all positive integers $a, b, c$ such that $a \geq 3$ and $c \geq 1$.

Proof. Let $s \geq 1$ be an integer. Consider the linear realizations $P_s$ of the list $\{1, 2^{2s-1}\}$ (given in Example 6): $P_s = (0, 2, \ldots, 2s, 2s - 1, 2s - 3, \ldots, 1)$. The following 4-extension $E_4(P_s, P_s^{rev}, P_s, P_s^{rev})$ is a linear extension of $\{1^3, 4^4, 8^{8s-7}\}$. By Remark 2 and Corollary 4.4 there exists a linear realization of the lists $\{1^a, 4^4, 8^c\}$, for all positive integers $a, b, c$ such that $a \geq 3$ and $c \geq 1$. □

Acknowledgment I would like to thank the anonymous referee for reading carefully the paper and for all comments and suggestions for improving the paper. Also, I would like to thank the Editor-in-Chief Professor Buratti for all his support and advice. The research was partially supported by SNI and CONACyT.

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