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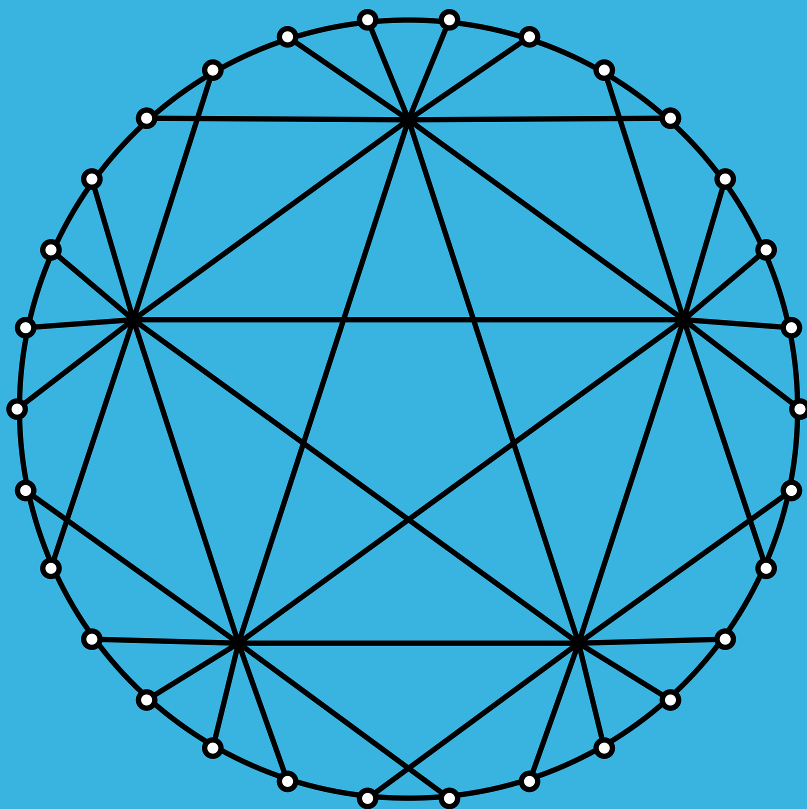

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An amalgamation of graceful and harmonious labelings

Alexis Byers* and Anita O'Mellan

Abstract. A graceful labeling of a graph G of size m is an injective function $f : V(G) \to \{0, 1, \ldots, m\}$ that induces an injective function $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$. If we want to induce the edge labeling via addition instead of subtraction, we would consider a harmonious labeling, which can be described as an injective function $f : V(G) \to \{0, 1, \ldots, m - 1\}$ that induces an injective function $f' : E(G) \to \{0, 1, \ldots, m - 1\}$ defined by $f'(uv) = f(u) + f(v) \pmod{m}$. We introduce a new concept that combines these labelings, and we present results and conjectures that build upon our knowledge of graceful and harmonious labelings.

1 Introduction

A *graceful labeling* of a graph G of order n and size m is an injective function $f : V(G) \to \{0, 1, \ldots, m\}$ that induces an injective function $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$. [9] A graph having a graceful labeling is called a *graceful graph*. Introduced by Alexander Rosa in 1967, graceful labelings have captured the attention of graph theorists for decades, spawning enough related graph labelings to fill over 500 pages of Joseph Gallian's most recent survey on the Electronic Journal of Combinatorics. [4]

One such labeling was introduced by Ronald Graham and Neil Sloane in their 1980 paper. [6] Rather than labeling edges by the positive difference of the labels of their incident vertices, the vertex labels are summed. More precisely, a *harmonious labeling* of a connected graph G of size m is an assignment f of distinct elements of $\mathbb{Z}_m$, the integers modulo m to the vertices of G so that the resulting edge labeling, in which each edge uv of G is labeled $f(u) + f(v) \pmod{m}$, is edge-distinguishing. Notice that if G

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is a tree, then this type of vertex labeling is not possible. Thus, if G is a tree, we allow one element of $\mathbb{Z}_m$ to be assigned to two vertices of G while all others are still used exactly once. A graph that admits a harmonious labeling is called a *harmonious graph*.

In the original 1967 paper, Rosa determined the gracefulness of several classes of graphs including cycles. [9]

Proposition 1.1. *A cycle C_n is graceful if and only if $n \equiv 0, 3 \pmod 4$.*

Solomon Golomb was the first to use the moniker "graceful" to describe the above graph labeling that was originally introduced by Rosa under the name "β-valuation". In the paper where this name was introduced, Golomb also characterized complete graphs that were graceful. [5]

Proposition 1.2. *A complete graph K_n is graceful if and only if $n \leq 4$.*

Another well-known class of graphs that contains many non-graceful members is the friendship graph. A *friendship graph*, denoted F_n, consists of n triangles which are all joined at a common vertex. Results from two papers, one authored by Bermond, Brouwer, and Germa and another authored by Bermond, Kotzig, and Turgeon, determined which friendship graphs are grapceful. [2], [3]

Proposition 1.3. *The friendship graph F_n is graceful if and only if $n \equiv 0, 1 \pmod 4$.*

Graham and Sloane proved analagous results to those listed above for harmonious labelings. [6]

Proposition 1.4. *A cycle C_n is harmonious if and only if n is odd.*

Proposition 1.5. *A complete graph K_n of order n is harmonious if and only if $n \leq 4$.*

Proposition 1.6. *The friendship graph F_n is harmonious if and only if $n \not\equiv 2$ (mod 4).*

Other interesting results from Graham and Sloane's paper involve the existence of graceful or harmonious labelings for graphs of order n, as n goes to infinity. [6]

Proposition 1.7. *Almost all graphs are not harmonious.*

Proposition 1.8. *Almost all graphs are not graceful.*

In the Gallian survey, there are many classes of graphs that have been proven to be not harmonious and not graceful, and there are even more for which we simply do not know if they are harmonious or graceful. Famously, we have the Graceful Tree Conjecture, spawned from the Ringel-Kotzig conjecture.

Graceful Tree Conjecture. Every tree is graceful.

Similar to the analogous results above, there is an analogous conjecture for harmonious trees posited by Graham and Sloane. [6]

Harmonious Tree Conjecture. Every tree is harmonious.

In this paper, we introduce a new graph labeling, an amalgamation of these two well-known labelings, and we investigate whether flexibility in the induced edge labeling allows non-graceful and non-harmonious graphs to satisfy this new labeling.

2 Graceful-harmonious labeling

Let G be a connected graph of order n and size m. Begin with an injective vertex labeling $f : V(G) \to \{0, 1, 2, \ldots, m\}$, assigning distinct labels to the vertices of G. For each edge uv in $E(G)$, we have the choice of either one of two edge labels: $|f(u) - f(v)|$ or $[f(u) + f(v)] \pmod m$. If we are able to label each edge using one of these two options such that each edge receives a distinct label from the set $\{0, 1, 2, \ldots, m\}$, then the resulting labeling is a *graceful-harmonious labeling.*

Example 2.1. *Consider the graph of order* 5 *and size* 6 *in Figure 1. This graph is neither graceful nor harmonious, but it has a graceful-harmonious labeling. Notice that the edge between vertices labeled* 5 *and* 6 *is labeled* $6 - 5 = 1$ *and the edge between the vertices labeled* 2 *and* 4 *is labeled* $2 + 4 = 0 \pmod 6$. *In all figures, solid edges indicate edge labels obtained by subtraction, and dashed edges indicate edge labels obtained by addition.*

Clearly, any graph that has a graceful labeling or a harmonious labeling has a graceful-harmonious labeling. Hence, we focused on graphs that were known to be neither graceful nor harmonious. Even though we have increased the number of labeling options in a graceful-harmonious labeling from those in either a graceful labeling or a harmonious labeling, the number of graphs of order n with such a labeling will still tend to zero as n tends to infinity. In fact, the proof given in Graham and Sloane's 1980 paper on harmonious labelings can be adapted to prove the following.

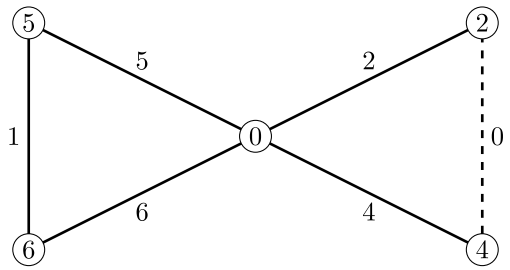

Figure 1: Graph from Example 2.1.

Theorem 2.2. *Almost all graphs are not graceful-harmonious.*

In the following pages, we show that several classes of graphs that fail to be both graceful and harmonious are indeed graceful-harmonious. We also provide examples of graphs that are not graceful-harmonious.

3 Cycles

Based on the results of Rosa and Graham and Sloane, we already know that cycles C_n of order $n \equiv 2 \pmod 4$ are the only cycles that are neither graceful nor harmonious.

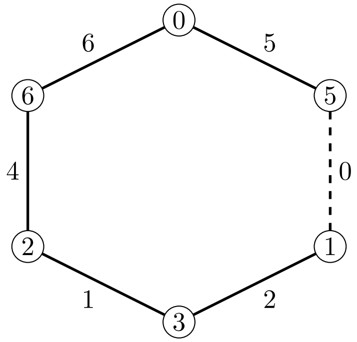

Figure 2: Graceful-harmonious labeling of C_6

Theorem 3.1. *All cycles have a graceful-harmonious labeling.*

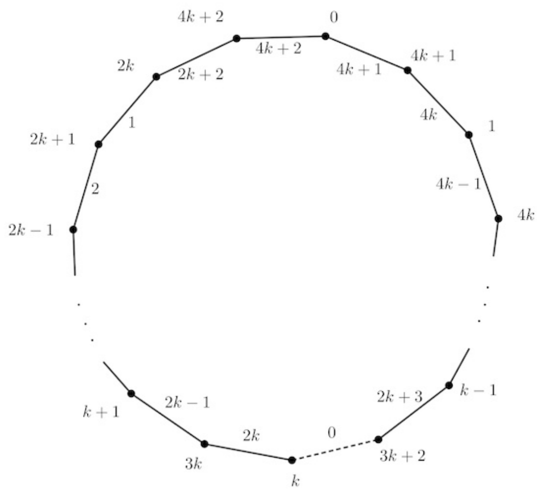

Figure 3: Graceful-harmonious labeling of a cycle of order $n = 4k + 2$ from Theorem 3.1

Proof. Let $n = 4k+2$, $k \geq 1$, and $C_n = (v_1, v_2, \ldots, v_n, v_1)$. First, label the odd-indexed vertices of C_n, starting at v_1, in one direction around the cycle with $0, 1, 2, \ldots, 2k$. Then label v_n with $n = 4k + 2$, and going in the same direction as the labeling of the odd-indexed vertices, continue to label the even-indexed vertices with: $4k+1, 4k, \ldots, 3k+3, 3k+2, 3k, 3k-1, \ldots, 2k+1$. To obtain the edge labels, subtract everywhere except for the edge $v_{2k}v_{2k+1}$, between the vertices labeled k and $3k+2$, where you add modulo $4k+2$ to get 0. The set of edge labels are described below.

- $\{1, 2, \ldots, 2k-1, 2k\}$ for edges v_iv_{i+1} with $2k+1 \leq i \leq n-2$
- $2k+2$ for the edge v_nv_{n-1}
- $\{2k+3, 2k+4, \ldots, 4k+1\}$ for edges v_iv_{i+1} with $1 \leq i \leq 2k-1$
- $4k+2$ for the edge v_nv_1
- 0 for the edge $v_{2k}v_{2k+1}$

Note: The skipped vertex label is $3k+1$, and the skipped edge label is $2k+1$. □

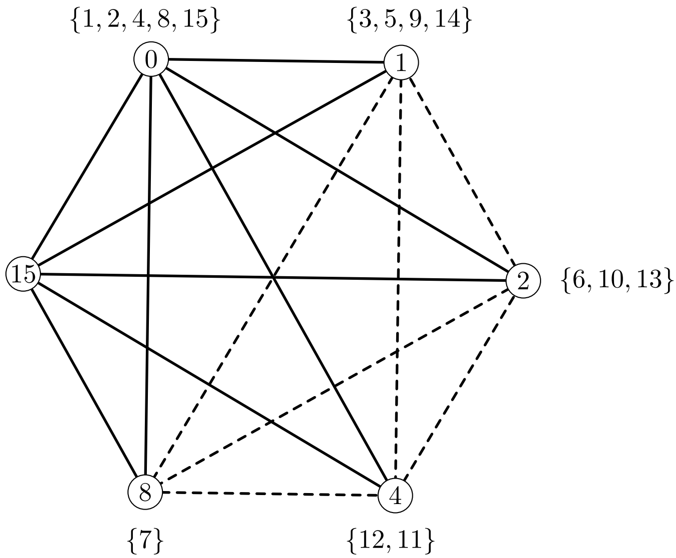

Figure 4: Graceful-harmonious labeling of K_6

The previous result shows that Rosa's parity condition for graceful graphs (i.e. if every vertex of a graph has even degree and the number of edges is congruent to 1 or 2 (mod 4), then the graph is not graceful) does not extend to graceful-harmonious labelings. However, we were able to use a parity argument in showing that some complete graphs do not have graceful-harmonious labelings.

4 Complete graphs

By Propositions 1.2 and 1.5, we know that all complete graphs of order greater than four are neither graceful nor harmonious, while complete graphs K_2, K_3, and K_4 are both graceful and harmonious. Graceful-harmonious labelings for K_6 and K_7 are seen in Figures 4 and 5. The labels of the clockwise incident edges of each vertex is denoted in the set next to the vertex. Jeremy Alm of Lamar University wrote a computer program to obtain graceful-harmonious labelings for K_9, K_{10}, and K_{11} after seeing our presentation on this topic at the 51st Southeastern International Conference on Combinatorics, Graph Theory & Computing. [1] These labelings are given in the Appendix.

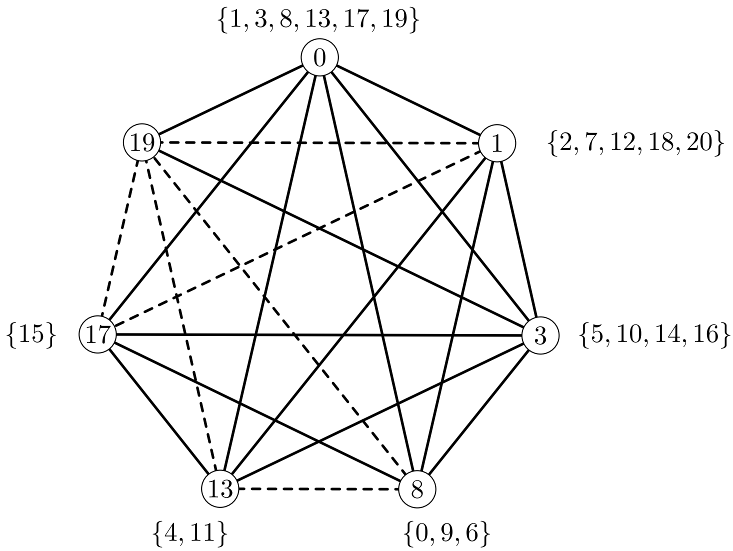

Figure 5: Graceful-harmonious Labeling of K_7

However, we know that not all complete graphs are graceful-harmonious. The first such complete graph that fails to have a graceful-harmonious labeling is K_5. The sets of possible vertex and edge labels for a graceful-harmonious labeling of K_5 contain six even numbers and five odd numbers. Since the modulus in this case would be even, the only allowable distribution of even and odd vertex labels would be four even vertices and one odd vertex, which forces 6 even-labeled edges and 4 odd-labeled edges. In particular, this means that there must be an edge uv labeled 10 and an edge xy labeled 0. (Necessarily, the vertices x, y, u, and v are distinct.) Say $f(u) = 0$ and $f(v) = 10$. We know that $[f(x) + f(y)] \equiv 0 \pmod{10}$, so $f(x) = 10 - f(y)$. However, this implies that ux and vy have the same label.

In fact, we can find infinite classes of complete graphs which do not have graceful-harmonious labelings. To see this, consider the following observations.

Lemma 4.1. *If G is the complete graph K_n, then any graceful-harmonious labeling of G will leave out either* 0 *or m as edge labels.*

Proof. Let f be a graceful-harmonious labeling of $G = K_n$, with induced edge labeling f'. Assume to the contrary that there are two edges $uv, xy \in$

$V(G)$ such that $f'(uv) = m$ and $f'(xy) = 0$. Necessarily, $m = f'(uv) = |f(u) - f(v)|$ and $f(u), f(v) \in \{0, m\}$. Let $f(u) = 0$ and $f(v) = m$. We can also conclude that $0 = f'(xy) = f(x) + f(y) \pmod{m}$. Thus, $f(x) = m - f(y)$. However, this implies that ux and vy have the same label, which is a contradiction:

$$f'(ux) = f(x) = m - f(y) = f(v) - f(y) = f'(vy)$$

□

Corollary 4.2. *Let G be the complete graph K_n of even size m. Then any graceful-harmonious labeling of G will use all odd edge labels and leave out exactly one even edge label.*

Proof. Let G have a graceful-harmonious labeling. Note that the induced edge labeling of G is an injective function from the set of edges to the set $\{0, 1, \ldots, m\}$. Since m is an even number, then this set contains exactly $\frac{m}{2}$ odd numbers and $\frac{m}{2} + 1$ even numbers. By Lemma 4.1 we will leave out exactly one of 0 or m, both even numbers. Therefore, we must use all of the odd edge labels. □

We use these observations to explore the possibility of a graceful-harmonious labeling of the complete graph K_n with an even number of edges. We consider two cases: $n = 0 \pmod 4$ and $n = 1 \pmod 4$.

Proposition 4.3. *Let G be the complete graph K_n with $n \geq 8$ and $n = 4k$. If G has a graceful-harmonious labeling, then k is a perfect square.*

Proof. Suppose that G has a graceful-harmonious labeling. We can suppose without loss of generality that there are $\frac{n}{2} + \ell$ vertices labeled with even numbers and $\frac{n}{2} - \ell$ vertices labeled with odd numbers, for some $\ell \geq 0$. We use the following calculation to determine the number of even-labeled edges.

$$\begin{aligned} \binom{\frac{n}{2}+\ell}{2} + \binom{\frac{n}{2}-\ell}{2} &= \binom{2k+\ell}{2} + \binom{2k-\ell}{2} && (1) \\ &= 4k^2 - 2k + \ell^2. && (2) \end{aligned}$$

Thus, there are $4k^2 - 2k + \ell^2$ even edges. This in turn implies that there are $m - (4k^2 - 2k + \ell^2) = 4k^2 - \ell^2$ edges with odd-numbered labels. And, by Corollary 4.2, the induced graceful-harmonious edge labeling of G produces exactly $\frac{m}{2} = 4k^2 - k$ odd numbers. Hence, $(4k^2 - k) = (4k^2 - \ell^2)$, which can be simplified to show $\ell^2 = k$. □

The same technique can be applied to prove the following proposition as well.

Proposition 4.4. *If K_n has a graceful-harmonious labeling when $n = 4k+1$, for $k \geq 1$, then $k = \ell^2 - \ell$ for some positive integer ℓ.*

The contrapositive of these propositions eliminates the possibility of a graceful-harmonious labeling for infinitely many complete graphs, including the following: K_8, K_{12}, K_{13}, K_{17}, K_{20}, K_{21}, K_{24}, K_{25}, K_{28}, K_{29}, etc. The smallest complete graph for which it is unknown whether or not there is a graceful-harmonious labeling is K_{14}.

5 Friendship graphs

By Propositions 1.3 and 1.6, we know that the friendship graph F_n is neither graceful nor harmonious if and only if $n \equiv 2$ $(mod\ 4)$. An example of a graceful-harmonious labeling of F_2 was given in Figure 1. We generalize this labeling below.

Theorem 5.1. *All friendship graphs have a graceful-harmonious labeling.*

Proof. It suffices to give a graceful-harmonious labeling for friendship graphs F_n for $n \equiv 2 \pmod 4$. Let $n = 4k+2$, $k \geq 1$. Denote the central vertex as c, and the ith triangle as (c, a_i, b_i, c) for $1 \leq i \leq n$. Label the central vertex with 0, and label each vertex a_i with the label $2k+i$ for $1 \leq i \leq n$. Then, label the vertices b_i with the label $m - i + 1 = 12k + 7 - i$ for $1 \leq i \leq 2k+1$. Label the rest of the b_i vertices with the sequence $10k+4, 10k+2, \ldots, 6k+6, 6k+4$ starting at b_{2k+2} and proceeding in ascending order of indices to b_n.

To obtain the edge labels, add all vertex labels modulo $12k+6$ between vertices a_i and b_i with $2k+2 \leq i \leq n$ and subtract elsewhere. The set of edge labels are described below.

- $\{0, 1, \ldots, 2k\}$, for edges $a_i b_i$ with $2k+2 \leq i \leq n$
- $\{2k+1, 2k+2, \ldots, 6k+2\}$, for edges ca_i with $1 \leq i \leq n$
- $\{6k+4, 6k+6, \ldots 10k+4\}$, for edges cb_i with $2k+2 \leq i \leq n$
- $\{6k+5, 6k+7, \ldots, 10k+5\}$, for edges $a_i b_i$ with $1 \leq i \leq 2k+1$
- $\{10k+6, 10k+7, \ldots, 12k+6\}$, for edges cb_i with $1 \leq i \leq 2k+1$

Note that we leave out the edge label $6k+3 = \frac{m}{2}$. □

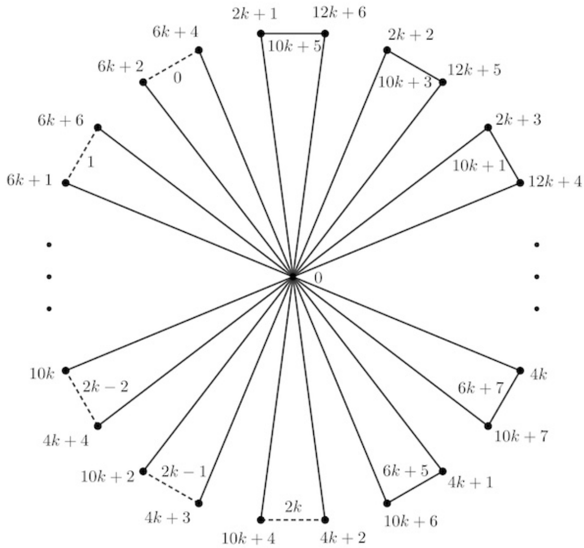

Figure 6: Graceful-harmonious labeling of F_n where $n \equiv 2$ $(mod\ 4)$ as described in the proof of Theorem 5.1

6 Double cones

The double cone $C_{n,2} = C_n + \overline{K_2}$ was shown to be graceful for $n = 3, 4, 5, 7, 8, 9, 11, \text{and } 12$, and not graceful for $n \equiv 2 \pmod 4$. [7] When n is odd, the double cone $C_{n,2}$ is harmonious. It remains to investigate the double cone $C_{n,2}$ when n is even. In half of these cases, we know the double cone has a graceful-harmonious labeling.

Proposition 6.1. *For $n = 4k$, $k \geq 1$, the double cone $C_{n,2}$ has a graceful-harmonious labeling.*

Proof. Suppose $n = 4k$, $k \geq 1$, and let $C_{n,2}$ be the cycle

$$C_n = (v_1, v_2, \ldots, v_n, v_1)$$

joined with two vertices u and w. First, label the odd-indexed vertices of C_n, starting at v_1, in one direction around the cycle with $0, 1, 2, \ldots, 2k-1$. Then, label v_2 with $2k$ and, going in the same direction as the labeling of the odd-indexed vertices, continue to label the rest of the even-indexed vertices with: $2k+1, 2k+2, \ldots, 4k-1$. (See Figure 7.) Finally, label the non-cycle vertices u and w with $6k-1$ and $10k-1$. To obtain the edge labels, add everywhere modulo $12k$ except for the edge $v_{2k}v_{2k+1}$ on C_n

where the vertex labels $3k-1$ and k are subtracted to get the edge label of $2k-1$. The set of edge labels are described below.

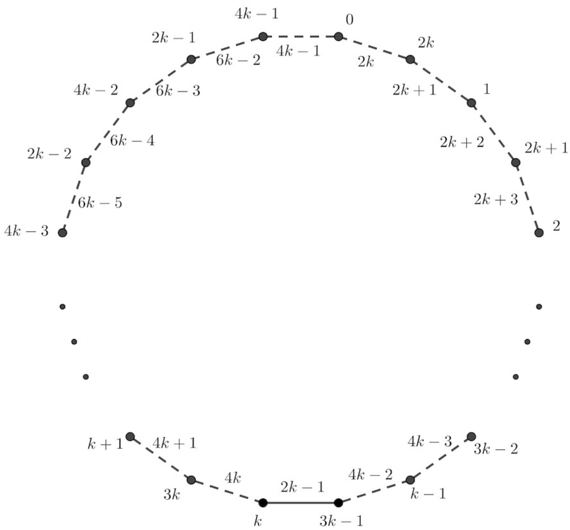

Figure 7: Graceful-harmonious labeling of the cycle subgraph of the double cone $C_{n,2}$ for $n = 4k$ as described in the proof of Proposition 6.1

- $\{2k-1, 2k, 2k+1, \ldots, 6k-2\}$ for edges on the cycle, $v_i v_{i+1} \pmod{n}$ for $1 \le i \le n$
- $\{6k-1, 6k, 6k+1, \ldots, 10k-2\}$ for edges uv_i for $1 \le i \le n$
- $\{10k-1, 10k, \ldots, 12k-1, 0\}$ for edges wv_i for $1 \le i \le n$

Note that the edge label $m = 12k$ is skipped. □

We have yet to determine whether or not double cones $C_{n,2}$ with $n = 4k+2$ have graceful-harmonious labelings. We were able to produce a graceful-harmonious labeling when $k = 1$, i.e. for $C_{6,2}$. But it remains unknown which double cones $C_{n,2}$ with $n = 4k+2$ and $k > 1$ are graceful-harmonious.

Proposition 6.2. *The double cone $C_{6,2}$ has a graceful-harmonious labeling.*

Proof. See Figure 8 for a graceful-harmonious labeling of $C_{6,2}$. □

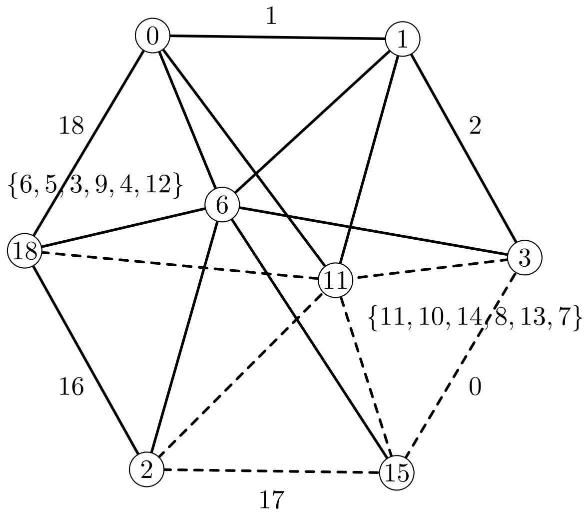

Figure 8: Graceful-harmonious labeling of the double cone $C_{6,2}$

7 Future work

We continue to work on graceful-harmonious labelings of complete graphs of order at least 14 as well as more general results for all complete graphs. There are many open questions presented in this paper, such as whether or not graceful-harmonious labelings for double cones of order $n = 4k + 2$, $k > 1$ exist in general, that can be investigated further. There are also other classes of graphs that are known to be neither graceful nor harmonious that would be interesting to study. One such class of graphs is that of windmill and Dutch windmill graphs, for which the friendship graph is a special case. We have partial results in this area and are working to generalize our results.

Of particular interest in any graph labeling is also the study of trees. Since the Graceful Tree Conjecture and Harmonious Tree Conjecture hypothesize that all trees are graceful and all trees are harmonious, then we conjecture that all trees have a graceful-harmonious labelings as well. However, we do hope to further investigate graceful-harmonious labelings of trees in hopes of gaining information and insights in the graceful and harmonious labelings of trees.

Acknowledgements

We greatly appreciate the contributions of Jeremy Alm, who was able to find the elusive graceful-harmonious labelings of the complete graphs of order 9, 10, and 11 using his expertise in computer programming.

We would also like to express our gratitude to Luke Nappi, a graduate student at Youngstown State University at the time of this research. We appreciate Mr. Nappi's collaboration in the beginning stages of this research and his contributions to the topic.

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8 Appendix

Below are graceful-harmonious labelings of K_9, K_{10}, and K_{11}. Recall that solid edges indicate edge labels obtained by subtraction, and dashed edges indicate edge labels obtained by addition.

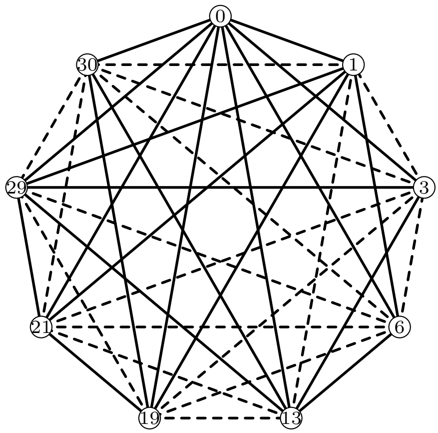

Figure 9: Graceful-harmonious labeling of K_9

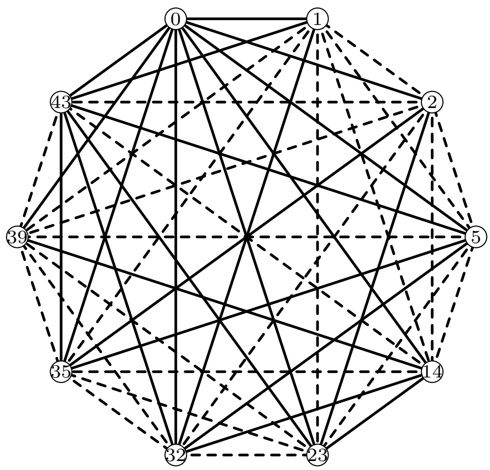

Figure 10: Graceful-harmonious labeling of K_{10}

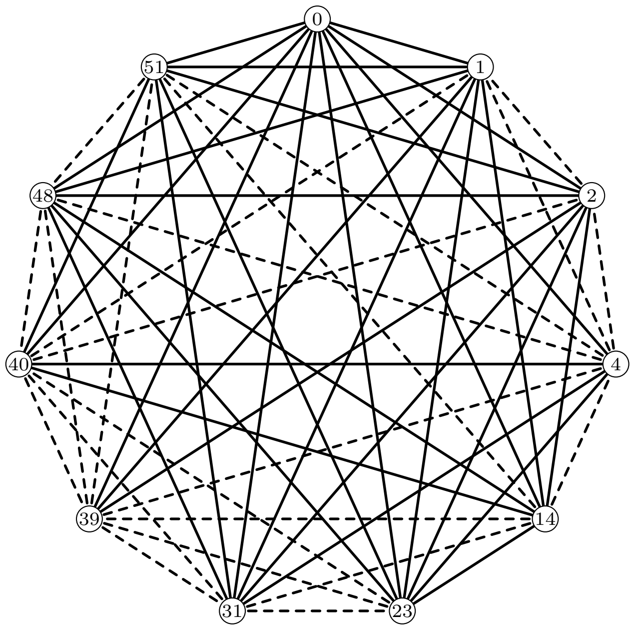

Figure 11: Graceful-harmonious labeling of K_{11}

Alexis Byers
Youngstown State University, Youngstown, Ohio, USA
abyers@ysu.edu

Anita O'Mellan
Youngstown State University, Youngstown, Ohio, USA
acomellan@ysu.edu