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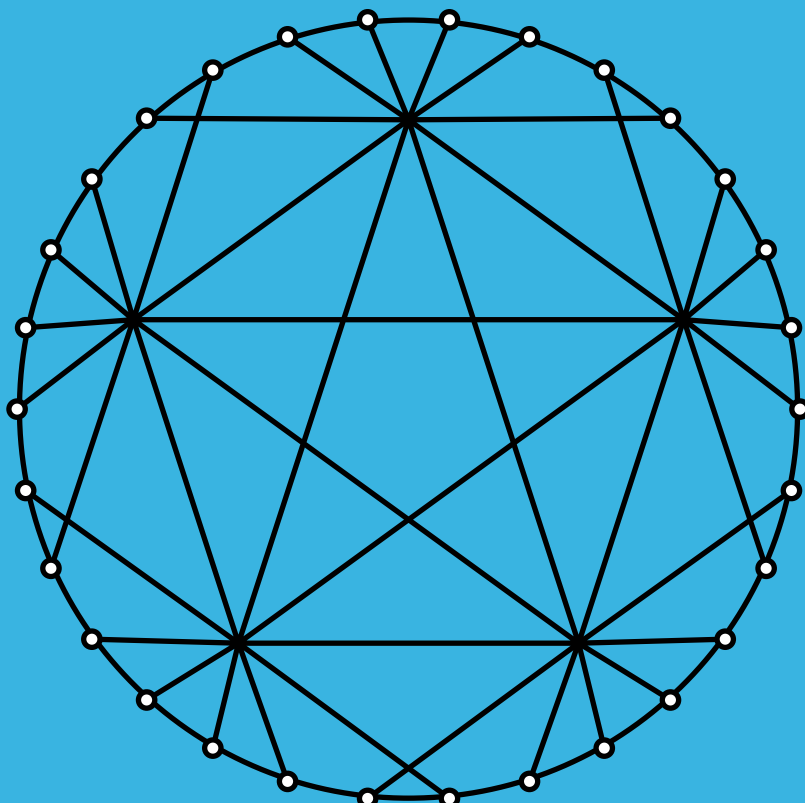

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$4t$-cycle decomposition of the 2-fold tensor product $(K_m \times K_n)(2)$

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Abstract. In this paper, it is shown that if t, m and n are positive integers with $t \geq 3$ is odd, $m \geq 3$, $n \geq 3$ and $mn \geq 4t$, then the 2-fold of the tensor product of complete graphs K_m and K_n, that is $(K_m \times K_n)(2)$, admits a decomposition into cycles of length $4t$, whenever $m \equiv 0, 1 \pmod{t}$, or $n \equiv 0, 1 \pmod{t}$. For any prime p, a necessary and sufficient condition for the existence of a $4p$-cycle decomposition of $(K_m \times K_n)(2)$ is also obtained.

1 Introduction and definitions

For a simple graph G and a positive integer λ, the graph $G(\lambda)$ is the graph obtained from G by replacing each of its edges by λ parallel edges. For a graph G and a positive integer λ, λG denotes λ mutually vertex disjoint copies of G. Let P_k (respectively, C_k) denote a path (respectively, cycle) on k vertices. The complete graph on m vertices is denoted by K_m. For a simple graph G, $\overline{G}$ denotes the *complement* of G.

If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of the graph G such that $E(G) = \bigcup_{i=1}^{k} E(H_i)$, then $H_1, H_2, \ldots, H_k$ *decompose* G and we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$. If for each i, $i \in \{1, 2, \ldots, k\}$, $H_i \cong H$, then G has a *H-decomposition* and we write $H|G$. A graph G has a *C_k-decomposition* or a *k-cycle decomposition* whenever $C_k|G$. A k-regular graph G is *Hamilton cycle decomposable* if G is decomposable into $\frac{k}{2}$ Hamilton cycles when k is even and into $\frac{k-1}{2}$ Hamilton cycles together with a 1-factor when k is odd. For simple graphs G and H, the *tensor product* of G and H, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$.

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For simple graphs G and H, and positive integers λ_1 and λ_2, the *tensor product* of $G(\lambda_1)$ and $H(\lambda_2)$, denoted by $G(\lambda_1) \times H(\lambda_2)$, is $(G \times H)(\lambda_1\lambda_2)$. In particular, for simple graphs G and H, and a positive integer λ, the tensor products $G(\lambda) \times H$ and $G \times H(\lambda)$ are $(G \times H)(\lambda)$.

Clearly, the tensor product is commutative and distributive over edge-disjoint union of graphs; that is, $G \times H \cong H \times G$, and if

$$G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$$

then

$$G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H).$$

Similarly, for simple graphs G and H, the *wreath product* of G and H, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ or, $g_1 = g_2$ and $h_1h_2 \in E(H)$.

Clearly, $K_m \circ \overline{K}_n$ is isomorphic to the complete m-partite graph in which each partite set has n vertices and $(K_m \circ \overline{K}_n) - E(nK_m) \cong K_m \times K_n$.

For graph theoretical terms not defined here see [12, 43].

For non-negative integers a and b with $a < b$, we denote the set

$$\{a, a+1, a+2, \ldots, b\}$$

by $[a, b]$.

Finding a C_k-decomposition of K_{2n+1} or $K_{2n} - F$, where F is a 1-factor of K_{2n}, is completely settled by Alspach et al. [2] and Šajna [36]. An alternate proof for a C_{2k+1}-decomposition of K_{2n+1} is obtained by Buratti [19]. Alspach et al. [3] obtained a necessary and sufficient condition for the existence of a k-cycle decomposition of $K_n(2)$. Smith [39] proved that the necessary conditions are sufficient for the existence of a p-cycle decomposition of $K_n(\lambda)$, where $p \geq 3$ is a prime. In [17, 18], it is proved that the necessary conditions are sufficient for the existence of $K_n(\lambda)$ to admit a decomposition into cycles of variable lengths, or into cycles of variable lengths and a 1-factor. In [41], Sotteau proved that $C_{2k}|K_{a,b}$ whenever the obvious necessary conditions are satisfied. Asplund et al. [8] proved that $K_{a,b}(\lambda)$ can be decomposed into cycles of different even lengths whenever the necessary conditions are satisfied. In [23], Hanani proved that $C_3|(K_m \circ \overline{K}_n)(\lambda)$ whenever the necessary conditions are satisfied. Billington et al. [14] proved that

$C_5|(K_m \circ \overline{K}_n)(\lambda)$ whenever the necessary conditions are satisfied. Further, for $k \in \{2,3,4\}$, Cavenagh [21], solved the C_{2k}-decomposition problem for complete multipartite graphs. Manikandan and Paulraja [28, 29] obtained a necessary and sufficient condition for the existence of a C_p-decomposition of $K_m \circ \overline{K}_n$, where $p \geq 5$ is a prime. In [37, 38, 40], it is proved that the necessary conditions for the existence of C_k-decomposition, $k \in \{2p, 3p, p^2\}$, of $K_m \circ \overline{K}_n$ are sufficient. Further, in [35], Muthusamy and Shanmuga Vadivu proved the existence of a C_k-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ whenever k is even. Irrespective of the pairity of k, Buratti et al. [20] actually solved the existence problem for a k-cycle decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ whose cycle-set can be partitioned into 2-regular graphs containing all the vertices except those belonging to one part. Horsley [24], studied the decompositions of various graphs into short even-length cycles. Recently, in [10], Bahmanian and Šajna, developed two techniques layering and detachment; using these techniques studied the existence of resolvable cycle decompositions of complete multigraphs and complete equipartite multigraphs. Decompositions of $(K_m \circ \overline{K}_n)(\lambda)$ into cycles of variable lengths are considered in [9].

A similar problem of decomposing $(K_m \times K_n)(\lambda)$, a proper spanning subgraph of $(K_m \circ \overline{K}_n)(\lambda)$, into cycles of length k is considered here. In the study of group divisible designs (respectively, modified group divisible designs), the edge sets of $K_m \circ \overline{K}_n$ (respectively, $K_m \times K_n$) is partitioned into complete subgraphs, see [4, 5, 6, 7, 15, 16, 23, 26]. Assaf [5] used modified group divisible designs to construct covering designs, packing designs and group divisible designs with block size 5. For prime $p \geq 5$, existence of a p-cycle decomposition of $K_m \times K_n$ is effectively used to obtain a p-cycle decomposition of $K_m \circ \overline{K}_n$, see [28, 29]. Further, Hamilton cycle decomposition of $K_m \times K_n$ is completely settled by Balakrishnan et al. [11]. Hence the graph $K_m \times K_n$ is an important regular subgraph of $K_m \circ \overline{K}_n$. For related developments of the study of Hamilton cycle decompositions in tensor products of complete multipartite graphs, or a complete graph and a complete bipartite graph, or a complete bipartite graph and a complete multipartite graph see [27, 30, 31]. Recently, Ganesamurthy et al. [22] obtained a necessary and sufficient condition for the existence of a C_{4p}-decomposition of $K_m \times K_n$, where $p \geq 3$ is a prime. In [34], Paulraja and Sivakaran obtained a necessary and sufficient condition for the graph $(K_m \times K_n)(2)$ to admit a k-cycle decomposition, where $k \in \{p, 2p, 3p, p^2\}$ and p is a prime.

The necessary conditions for the existence of a C_{4t}-decomposition of $(K_m \times K_n)(\lambda)$ is that $4t$ divides $\frac{\lambda mn(m-1)(n-1)}{2}$ and $\lambda(m-1)(n-1)$, the degree of each vertex of $(K_m \times K_n)(\lambda)$, is divisible by 2, the degree of each vertex of C_{4t}.

In this paper, we obtain the following results.

Theorem 1.1. *Let t, m and n be positive integers with $t \geq 3$ is odd, $m \geq 3$, $n \geq 3$ and $mn \geq 4t$. Then, the 2-fold of the tensor product of complete graphs K_m and K_n, that is, $(K_m \times K_n)(2)$, has a $4t$-cycle decomposition, whenever $m \equiv 0, 1 \pmod{t}$ or $n \equiv 0, 1 \pmod{t}$*

Theorem 1.2. *Let p, m and n be positive integers with $p \geq 2$ is prime, $m \geq 3$, $n \geq 3$ and $mn \geq 4p$. Then, $C_{4p}|(K_m \times K_n)(2)$ if and only if $4p|m(m-1)n(n-1)$.*

2 Preliminary results

We use the following notation for the vertices of $G \times H$. Let $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(H) = \{y_1, y_2, \ldots, y_n\}$. Then, $V(G \times H) = \{v_i^j : i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n\}$, where $v_i^j = (x_i, y_j)$.

Write, for $i \in \{1, 2, \ldots, m\}$, $\{x_i\} \times V(H) = \{v_i^1, v_i^2, \ldots, v_i^n\}$ by V_i, the i^{th}-*layer* of vertices of $G \times H$ corresponding to x_i.

Consider the complete bipartite graph $K_{n,n}$ with bipartition (V_r, V_s), where $r, s \in \{1, 2, \ldots, m\}$, $r \neq s$, $V_r = \{v_r^1, v_r^2, \ldots, v_r^n\}$ and $V_s = \{v_s^1, v_s^2, \ldots, v_s^n\}$. For $\ell \in \{0, 1, \ldots, n-1\}$, let $F_\ell(V_r, V_s) = \{v_r^t v_s^{t+\ell} | t = 1, 2, \ldots, n\}$, where addition $t + \ell$ in the superscript of $v_s^{t+\ell}$ is taken modulo n with residues $1, 2, \ldots, n$. The edge $v_r^t v_s^{t+\ell} \in F_\ell(V_r, V_s)$ is called an *edge of length ℓ from V_r to V_s*. Note that, $F_\ell(V_r, V_s) = F_{n-\ell}(V_s, V_r)$. So, the edge $v_r^t v_s^{t+\ell}$ is also called an *edge of length $n - \ell$ from V_s to V_r*. The *rotation-distance* of two edges $v_r^{t_1} v_s^{t_1+\ell}$, $v_r^{t_2} v_s^{t_2+\ell}$ in $F_\ell(V_r, V_s)$, where $t_1, t_2 \in \{1, 2, \ldots, n\}$, of same length ℓ from V_r to V_s is defined as $\min\{|t_1 - t_2|, n - |t_1 - t_2|\}$. Note that, rotation-distances are in $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

Define a permutation σ on $V(G \times H)$ $(= V((G \times H)(2)))$ as follows: for every $i \in \{1, 2, \ldots, m\}$, $\sigma(v_i^j) = v_i^{j+1}$ if $j \in \{1, 2, \ldots, n-1\}$ and $\sigma(v_i^n) = v_i^1$.

2.1 $P_{2t+1} \times K_3$

Lemma 2.1. *If $t \geq 3$ is an odd integer, then $C_{4t}|(P_{2t+1} \times K_3)$.*

Proof. Let the path P_{2t+1} be $x_1x_2x_3 \ldots x_{2t+1}$ and let $V(K_3) = \{y_1, y_2, y_3\}$. Then

$$C = v_1^1 - v_2^3v_3^2v_4^3v_5^2v_6^3v_7^2 \cdots v_{2t-2}^3v_{2t-1}^2v_{2t}^3 - v_{2t+1}^1 - v_{2t}^2v_{2t-1}^3v_{2t-2}^2v_{2t-3}^3v_{2t-4}^2v_{2t-5}^3 \cdots v_4^2v_3^3v_2^2 - v_1^1$$

is a cycle of length $4t$ in $P_{2t+1} \times K_3$ containing: for each $i \in [1, 2t]$ and for each $\ell \in \{1, 2\}$, one edge of length ℓ from V_i to V_{i+1}. Hence, $\{C, \sigma(C), \sigma^2(C)\}$ is a decomposition of $P_{2t+1} \times K_3$. □

2.2 $(P_{t+1} \times K_6)(2)$

Lemma 2.2. *If $t \geq 3$ is an odd integer, then $C_{4t}|(P_{t+1} \times K_6)(2)$.*

Proof. Let the path P_{t+1} be $x_1x_2x_3 \ldots x_{t+1}$. First, we find five $4t$-cycles of $(P_{t+1} \times K_6)(2)$ as follows:

$$\begin{aligned} C_{4t}^1 = v_1^1 - v_2^2v_3^3v_4^2v_5^3v_6^2v_7^3 \ldots v_{t-1}^2v_t^3 - v_{t+1}^4 \\ - v_t^6v_{t-1}^4v_{t-2}^6v_{t-3}^4v_{t-4}^6v_{t-5}^4 \ldots v_3^6v_2^4 - v_1^2 \\ - v_2^6v_3^4v_4^6v_5^4v_6^6v_7^4 \ldots v_{t-1}^6v_t^4 - v_{t+1}^3 \\ - v_t^2v_{t-1}^3v_{t-2}^2v_{t-3}^3v_{t-4}^2v_{t-5}^3 \ldots v_3^2v_2^3 - v_1^1 \end{aligned}$$

(C_{4t}^1 contains: one edge of length 1, two edges of length 2 and one edge of length 4 from V_1 to V_2; for each $i \in [2, t-1]$ and for each $\ell \in \{1, 2, 4, 5\}$, one edge of length ℓ from V_i to V_{i+1}; two edges of length 1, one edge of length 4 and one edge of length 5 from V_t to V_{t+1}),

$$\begin{aligned} C_{4t}^2 = v_1^2 - v_2^1v_3^3v_4^1v_5^3v_6^1v_7^3 \ldots v_{t-1}^1v_t^3 - v_{t+1}^4 \\ - v_t^5v_{t-1}^6v_{t-2}^5v_{t-3}^6v_{t-4}^5v_{t-5}^6 \ldots v_3^5v_2^6v_1^1 \\ - v_2^5v_3^6v_4^5v_5^6v_6^5v_7^6 \ldots v_{t-1}^5v_t^6 - v_{t+1}^3 \\ - v_t^1v_{t-1}^3v_{t-2}^1v_{t-3}^3v_{t-4}^1v_{t-5}^3 \ldots v_3^1v_2^3\text{-}v_1^2 \end{aligned}$$

(C_{4t}^2 contains: one edge of length 1, one edge of length 4 and two edges of length 5 from V_1 to V_2; for each $i \in [2, t-1]$ and for each $\ell \in \{1, 2, 4, 5\}$, one edge of length ℓ from V_i to V_{i+1}; for each $\ell \in \{1, 2, 3, 5\}$, one edge of length ℓ from V_t to V_{t+1}),

$$\begin{aligned} C_{4t}^3 = v_1^1 - v_2^4 v_3^6 v_4^4 v_5^6 v_6^4 v_7^6 \dots v_{t-1}^4 v_t^6 - v_{t+1}^3 \\ - v_t^5 v_{t-1}^3 v_{t-2}^5 v_{t-3}^3 v_{t-4}^5 v_{t-5}^3 \dots v_3^5 v_2^3 - v_1^6 \\ - v_2^5 v_3^3 v_4^5 v_5^3 v_6^5 v_7^3 \dots v_{t-1}^5 v_t^3 - v_{t+1}^5 \\ - v_t^4 v_{t-1}^6 v_{t-2}^4 v_{t-3}^6 v_{t-4}^4 v_{t-5}^6 \dots v_3^4 v_2^6 - v_1^1 \end{aligned}$$

(C_{4t}^3 contains: for each $\ell \in \{3, 5\}$, two edges of length ℓ from V_1 to V_2; for each $i \in [2, t-1]$ and for each $\ell \in \{2, 4\}$, two edges of length ℓ from V_i to V_{i+1}; for each $\ell \in [1, 4]$, one edge of length ℓ from V_t to V_{t+1}),

$$\begin{aligned} C_{4t}^4 = v_1^1 - v_2^3 v_3^4 v_4^3 v_5^4 v_6^3 v_7^4 \dots v_{t-1}^3 v_t^4 - v_{t+1}^2 \\ - v_t^6 v_{t-1}^1 v_{t-2}^6 v_{t-3}^1 v_{t-4}^6 v_{t-5}^1 \dots v_3^6 v_2^1 - v_1^4 \\ - v_2^6 v_3^1 v_4^6 v_5^1 v_6^6 v_7^1 \dots v_{t-1}^6 v_t^1 - v_{t+1}^6 \\ - v_t^3 v_{t-1}^4 v_{t-2}^3 v_{t-3}^4 v_{t-4}^3 v_{t-5}^4 \dots v_3^3 v_2^4 - v_1^1 \end{aligned}$$

(C_{4t}^4 contains: for each $\ell \in [2, 3]$, two edges of length ℓ from V_1 to V_2; for each $i \in [2, t-1]$ and for each $\ell \in \{1, 5\}$, two edges of length ℓ from V_i to V_{i+1}; for each $\ell \in [2, 5]$, one edge of length ℓ from V_t to V_{t+1}),
and

$$\begin{aligned} C_{4t}^5 = v_1^1 - v_2^2 v_3^5 v_4^2 v_5^5 v_6^2 v_7^5 \dots v_{t-1}^2 v_t^5 - v_{t+1}^2 \\ - v_t^6 v_{t-1}^3 v_{t-2}^6 v_{t-3}^3 v_{t-4}^6 v_{t-5}^3 \dots v_3^6 v_2^3 - v_1^2 \\ - v_2^6 v_3^3 v_4^6 v_5^3 v_6^3 v_7^6 \dots v_{t-1}^6 v_t^3 - v_{t+1}^1 \\ - v_t^2 v_{t-1}^5 v_{t-2}^2 v_{t-3}^5 v_{t-4}^2 v_{t-5}^5 \dots v_3^2 v_2^5 - v_1^1 \end{aligned}$$

(C_{4t}^5 contains: for each $\ell \in \{1, 4\}$, two edges of length ℓ from V_1 to V_2; for each $i \in [2, t-1]$, four edges of length 3 from V_i to V_{i+1}; for each $\ell \in [2, 5]$, one edge of length ℓ from V_t to V_{t+1}).

(For each $\ell \in [1, 5]$, we pair the four edges of length ℓ from V_1 to V_2 as follows: For $\ell = 1$, the four edges are: $v_1^1 v_2^2 \in C_{4t}^1$, $v_1^2 v_2^3 \in C_{4t}^2$ and $v_1^1 v_2^2$, $v_1^2 v_2^3 \in C_{4t}^5$; pair these edges as $(v_1^1 v_2^2, v_1^2 v_2^3)$ and $(v_1^1 v_2^2, v_1^2 v_2^3)$; both pairs have rotation-distance 1. For $\ell = 2$, the four edges are: $v_1^1 v_2^3$, $v_1^2 v_2^4 \in C_{4t}^1$ and $v_1^1 v_2^3$, $v_1^4 v_2^6 \in C_{4t}^4$; pair these edges as $(v_1^1 v_2^3, v_1^2 v_2^4)$ and $(v_1^1 v_2^3, v_1^4 v_2^6)$; first pair is of rotation-distance 1 and that for the second pair is 3. For $\ell = 3$, the four edges are: $v_1^1 v_2^4$, $v_1^6 v_2^3 \in C_{4t}^3$ and $v_1^1 v_2^4$, $v_1^4 v_2^1 \in C_{4t}^4$; pair these edges as

$(v_1^1v_2^4, v_1^6v_2^3)$ and $(v_1^1v_2^4, v_1^4v_2^1)$; first pair is of rotation-distance 1 and that for the second pair is 3. For $\ell = 4$, the four edges are: $v_1^2v_2^6 \in C_{4t}^1$, $v_1^1v_2^5 \in C_{4t}^2$ and $v_1^1v_2^5$, $v_1^2v_2^6 \in C_{4t}^5$; pair these edges as $(v_1^2v_2^6, v_1^1v_2^5)$ and $(v_1^1v_2^5, v_1^2v_2^6)$; both pairs have rotation-distance 1. For $\ell = 5$, the four edges are: $v_1^1v_2^6$, $v_1^2v_2^1 \in C_{4t}^2$ and $v_1^1v_2^6$, $v_1^6v_2^5 \in C_{4t}^3$; pair these edges as $(v_1^1v_2^6, v_1^2v_2^1)$ and $(v_1^1v_2^6, v_1^6v_2^5)$; both pairs have rotation-distance 1.

For $i \in [2, t-1]$ and for each $\ell \in [1, 5]$, we pair the four edges of length ℓ from V_i to V_{i+1} as follows: For $\ell = 1$, the four edges are: $v_i^2v_{i+1}^3 \in C_{4t}^1$, $v_i^5v_{i+1}^6 \in C_{4t}^2$ and $v_i^3v_{i+1}^4$, $v_i^6v_{i+1}^1 \in C_{4t}^4$; pair these edges as $(v_i^2v_{i+1}^3, v_i^5v_{i+1}^6)$ and $(v_i^3v_{i+1}^4, v_i^6v_{i+1}^1)$; both pairs have rotation-distance 3. For $\ell = 2$, the four edges are: $v_i^4v_{i+1}^6 \in C_{4t}^1$, $v_i^1v_{i+1}^3 \in C_{4t}^2$ and $v_i^3v_{i+1}^5$, $v_i^4v_{i+1}^6 \in C_{4t}^3$; pair these edges as $(v_i^4v_{i+1}^6, v_i^1v_{i+1}^3)$ and $(v_i^3v_{i+1}^5, v_i^4v_{i+1}^6)$; first pair is of rotation-distance 3 and that for the second pair is 1. For $\ell = 3$, the four edges are: $v_i^2v_{i+1}^5$, $v_i^3v_{i+1}^6$, $v_i^5v_{i+1}^2$, $v_i^6v_{i+1}^3 \in C_{4t}^5$; pair these edges as $(v_i^2v_{i+1}^5, v_i^3v_{i+1}^6)$ and $(v_i^5v_{i+1}^2, v_i^6v_{i+1}^3)$; both pairs have rotation-distance 1. For $\ell = 4$, the four edges are: $v_i^6v_{i+1}^4 \in C_{4t}^1$, $v_i^3v_{i+1}^1 \in C_{4t}^2$ and $v_i^5v_{i+1}^3$, $v_i^6v_{i+1}^4 \in C_{4t}^3$; pair these edges as $(v_i^6v_{i+1}^4, v_i^3v_{i+1}^1)$ and $(v_i^5v_{i+1}^3, v_i^6v_{i+1}^4)$; first pair is of rotation-distance 3 and that for the second pair is 1. For $\ell = 5$, the four edges are: $v_i^3v_{i+1}^2 \in C_{4t}^1$, $v_i^6v_{i+1}^5 \in C_{4t}^2$ and $v_i^1v_{i+1}^6$, $v_i^4v_{i+1}^3 \in C_{4t}^4$; pair these edges as $(v_i^3v_{i+1}^2, v_i^6v_{i+1}^5)$ and $(v_i^1v_{i+1}^6, v_i^4v_{i+1}^3)$; both pairs have rotation-distance 3.

For each $\ell \in [1, 5]$, we pair the four edges of length ℓ from V_t to V_{t+1} as follows: For $\ell = 1$, the four edges are: $v_t^2v_{t+1}^3$, $v_t^3v_{t+1}^4 \in C_{4t}^1$, $v_t^3v_{t+1}^4 \in C_{4t}^2$ and $v_t^4v_{t+1}^5 \in C_{4t}^3$; pair these edges as $(v_t^2v_{t+1}^3, v_t^3v_{t+1}^4)$ and $(v_t^3v_{t+1}^4, v_t^4v_{t+1}^5)$; both pairs have rotation-distance 1. For $\ell = 2$, the four edges are: $v_t^1v_{t+1}^3 \in C_{4t}^2$, $v_t^3v_{t+1}^5 \in C_{4t}^3$, $v_t^6v_{t+1}^2 \in C_{4t}^4$ and $v_t^6v_{t+1}^2 \in C_{4t}^5$; pair these edges as $(v_t^1v_{t+1}^3, v_t^6v_{t+1}^2)$ and $(v_t^3v_{t+1}^5, v_t^6v_{t+1}^2)$; first pair is of rotation-distance 1 and that for the second pair is 3. For $\ell = 3$, the four edges are: $v_t^6v_{t+1}^3 \in C_{4t}^2$, $v_t^6v_{t+1}^3 \in C_{4t}^3$, $v_t^3v_{t+1}^6 \in C_{4t}^4$ and $v_t^5v_{t+1}^2 \in C_{4t}^5$; pair these edges as $(v_t^6v_{t+1}^3, v_t^3v_{t+1}^6)$ and $(v_t^6v_{t+1}^3, v_t^5v_{t+1}^2)$; first pair is of rotation-distance 3 and that for the second pair is 1. For $\ell = 4$, the four edges are: $v_t^6v_{t+1}^4 \in C_{4t}^1$, $v_t^5v_{t+1}^3 \in C_{4t}^3$, $v_t^4v_{t+1}^2 \in C_{4t}^4$ and $v_t^3v_{t+1}^1 \in C_{4t}^5$; pair these edges as $(v_t^6v_{t+1}^4, v_t^5v_{t+1}^3)$ and $(v_t^4v_{t+1}^2, v_t^3v_{t+1}^1)$; both pairs have rotation-distance 1. For $\ell = 5$, the four edges are: $v_t^4v_{t+1}^3 \in C_{4t}^1$, $v_t^5v_{t+1}^4 \in C_{4t}^2$, $v_t^1v_{t+1}^6 \in C_{4t}^4$ and $v_t^2v_{t+1}^1 \in C_{4t}^5$; pair these edges as $(v_t^4v_{t+1}^3, v_t^5v_{t+1}^4)$ and $(v_t^1v_{t+1}^6, v_t^2v_{t+1}^1)$; both pairs have rotation-distance 1.)

Consider the sets $\mathscr{F} = \{C_{4t}^k \mid k = 1, 2, 3, 4, 5\}$ and $\mathscr{D} = \{C_{4t}^k, \sigma^2(C_{4t}^k), \sigma^4(C_{4t}^k) \mid k = 1, 2, 3, 4, 5\}$ of cycles of length $4t$ in $(P_{t+1} \times K_6)(2)$. (For every $i \in [1, t]$ and for every $\ell \in [1, 5]$, the union of the cycles in $\mathscr{F}$ contains four edges of length ℓ from V_i to V_{i+1} and we have paired the edges in such a way that no rotation-distance is 2.) It follows that $\mathscr{D}$ is a decomposition of $(P_{t+1} \times K_6)(2)$. $\square$

2.3 $(K_{t+1} \times K_6)(2)$

Lemma 2.3. *If $t \geq 3$ is an odd integer, then $C_{4t}|(K_{t+1} \times K_6)(2)$.*

Proof. As t is odd, K_{t+1} is P_{t+1}-decomposable, and hence

$$K_{t+1} = P_{t+1} \oplus P_{t+1} \oplus \cdots \oplus P_{t+1}.$$

Therefore,

$$(K_{t+1} \times K_6)(2) = (P_{t+1} \times K_6)(2) \oplus (P_{t+1} \times K_6)(2) \oplus \cdots \oplus (P_{t+1} \times K_6)(2).$$

By Lemma 2.2, $C_{4t}|(P_{t+1} \times K_6)(2)$. Thus, $C_{4t}|(K_{t+1} \times K_6)(2)$. □

2.4 $(P_{t+1} \times K_7)(2)$

Lemma 2.4. *If $t \geq 3$ is an odd integer, then $C_{4t}|(P_{t+1} \times K_7)(2)$.*

Proof. Let the path P_{t+1} be $x_1x_2x_3 \ldots x_{t+1}$. First, we find three $4t$-cycles of $(P_{t+1} \times K_7)(2)$ as follows:

$$\begin{aligned} C_{4t}^1 = v_1^2 - v_2^3v_3^1v_4^3v_5^1v_6^3v_7^1 \ldots v_{t-2}^1v_{t-1}^3v_t^1 \\ - v_{t+1}^6 - v_t^5v_{t-1}^7v_{t-2}^5v_{t-3}^7v_{t-4}^5v_{t-5}^7 \ldots v_4^7v_3^5v_2^7 \\ - v_1^6 - v_2^5v_3^7v_4^5v_5^7v_6^5v_7^7 \ldots v_{t-2}^7v_{t-1}^5v_t^7 - v_{t+1}^2 \\ - v_t^3v_{t-1}^1v_{t-2}^3v_{t-3}^1v_{t-4}^3v_{t-5}^1 \ldots v_4^1v_3^3v_2^1\text{-}v_1^2 \end{aligned}$$

(C_{4t}^1 contains: for each $\ell \in \{1, 6\}$, two edges of length ℓ from V_1 to V_2; for each $i \in [2, t-1]$ and for each $\ell \in \{2, 5\}$, two edges of length ℓ from V_i to V_{i+1}; for each $\ell \in \{1, 2, 5, 6\}$, one edge of length ℓ from V_t to V_{t+1}),

$$\begin{aligned} C_{4t}^2 = v_1^1 - v_2^5v_3^4v_4^5v_5^4v_6^5v_7^4 \ldots v_{t-2}^4v_{t-1}^5v_t^4 \\ - v_{t+1}^5 - v_t^1v_{t-1}^2v_{t-2}^1v_{t-3}^2v_{t-4}^1v_{t-5}^2 \ldots v_4^2v_3^1v_2^2 \\ - v_1^5 - v_2^1v_3^2v_4^1v_5^2v_6^1v_7^2 \ldots v_{t-2}^2v_{t-1}^1v_t^2 - v_{t+1}^1 \\ - v_t^5v_{t-1}^4v_{t-2}^5v_{t-3}^4v_{t-4}^5v_{t-5}^4 \ldots v_4^4v_3^5v_2^4 - v_1^1 \end{aligned}$$

(C_{4t}^2 contains: for each $\ell \in \{3,4\}$, two edges of length ℓ from V_1 to V_2; for each $i \in [2, t-1]$ and for each $\ell \in \{1,6\}$, two edges of length ℓ from V_i to V_{i+1}; for each $\ell \in \{1,3,4,6\}$, one edge of length ℓ from V_t to V_{t+1}),
and

$$\begin{aligned} C_{4t}^3 = v_1^1 - v_2^6 v_3^3 v_4^6 v_5^3 v_6^6 v_7^3 \dots v_{t-2}^3 v_{t-1}^6 v_t^3 - v_{t+1}^1 \\ - v_t^4 v_{t-1}^7 v_{t-2}^4 v_{t-3}^7 v_{t-4}^4 v_{t-5}^7 \dots v_4^7 v_3^4 v_2^7 - v_1^2 \\ - v_2^4 v_3^7 v_4^4 v_5^7 v_6^4 v_7^7 \dots v_{t-2}^7 v_{t-1}^4 v_t^7 - v_{t+1}^2 \\ - v_t^6 v_{t-1}^3 v_{t-2}^6 v_{t-3}^3 v_{t-4}^6 v_{t-5}^3 \dots v_4^3 v_3^6 v_2^3 \text{-} v_1^1 \end{aligned}$$

(C_{4t}^3 contains: for each $\ell \in \{2,5\}$, two edges of length ℓ from V_1 to V_2; for each $i \in [2, t-1]$ and for each $\ell \in \{3,4\}$, two edges of length ℓ from V_i to V_{i+1}; for each $\ell \in \{2,3,4,5\}$, one edge of length ℓ from V_t to V_{t+1}).

Consider the sets $\mathscr{F} = \{C_{4t}^k | k = 1,2,3\}$ and $\mathscr{D} = \{C_{4t}^k, \sigma(C_{4t}^k), \sigma^2(C_{4t}^k), \sigma^3(C_{4t}^k), \sigma^4(C_{4t}^k), \sigma^5(C_{4t}^k), \sigma^6(C_{4t}^k) | k = 1,2,3\}$ of cycles of length $4t$ in $(P_{t+1} \times K_7)(2)$. (For every $i \in [1,t]$ and for every $\ell \in [1,6]$, the union of the cycles in $\mathscr{F}$ contains two edges of length ℓ from V_i to V_{i+1}.) It follows that $\mathscr{D}$ is a decomposition of $(P_{t+1} \times K_7)(2)$. □

2.5 $(K_{t+1} \times K_7)(2)$

Lemma 2.5. *If $t \geq 3$ is an odd integer, then $C_{4t} | (K_{t+1} \times K_7)(2)$.*

Proof. As t is odd, K_{t+1} is P_{t+1}-decomposable, and hence

$$K_{t+1} = P_{t+1} \oplus P_{t+1} \oplus \cdots \oplus P_{t+1}.$$

Hence,

$$(K_{t+1} \times K_7)(2) = (P_{t+1} \times K_7)(2) \oplus (P_{t+1} \times K_7)(2) \oplus \cdots \oplus (P_{t+1} \times K_7)(2).$$

By Lemma 2.4, $C_{4t} | (P_{t+1} \times K_7)(2)$. Thus, $C_{4t} | (K_{t+1} \times K_7)(2)$. □

2.6 $(C_t \times K_n)(2)$

Let G be a simple graph with vertex set $\{x_1, x_2, \dots, x_n\}$. For convenience, we denote an edge e of G with ends x_i and x_j, $i < j$, as $x_i x_j$ (instead of $x_j x_i$). Consider its 4-fold $G(4)$. For any odd integer t, $t \geq 3$, our aim is to find a C_{4t}-decomposition $\mathscr{D}$ of the 2-fold tensor product $(G \times C_t)(2)$ from a specific C_4-decomposition $\mathscr{D}_0$ of $G(4)$. For this, first write $G(4)$ as $H_1(2) \oplus H_{t-1}(2)$ with $H_1 \cong H_{t-1} \cong G$. If an edge e' of $G(4)$ is in $H_i(2)$, for some i, $i \in \{1, t-1\}$, then we say that e' is of *length* i. Hence, each edge e of G duplicates in $G(4)$ with two edges of length 1 and two edges of length $t-1$. Suppose there is a C_4-decomposition $\mathscr{D}_0$ of $G(4)$.

Construction: Let C_0 be any cycle of length 4 in $\mathscr{D}_0$ and let C be the subgraph of $(G \times C_t)(2)$, arise out of C_0, by the procedure given below. Let e' be any edge of C_0 and let e be the edge corresponding to e' in G with ends, say, x_i and x_j, $i < j$. If e' is of length 1, then, for C, we take the t edges in $F_1(V_i, V_j)$. If the length of e' is $t-1$, then, for C, we take the t edges in $F_{t-1}(V_i, V_j)$.

This construction yields for each cycle C_0 in $\mathscr{D}_0$, a subgraph C of $(G \times C_t)(2)$ with $4t$ edges, and hence, we have a decomposition of $(G \times C_t)(2)$ into subgraphs of size $4t$.

Let $C = x_{i_1}(\ell_1)x_{i_2}(\ell_2)x_{i_3}(\ell_3)x_{i_4}(\ell_4)x_{i_1}$ be any cycle in $\mathscr{D}_0$; here the edge $x_{i_j}x_{i_{j+1}}$, $j \in [1,4]$, is of length ℓ_j and $x_{i_5} = x_{i_1}$, i.e., $i_5 = i_1$. For $j \in [1,4]$, if $i_j < i_{j+1}$, then let $\alpha_j = \ell_j$; otherwise $i_j > i_{j+1}$, let $\alpha_j = t - \ell_j$. As $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{1, t-1\}$, $\sum_{j=1}^{4} \alpha_j \in \{4, t+2, 2t, 3t-2, 4t-4\}$. If $\sum_{j=1}^{4} \alpha_j \neq 2t$, then, as t is odd, the subgraph C is a cycle of length $4t$. Otherwise $\sum_{j=1}^{4} \alpha_j = 2t$, then, C is tC_4.

Examples: First we take $G = K_6$ with $V(K_6) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Set $\mathscr{D}_0 = \{x_2(1)x_6(1)x_4(1)x_5(t-1)x_2,\ x_3(1)x_6(1)x_5(1)x_1(t-1)x_3,\ x_4(1)x_6(1)x_1(1)x_2(1)x_4,\ x_3(1)x_5(t-1)x_2(1)x_4(1)x_3,\ x_1(t-1)x_3(t-1)x_5(1)x_2(t-1)x_1,\ x_2(t-1)x_3(1)x_4(t-1)x_6(1)x_2,\ x_4(1)x_5(1)x_1(1)x_6(t-1)x_4,\ x_5(t-1)x_1(t-1)x_2(t-1)x_6(1)x_5,\ x_3(t-1)x_2(t-1)x_6(t-1)x_5(t-1)x_3,\ x_4(t-1)x_3(1)x_6(t-1)x_1(t-1)x_4,\ x_1(1)x_4(t-1)x_5(1)x_3(1)x_1,\ x_2(1)x_3(1)x_1(1)x_4(t-1)x_2,\ x_4(t-1)x_2(1)x_5(t-1)x_1(t-1)x_4,\ x_5(t-1)x_6(t-1)x_3(t-1)x_4(t-1)x_5,\ x_1(t-1)x_6(t-1)x_3(1)x_2(1)x_1\}$. Then, $\mathscr{D}_0$ is a 4-cycle decomposition of $K_6(4) = H_1(2) \oplus H_{t-1}(2)$ with the condition that $\sum_{j=1}^{4} \alpha_j \in \{t+2, 3t-2\}$. Hence, by the above construction, $C_{4t} | (K_6 \times C_t)(2)$.

Next we take $G = K_7$ with $V(K_7) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Set $\mathscr{D}_0 = \{x_2(1)x_6(1)x_3(1)x_7(t-1)x_2,\ x_3(t-1)x_7(1)x_4(1)x_1(t-1)x_3,\ x_4(1)x_2(1)x_5(t-1)x_1(1)x_4,\ x_5(1)x_2(1)x_6(1)x_3(t-1)x_5,\ x_6(t-1)x_4(1)x_7(t-1)x_3$

$(t-1)x_6$, $x_7(t-1)x_4(t-1)x_1(t-1)x_5(1)x_7$, $x_1(t-1)x_6(t-1)x_2(t-1)$ $x_5(1)x_1$, $x_7(1)x_6(t-1)x_5(1)x_4(t-1)x_7$, $x_1(1)x_5(1)x_6(t-1)x_7(t-1)x_1$, $x_4(t-1)x_3(t-1)x_2(t-1)x_1(t-1)x_4$, $x_5(t-1)x_2(1)x_3(t-1)x_4(1)x_5$, $x_5(1)$ $x_7(t-1)x_2(1)x_4(t-1)x_5$, $x_6(t-1)x_1(t-1)x_3(t-1)x_5(t-1)x_6$, $x_7(1)x_6(t-1)$ $x_4(t-1)x_2(1)x_7$, $x_1(t-1)x_7(t-1)x_5(1)x_3(1)x_1$, $x_2(t-1)x_4(1)x_6(1)x_1$ $(t-1)x_2$, $x_3(t-1)x_2(1)x_7(t-1)x_5(1)x_3$, $x_3(t-1)x_6(1)x_5(t-1)x_4(1)x_3$, $x_6(t-1)x_7(1)x_1(1)x_2(t-1)x_6$, $x_2(1)x_1(1)x_7(1)x_3(1)x_2$, $x_4(1)x_3(1)x_1(1)$ $x_6(1)x_4\}$. Then, $\mathscr{D}_0$ is a 4-cycle decomposition of $K_7(4) = H_1(2) \oplus H_{t-1}(2)$ with $\sum_{j=1}^{4} \alpha_j \in \{4, t+2, 3t-2, 4t-4\}$. Once again, by the above construction, $C_{4t}|(K_7 \times C_t)(2)$.

The following theorems are used in the proof of Lemma 2.7.

Theorem 2.1. [41]. *The bipartite graph $K_{r,s}$ can be decomposed into cycles of length $2k$ if and only if r and s are even, $r \geq k$, $s \geq k$, and $2k$ divides rs.*

Theorem 2.2. [3]. *Suppose n and k are positive integers with $3 \leq k \leq n$. Then the complete multigraph $K_n(2)$ has a decomposition into k-cycles if and only if $k|n(n-1)$.*

Theorem 2.3. [13]. *The graph $C_r \times C_s$ can be decomposed into two Hamilton cycles if and only if at least one of r and s is odd.*

Lemma 2.6. *If $n \geq 4$ is an integer, then*

$$K_n(2) = \begin{cases} C_4 \oplus C_4 \oplus \cdots \oplus C_4, & \text{if } n \equiv 0 \text{ or } 1 \pmod 4; \\ C_4 \oplus C_4 \oplus \cdots \oplus C_4 \oplus K_6(2), & \text{if } n \equiv 2 \pmod 4; \\ C_4 \oplus C_4 \oplus \cdots \oplus C_4 \oplus K_7(2), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. If $n \equiv 0$ or $1 \pmod 4$, then, by Theorem 2.2, $C_4|K_n(2)$.

If $n \equiv 2 \pmod 4$, then $n = 4k+2$ for some integer $k \geq 1$. Therefore

$$\begin{aligned} K_n(2) &= K_{4k+2}(2) \\ &= K_6(2) \oplus \underbrace{K_4(2) \oplus K_4(2) \oplus \cdots \oplus K_4(2)}_{k-1 \text{ times}} \\ &\quad \oplus \underbrace{K_{6,4}(2) \oplus K_{6,4}(2) \oplus \cdots \oplus K_{6,4}(2)}_{k-1 \text{ times}} \\ &\quad \oplus \underbrace{K_{4,4}(2) \oplus K_{4,4}(2) \oplus \cdots \oplus K_{4,4}(2)}_{(k-1)(k-2)/2 \text{ times}}. \end{aligned}$$

By Theorem 2.1, $C_4|K_{6,4}$ and $C_4|K_{4,4}$, and hence $C_4|K_{6,4}(2)$ and $C_4|K_{4,4}(2)$. Thus, $K_n(2) = K_6(2) \oplus C_4 \oplus C_4 \oplus \cdots \oplus C_4$.

If $n \equiv 3 \pmod 4$, then $n = 4k+3$ for some integer $k \geq 1$. Therefore

$$\begin{aligned} K_n(2) &= K_{4k+3}(2) \\ &= K_7(2) \oplus \underbrace{K_4(2) \oplus K_4(2) \oplus \cdots \oplus K_4(2)}_{k-1 \text{ times}} \\ &\quad \oplus \underbrace{K_{7,4}(2) \oplus K_{7,4}(2) \oplus \cdots \oplus K_{7,4}(2)}_{k-1 \text{ times}} \\ &\quad \oplus \underbrace{K_{4,4}(2) \oplus K_{4,4}(2) \oplus \cdots \oplus K_{4,4}(2)}_{(k-1)(k-2)/2 \text{ times}} \\ &= K_7(2) \oplus \underbrace{K_4(2) \oplus K_4(2) \oplus \cdots \oplus K_4(2)}_{k-1 \text{ times}} \\ &\quad \oplus \underbrace{K_{3,4}(2) \oplus K_{3,4}(2) \oplus \cdots \oplus K_{3,4}(2)}_{k-1 \text{ times}} \\ &\quad \oplus \underbrace{K_{4,4}(2) \oplus K_{4,4}(2) \oplus \cdots \oplus K_{4,4}(2)}_{(k-1)+(k-1)(k-2)/2 \text{ times}}. \end{aligned}$$

By Theorem 2.2, $C_4|K_4(2)$. By Theorem 2.1, $C_4|K_{4,4}$. Hence $C_4|K_{4,4}(2)$. Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2\}$ be the partite sets of the bipartite graph $K_{3,2}(2)$. Then the 4-cycles $a_1b_1a_2b_2a_1$, $a_2b_1a_3b_2a_2$ and $a_3b_1a_1b_2a_3$ decomposes $K_{3,2}(2)$. Thus, $C_4|K_{3,2}(2)$. Since $K_{3,4}(2) = K_{3,2}(2) \oplus K_{3,2}(2)$, we have $C_4|K_{3,4}(2)$. Thus, $K_n(2) = K_7(2) \oplus C_4 \oplus C_4 \oplus \cdots \oplus C_4$. □

Lemma 2.7. *If $t \geq 3$ is an odd integer and $n \geq 4$, then $C_{4t}|(C_t \times K_n)(2)$.*

Proof. We consider three cases.

Case 1. $n \equiv 0$ or $1 \pmod 4$.
Then, by Lemma 2.6, $K_n(2) = C_4 \oplus C_4 \oplus \cdots \oplus C_4$. Now,

$$\begin{aligned} (C_t \times K_n)(2) &= C_t \times K_n(2) \\ &= C_t \times (C_4 \oplus C_4 \oplus \cdots \oplus C_4) \\ &= (C_t \times C_4) \oplus (C_t \times C_4) \oplus \cdots \oplus (C_t \times C_4). \end{aligned}$$

By Theorem 2.3, $C_{4t}|(C_t \times C_4)$, and hence $C_{4t}|(C_t \times K_n)(2)$.

Case 2. $n \equiv 2 \pmod 4$.

Then, by Lemma 2.6, $K_n(2) = C_4 \oplus C_4 \oplus \cdots \oplus C_4 \oplus K_6(2)$. Hence,

$$\begin{aligned}(C_t \times K_n)(2) &= C_t \times K_n(2)\\ &= (C_t \times C_4) \oplus (C_t \times C_4) \oplus \cdots \oplus (C_t \times C_4) \oplus (C_t \times K_6(2)).\end{aligned}$$

By Theorem 2.3, $C_{4t}|(C_t \times C_4)$. By the above example, $C_{4t}|(K_6 \times C_t)(2)$. Since the tensor product is commutative, $K_6 \times C_t \cong C_t \times K_6$, and hence, $C_{4t}|(C_t \times K_6)(2)$, equivalently, $C_{4t}|(C_t \times K_6(2))$.

Hence, $C_{4t}|(C_t \times K_n)(2)$.

Case 3. $n \equiv 3 \pmod 4$.

Then, by Lemma 2.6, $K_n(2) = C_4 \oplus C_4 \oplus \cdots \oplus C_4 \oplus K_7(2)$. Hence,

$$\begin{aligned}(C_t \times K_n)(2) &= C_t \times K_n(2)\\ &= (C_t \times C_4) \oplus (C_t \times C_4) \oplus \cdots \oplus (C_t \times C_4) \oplus (C_t \times K_7(2)).\end{aligned}$$

By Theorem 2.3, $C_{4t}|(C_t \times C_4)$. By the above example, $C_{4t}|(K_7 \times C_t)(2)$. Since the tensor product is commutative, $K_7 \times C_t \cong C_t \times K_7$, and hence, $C_{4t}|(C_t \times K_7)(2)$, equivalently, $C_{4t}|(C_t \times K_7(2))$.

Hence, $C_{4t}|(C_t \times K_n)(2)$. □

2.7 $(K_2 \times K_n)(2)$

Lemma 2.8. *If $n \geq 4$ and $t \geq 2$ are integers, and $n \equiv 0 \pmod{2t}$, then $C_{4t}|(K_2 \times K_n)(2)$.*

Proof. Then, $n = 2tk$, where $k \geq 1$ is an integer. We consider two cases:

Case 1. $k = 1$.

First, write $(K_2 \times K_{2t})(2)$ as $(K_2 \times K_{2t}) \oplus (K_2 \times K_{2t})$. Next, write the first $K_2 \times K_{2t}$ as $(F_1(V_1, V_2) \cup F_2(V_1, V_2)) \oplus (F_3(V_1, V_2) \cup F_4(V_1, V_2)) \oplus \ldots \oplus (F_{2t-3}(V_1, V_2) \cup F_{2t-2}(V_1, V_2)) \oplus F_{2t-1}(V_1, V_2)$ and the next $K_2 \times K_{2t}$ as $(F_2(V_1, V_2) \cup F_3(V_1, V_2)) \oplus (F_4(V_1, V_2) \cup F_5(V_1, V_2)) \oplus \ldots \oplus (F_{2t-2}(V_1, V_2) \cup F_{2t-1}(V_1, V_2)) \oplus F_1(V_1, V_2)$. For $i \in \{1, 2, 3, \ldots, t-1\}$, both $F_{2i-1}(V_1, V_2) \cup F_{2i}(V_1, V_2)$ and $F_{2i}(V_1, V_2) \cup F_{2i+1}(V_1, V_2)$ are isomorphic to C_{4t}. Also, $F_{2t-1}(V_1, V_2) \cup F_1(V_1, V_2)$ is isomorphic to C_{4t}. Hence, we have $C_{4t}|(K_2 \times K_{2t})(2)$.

Case 2. $k \geq 2$.

Clearly, $K_2 \times K_{2kt}$ can be decomposed into k copies each isomorphic to $K_2 \times K_{2t}$ and $k(k-1)$ copies each isomorphic to $K_{2t,2t}$. Hence,

$$K_2 \times K_{2kt} = ((K_2 \times K_{2t}) \oplus \cdots \oplus (K_2 \times K_{2t})) \ \oplus \ (K_{2t,2t} \oplus \cdots \oplus K_{2t,2t}),$$

and therefore,

$$(K_2 \times K_{2kt})(2) = ((K_2 \times K_{2t})(2) \oplus \cdots \oplus (K_2 \times K_{2t})(2)) \oplus \\ (K_{2t,2t})(2) \oplus \cdots \oplus (K_{2t,2t})(2)).$$

By Case 1, $C_{4t}|(K_2 \times K_{2t})(2)$. By Theorem 2.1, $C_{4t}|K_{2t,2t}$, and so $C_{4t}|(K_{2t,2t})(2)$. Hence, $C_{4t}|(K_2 \times K_{2kt})(2)$. □

Lemma 2.9. $C_8|(K_2 \times K_5)(2)$ *and* $C_{12}|(K_2 \times K_7)(2)$.

Proof. Let $V(K_2) = \{x_1, x_2\}$, $V(K_5) = \{y_1, y_2, \ldots, y_5\}$ and $V(K_7) = \{y_1, y_2, \ldots, y_7\}$.

In $(K_2 \times K_5)(2)$, $C' = v_1^1 v_2^3 v_1^5 v_2^4 v_1^3 v_2^1 v_1^4 v_2^5 v_1^1$ is a cycle of length 8 and it contains: for each $\ell \in [1, 4]$, two edges of length ℓ from V_1 to V_2. Hence, $\{C', \sigma(C'), \sigma^2(C'), \sigma^3(C'), \sigma^4(C')\}$ is a decomposition of $(K_2 \times K_5)(2)$.

In $(K_2 \times K_7)(2)$, $C'' = v_1^1 v_2^4 v_1^2 v_2^3 v_1^4 v_2^2 v_1^5 v_2^6 v_1^3 v_2^1 v_1^6 v_2^5 v_1^1$ is a cycle of length 12 and it contains: for each $\ell \in [1, 6]$, two edges of length ℓ from V_1 to V_2. Hence, $\{C'', \sigma(C''), \sigma^2(C''), \sigma^3(C''), \sigma^4(C''), \sigma^5(C''), \sigma^6(C'')\}$ is a decomposition of $(K_2 \times K_7)(2)$. □

2.8 $K_m \circ \overline{K}_n$

The following theorems are used in the proof of Lemma 2.10.

Theorem 2.4. (see [25]). *Let $m \geq 3$ be an odd integer.*
(1) If $m \equiv 1 or\ 3 \pmod 6$, then $C_3|K_m$.
(2) If $m \equiv 5 \pmod 6$, then K_m can be decomposed into $(m(m-1)-20)/6$ 3-cycles and a K_5.

Theorem 2.5 is proven in [1] when m is an odd prime, but one can easily see that the same proof works for any odd integer m.

Theorem 2.5. [1]. *If m and k are at least 3, both of them are odd and $3 \leq k \leq m$, then $C_k \circ \overline{K}_m$ admits a C_m-factorization.*

Theorem 2.6. [23]. *If m and n are at least 3, then $C_3|(K_m \circ \overline{K}_n)$ if and only if (1) $(m-1)n$ is even and (2) $3|m(m-1)n^2$.*

Lemma 2.10. *If $m \geq 3$ and $n \geq 3$ are odd integers, then $C_n|(K_m \circ \overline{K}_n)$.*

Proof. By Theorem 2.4, $K_m = K_3 \oplus K_3 \oplus \cdots \oplus K_3$, if $m \equiv 1$or 3 (mod 6) and $K_m = K_3 \oplus K_3 \oplus \cdots \oplus K_3 \oplus K_5$, if $m \equiv 5$ (mod 6). Hence, $K_m \circ \overline{K}_n = (K_3 \circ \overline{K}_n) \oplus (K_3 \circ \overline{K}_n) \oplus \cdots \oplus (K_3 \circ \overline{K}_n)$, if $m \equiv 1$or 3 (mod 6) and $K_m \circ \overline{K}_n = (K_3 \circ \overline{K}_n) \oplus (K_3 \circ \overline{K}_n) \oplus \cdots \oplus (K_3 \circ \overline{K}_n) \oplus (K_5 \circ \overline{K}_n)$, if $m \equiv 5$ (mod 6). To prove the lemma, it is enough to prove that $C_n|(K_3 \circ \overline{K}_n)$ and $C_n|(K_5 \circ \overline{K}_n)$. By Theorem 2.5, $C_n|(K_3 \circ \overline{K}_n)$. By Theorem 2.6, $C_3|(K_5 \circ \overline{K}_3)$. Hence, it is enough to prove that $C_n|(K_5 \circ \overline{K}_n)$, for $n \geq 5$. As $C_5|K_5$, we have $K_5 \circ \overline{K}_n = (C_5 \circ \overline{K}_n) \oplus (C_5 \circ \overline{K}_n)$. By Theorem 2.5, $C_n|(C_5 \circ \overline{K}_n)$. This completes the proof. □

3 Proof of Theorem 1.1

We need following theorems and a lemma for the proof of Theorem 1.1.

Theorem 3.1. [2, 36]. *Suppose $n \geq 3$ and $k \geq 3$ are positive integers. Then the complete graph K_n admits a decomposition into k-cycles if and only if $n \geq k$, n is odd and $k|\binom{n}{2}$.*

Theorem 3.2. [42]. *Let λ, k and n be positive integers. There exists a P_{k+1}-decomposition of $K_n(\lambda)$ if and only if $n \geq k+1$ and $\lambda n(n-1) \equiv 0 \pmod{2k}$.*

Lemma 3.1. [32]. *If $s \geq 3$ is an odd integer, $r \geq 3$ and $C_r|G$, then $C_{rs}|(G \times K_{s+1})$.*

Proof of Theorem 1.1.
By hypothesis, $m \equiv 0 \pmod t$, $m \equiv 1 \pmod t$, $n \equiv 0 \pmod t$ or $n \equiv 1 \pmod t$. Since the tensor product is commutative, we assume that $m \equiv 0$ or 1 (mod t). As $t \geq 3$ and $mn \geq 4t$, we have $(m,n) \neq (3,3)$. We consider four cases.

Case 1. $m \geq 5$ is odd and $n \geq 4$.
As $m \equiv 0$ or 1 (mod t), we have, by Theorem 3.1, $C_t|K_m$. Thus, $K_m = C_t \oplus C_t \oplus \cdots \oplus C_t$. Hence, $(K_m \times K_n)(2) = ((C_t \oplus C_t \oplus \cdots \oplus C_t) \times K_n)(2) = (C_t \times K_n)(2) \oplus (C_t \times K_n)(2) \oplus \cdots \oplus (C_t \times K_n)(2)$. By Lemma 2.7, $C_{4t}|(C_t \times K_n)(2)$. Thus, we have $C_{4t}|(K_m \times K_n)(2)$.

Case 2. $m \geq 4$ is even and $n \geq 4$.
We consider two subcases.

Subcase 2.1. $m \equiv 0 \pmod{t}$.
As t is odd and m is even, we have $m \equiv 0 \pmod{2t}$. Then, $m = 2tk$ for some integer $k \geq 1$.

If $k = 1$, then $K_m = K_{2t}$. Also, $K_n = K_2 \oplus K_2 \oplus \cdots \oplus K_2$. Hence, $(K_m \times K_n)(2) = (K_{2t} \times K_2)(2) \oplus (K_{2t} \times K_2)(2) \oplus \cdots \oplus (K_{2t} \times K_2)(2)$. By Lemma 2.8, $C_{4t}|(K_2 \times K_{2t})(2)$, equivalently, $C_{4t}|(K_{2t} \times K_2)(2)$. Hence, $C_{4t}|(K_m \times K_n)(2)$.

So, assume that $k \geq 2$. Then, $K_m = K_{2tk} = kK_{2t} \oplus (K_k \circ \overline{K}_{2t})$. Hence, $(K_m \times K_n)(2) = k(K_{2t} \times K_n)(2) \oplus ((K_k \circ \overline{K}_{2t}) \times K_n)(2)$. By the above particular value for k, i.e., $k = 1$, we have $C_{4t}|(K_{2t} \times K_n)(2)$. To show that $C_{4t}|(K_m \times K_n)(2)$, it is enough if we show that $C_{4t}|((K_k \circ \overline{K}_{2t}) \times K_n)(2)$. First, write $K_k \circ \overline{K}_{2t}$ as an edge-disjoint union of $k(k-1)/2$ copies of $K_{2t,2t}$. By Theorem 2.1, $C_{4t}|K_{2t,2t}$. Now, write each copy of $K_{2t,2t}$ as an edge-disjoint union of t copies of C_{4t}. Finally, write K_n as the edge-disjoint union of $n(n-1)/2$ copies of K_2. Hence, it is enough if we show that $C_{4t}|(C_{4t} \times K_2)(2)$. Since $C_{4t} \times K_2$ is the disjoint union of two copies of C_{4t}, $C_{4t}|(C_{4t} \times K_2)$, and hence $C_{4t}|(C_{4t} \times K_2)(2)$.

Subcase 2.2. $m \equiv 1 \pmod{t}$.
Then, $m = tk + 1$ for some integer $k \geq 1$. As t is odd and m is even, we have k is odd.

If $k = 1$, then $(K_{t+1} \times K_n)(2) = K_{t+1} \times K_n(2)$. By Lemma 2.6,

$$K_n(2) = \begin{cases} C_4 \oplus C_4 \oplus \cdots \oplus C_4, & \text{if } n \equiv 0 \text{ or } 1 \pmod{4}; \\ C_4 \oplus C_4 \oplus \cdots \oplus C_4 \oplus K_6(2), & \text{if } n \equiv 2 \pmod{4}; \\ C_4 \oplus C_4 \oplus \cdots \oplus C_4 \oplus K_7(2), & \text{if } n \equiv 3 \pmod{4}; \end{cases}$$

To show that $C_{4t}|(K_{t+1} \times K_n)(2)$, it is enough if we show that $C_{4t}|(K_{t+1} \times C_4)$, $C_{4t}|(K_{t+1} \times K_6(2))$ and $C_{4t}|(K_{t+1} \times K_7(2))$. As $C_4|C_4$ and $t \geq 3$ is odd, we have, by Lemma 3.1, $C_{4t}|(C_4 \times K_{t+1})$. As $C_4 \times K_{t+1} \cong K_{t+1} \times C_4$, $C_{4t}|(K_{t+1} \times C_4)$.

By Lemmas 2.3 and 2.5, we have, respectively, $C_{4t}|(K_{t+1} \times K_6)(2)$ and $C_{4t}|(K_{t+1} \times K_7)(2)$. Hence, $C_{4t}|(K_{t+1} \times K_6(2))$ and $C_{4t}|(K_{t+1} \times K_7(2))$.

So, assume that $k \geq 3$. We can write $K_m = K_{tk+1}$ as

$$\underbrace{K_{t+1} \oplus K_{t+1} \oplus \cdots \oplus K_{t+1}}_{k \text{ times}} \oplus (K_k \circ \overline{K}_t),$$

and hence,

$$\begin{aligned}(K_m \times K_n)(2) = \\ &\underbrace{(K_{t+1} \times K_n)(2) \oplus (K_{t+1} \times K_n)(2) \oplus \cdots \oplus (K_{t+1} \times K_n)(2)}_{k \text{ times}} \\ &\oplus ((K_k \circ \overline{K}_t) \times K_n)(2).\end{aligned}$$

By the above particular value for k, i.e., $k = 1$, we have $C_{4t}|(K_{t+1} \times K_n)(2)$. To show that $C_{4t}|(K_m \times K_n)(2)$, it is enough if we show that $C_{4t}|((K_k \circ \overline{K}_t) \times K_n)(2)$. By Lemma 2.10, $C_t|(K_k \circ \overline{K}_t)$. Hence, it is enough if we show that $C_{4t}|(C_t \times K_n)(2)$. This follows from Lemma 2.7.

Case 3. $m = 3$ and $n \geq 4$.
As $3 = m \equiv 0$ or $1 \pmod t$ and $t \geq 3$, we have $t = 3$. Hence, we need to show $C_{12}|(K_3 \times K_n)(2)$. Equivalently, we have to show $C_{12}|(K_3 \times K_n(2))$. Now, by Lemma 2.6,

$$K_n(2) = \begin{cases} C_4 \oplus C_4 \oplus \cdots \oplus C_4, & \text{if } n \equiv 0 \text{ or } 1 \pmod 4; \\ C_4 \oplus C_4 \oplus \cdots \oplus C_4 \oplus K_6(2), & \text{if } n \equiv 2 \pmod 4; \\ C_4 \oplus C_4 \oplus \cdots \oplus C_4 \oplus K_7(2), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

To show that $C_{12}|(K_3 \times K_n(2))$, we have to show that $C_{12}|(K_3 \times C_4)$, $C_{12}|(K_3 \times K_6(2))$ and $C_{12}|(K_3 \times K_7(2))$.

By Theorem 2.3, $C_{12}|(C_3 \times C_4)$, i.e., $C_{12}|(K_3 \times C_4)$.

Since $K_3 = K_2 \oplus K_2 \oplus K_2$, to show that $C_{12}|(K_3 \times K_6(2))$ (respectively, $C_{12}|(K_3 \times K_7(2)))$, it is enough if we show that $C_{12}|(K_2 \times K_6(2))$ (respectively, $C_{12}|(K_2 \times K_7(2)))$. By Lemma 2.8 (respectively, 2.9), $C_{12}|(K_2 \times K_6)(2)$ (respectively, $C_{12}|(K_2 \times K_7)(2))$, equivalently, $C_{12}|(K_2 \times K_6(2))$ (respectively, $C_{12}|(K_2 \times K_7(2)))$.

Case 4. $m \geq 4$ and $n = 3$.
We have to show that $C_{4t}|(K_m \times K_3)(2)$; equivalently, we have to show that $C_{4t}|(K_m \times K_3(2))$.

If $m \geq 2t + 1$ and $4t|2m(m-1)$, then, by Theorem 3.2, $P_{2t+1}|K_m(2)$. So, $K_m(2) = P_{2t+1} \oplus P_{2t+1} \oplus \cdots \oplus P_{2t+1}$. Hence, $(K_m \times K_3)(2) = (K_m(2) \times$

$K_3) = (P_{2t+1} \times K_3) \oplus (P_{2t+1} \times K_3) \oplus \cdots \oplus (P_{2t+1} \times K_3)$. By Lemma 2.1, $C_{4t}|(P_{2t+1} \times K_3)$, and hence, $C_{4t}|(K_m \times K_3)(2)$. Observe that $4t|2m(m-1)$ is same as $2t|m(m-1)$; since $m \equiv 0$ or $1 \pmod t$ and t is odd, this divisibility is again same as $2|m(m-1)$, which is clearly true. As $m \equiv 0$ or $1 \pmod t$, m equals kt or $kt+1$ for some integer $k \geq 1$. The inequality $m \geq 2t+1$ fails only for $m \in \{t, t+1, 2t\}$. So, assume that $m \in \{t, t+1, 2t\}$.

If $m = 2t$, then $(K_m \times K_3)(2) = (K_{2t} \times K_3)(2) = (K_{2t} \times K_2)(2) \times (K_{2t} \times K_2)(2) \times (K_{2t} \times K_2)(2)$. By Lemma 2.8, $C_{4t}|(K_2 \times K_{2t})(2)$, and hence $C_{4t}|(K_{2t} \times K_2)(2)$. Thus, $C_{4t}|(K_m \times K_3)(2)$. Hence, assume that $m \in \{t, t+1\}$. As $mn \geq 4t$, we have $3m \geq 4t$, and hence $m \neq t$; also $m = t+1$ only when $m = 4$ and $t = 3$.

For $m = 4$ and $t = 3$, $(K_4 \times K_3)(2) = K_4(2) \times K_3 = (C_4 \times K_3) \times (C_4 \times K_3) \times (C_4 \times K_3)$; since $C_4|K_4(2)$, by Theorem 2.2. By Theorem 2.3, $C_{12}|(C_4 \times C_3)$. Hence, $C_{12}|(K_4 \times K_3)(2)$.

This completes the proof.

4 Proof of Theorem 1.2

The proof of the necessity of Theorem 1.2 is obvious, and we prove the sufficiency. We consider two cases.

Case 1. $p \geq 3$.
As p is an odd prime, the hypothesis, $4p|m(m-1)n(n-1)$, implies that $m \equiv 0 \pmod p$, $m \equiv 1 \pmod p$, $n \equiv 0 \pmod p$ or $n \equiv 1 \pmod p$. Hence, by Theorem 1.1, $C_{4p}|(K_m \times K_n)(2)$.

Case 2. $p = 2$.
We have to show that $C_8|(K_m \times K_n)(2)$. As $8|m(m-1)n(n-1)$, we have, $4|m(m-1)$ or $4|n(n-1)$. Since the tensor product is commutative, we assume that $4|m(m-1)$. Hence, $4|m$ or $4|(m-1)$. We consider two subcases. First, we claim the following.

Claim 1. For $k \geq 2$, $C_8|((K_k \circ \overline{K}_4) \times K_n)$.

First, write $K_k \circ \overline{K}_4$ as an edge-disjoint union of $k(k-1)/2$ copies of $K_{4,4}$. By Theorem 2.1, $C_8|K_{4,4}$. Now, write each copy of $K_{4,4}$ as an edge-disjoint union of 2 copies of C_8. Finally, write K_n as the edge-disjoint union of $n(n-1)/2$ copies of K_2. Hence, to prove the claim, it is enough if we show that $C_8|(C_8 \times K_2)$. Since $C_8 \times K_2 = 2C_8$, $C_8|(C_8 \times K_2)$.

It follows from Claim 1 that

Claim 2. For $k \geq 2$, $C_8|((K_k \circ \overline{K}_4) \times K_n)(2)$.

Subcase 2.1. $4|m$.
Then, $m = 4k$ for some integer $k \geq 1$.

If $k = 1$, then

$$\begin{aligned}(K_m \times K_n)(2) &= (K_4 \times K_n)(2) = (K_4 \times (K_2 \oplus K_2 \oplus \cdots \oplus K_2))(2) \\ &= (K_4 \times K_2)(2) \oplus (K_4 \times K_2)(2) \oplus \cdots \oplus (K_4 \times K_2)(2).\end{aligned}$$

By Lemma 2.8, $C_8|(K_2 \times K_4)(2)$, and hence $C_8|(K_4 \times K_2)(2)$. Thus, $C_8|(K_4 \times K_n)(2)$. So, assume that $k \geq 2$. Then

$$K_m = K_{4k} = kK_4 \oplus (K_k \circ \overline{K}_4),$$

and hence,

$$(K_m \times K_n)(2) = k(K_4 \times K_n)(2) \oplus ((K_k \circ \overline{K}_4) \times K_n)(2).$$

By the above particular value for k, i.e., $k = 1$, we have $C_8|(K_4 \times K_n)(2)$. Also, by Claim 2, $C_8|((K_k \circ \overline{K}_4) \times K_n)(2)$.

Subcase 2.2. $4|(m-1)$.
Then $m = 4k+1$ for some integer $k \geq 1$. If $k = 1$, then

$$\begin{aligned}(K_m \times K_n)(2) &= (K_5 \times K_n)(2) = (K_5 \times (K_2 \oplus K_2 \oplus \cdots \oplus K_2))(2) \\ &= (K_5 \times K_2)(2) \oplus (K_5 \times K_2)(2) \oplus \cdots \oplus (K_5 \times K_2)(2).\end{aligned}$$

By Lemma 2.9, $C_8|(K_2 \times K_5)(2)$, and hence, $C_8|(K_5 \times K_2)(2)$. Thus, $C_8|(K_5 \times K_n)(2)$. So, assume that $k \geq 2$. We can write $K_m = K_{4k+1}$ as

$$\underbrace{K_5 \oplus K_5 \oplus \cdots \oplus K_5}_{k \text{ times}} \oplus (K_k \circ \overline{K}_4),$$

and hence,

$$(K_m \times K_n)(2) = \underbrace{(K_5 \times K_n)(2) \oplus (K_5 \times K_n)(2) \oplus \cdots \oplus (K_5 \times K_n)(2)}_{k \text{ times}} \oplus ((K_k \circ \overline{K}_4) \times K_n)(2).$$

By the above particular value for k, i.e., $k = 1$, we have $C_8|(K_5 \times K_n)(2)$. Again, by Claim 2, $C_8|((K_k \circ \overline{K}_4) \times K_n)(2)$.

This completes the proof.

5 Conclusion

The following theorems are used in the proof of Corollary 5.1.

Theorem 5.1. [34]. *If $p \geq 3$ is a prime, $m, n \geq 3$ and $k \in \{p, 2p, 3p, p^2\}$, then $C_k|(K_m \times K_n)(2)$ if and only if $k|m(m-1)n(n-1)$ and $k \leq mn$.*

Theorem 5.2. [33]. *If $m, n \geq 3$, then $C_4|(K_m \times K_n)(\lambda)$ if and only if $4|\lambda\binom{m}{2}n(n-1)$ and $(K_m \times K_n)(\lambda)$ is an even regular graph.*

By Theorems 5.1, 5.2 and 1.2, we have:

Corollary 5.1. *If $m, n \geq 3$ and $3 \leq k \leq 15$, then $C_k|(K_m \times K_n)(2)$ if and only if $k|m(m-1)n(n-1)$ and $k \leq mn$.*

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