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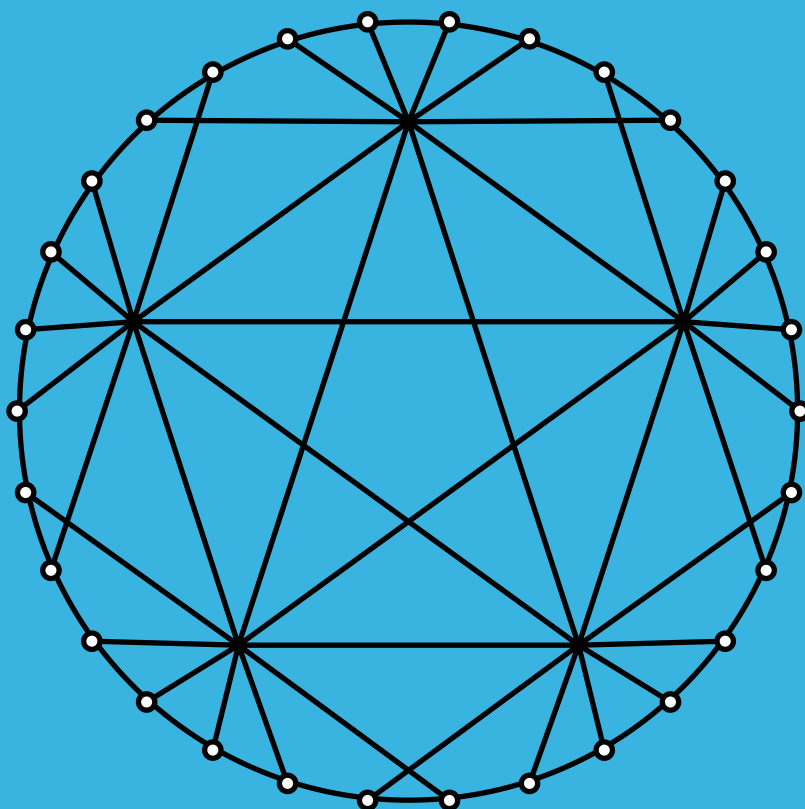

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Matrix approaches to constructions of group divisible designs

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Abstract. Saurabh and Sinha [30] obtained some series of L_2-type Latin square designs using certain combinatorial matrices. These constructions cover all the L_2-type Latin square designs listed in Clatworthy [6] except one. Here by using matrix approaches, solutions of the semi-regular group divisible (SRGD) and symmetric regular group divisible (RGD) designs listed in Clatworthy [6] and elsewhere in the range of $r, k \leq 10$ are obtained except few. In the process non-isomorphic solutions of some SRGD designs are also obtained.

1 Introduction

1.1 Group divisible designs

Let $v = mn$ elements be arranged in an $m \times n$ array. A *group divisible* (GD) *design* is an arrangement of the $v = mn$ elements in b blocks each of size k such that:

1. Every element occurs at most once in a block;
2. Every element occurs in r blocks;
3. Every pair of elements, which are in the same row of the $m \times n$ array, occur together in λ_1 blocks; while every other pair of elements occur together in λ_2 blocks.

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The integers $v = mn$, b, r, k, λ_1 and λ_2 are known as *parameters* of the GD design and they satisfy the relations: $bk = vr$ and $(n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1)$. [1]. Furthermore, if $r - \lambda_1 = 0$, then the GD design is singular (S); if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$, then it is semi-regular (SR); and if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$, the design is regular (R). For $\lambda_1 = 0$ and $\lambda_2 = \lambda$, the above definition is equivalent to uniform (k, λ)-GD design of type n^m, see Furino et al. [13] and Abel et al. [1]. Let N be the incidence matrix of a GD design then the structure of NN' is given as:

$$\text{(i) } NN' = \begin{pmatrix} (r-\lambda_1)I_n + \lambda_1 J_n & \lambda_2 J_n & \cdots & \lambda_2 J_n \\ \lambda_2 J_n & (r-\lambda_1)I_n + \lambda_1 J_n & \cdots & \lambda_2 J_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_2 J_n & \lambda_2 J_n & \cdots & (r-\lambda_1)I_n + \lambda_1 J_n \end{pmatrix}$$

$$= (r-\lambda_1)(I_m \otimes I_n) + (\lambda_1 - \lambda_2)(I_m \otimes J_n) + \lambda_2(J_m \otimes J_n)$$

The $m \times n$ array is given as:

$$\begin{matrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (m-1)n+1 & (m-1)n+2 & (m-1)n+3 & \cdots & (m-1)n \end{matrix}$$

$$\text{or (ii) } NN' = \begin{pmatrix} (r-\lambda_2)I_m + \lambda_2 J_m & (\lambda_1-\lambda_2)I_m + \lambda_2 J_m & \cdots & (\lambda_1-\lambda_2)I_m + \lambda_2 J_m \\ (\lambda_1-\lambda_2)I_m + \lambda_2 J_m & (r-\lambda_2)I_m + \lambda_2 J_m & \cdots & (\lambda_1-\lambda_2)I_m + \lambda_2 J_m \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1-\lambda_2)I_m + \lambda_2 J_m & (\lambda_1-\lambda_2)I_m + \lambda_2 J_m & \cdots & (r-\lambda_2)I_m + \lambda_2 J_m \end{pmatrix}$$

$$= (r-\lambda_2)(I_n \otimes I_m) + \lambda_2(J_n \otimes J_m) + (\lambda_1-\lambda_2)\{(J_n - I_n) \otimes J_m\}.$$

In this case, the $m \times n$ array is given as:

$$\begin{matrix} 1 & m+1 & 2m+1 & \cdots & (n-1)m+1 \\ 2 & m+2 & 2m+2 & \cdots & (n-1)m+2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m & 2m & 3m & \cdots & mn \end{matrix}$$

The GD design whose incidence matrix is N' is called the dual of the design with incidence matrix N and the GD design whose incidence matrix is $J_{v\times b} - N$ is called the complement of the design with incidence matrix N. Let D be a GD design with parameters: $v = mn$, b, r, k, λ_1, λ_2, m, n. Then the complement of D is again a GD design with parameters: $v^* = v$, $b^* = b$, $r^* = b - r$, $k^* = v - k$, $\lambda_1^* = b - 2r + \lambda_1$, $\lambda_2^* = b - 2r + \lambda_2$, $m^* = m$, $n^* = n$.

Further let the incidence matrix of a GD design with parameters: $v = mn$, b, r, k, λ_1, λ_2, m, n, be partitioned into m, $n \times b$, submatrices using suitable permutations of rows and columns of N such that each column sum of the partitioned submatrix is θ. Then removing t rows of blocks of N we obtain another GD design with parameters: $v^* = v - nt = n(m - t)$, $b^* = b$, $r^* = r$, $k^* = k - t\theta$, $\lambda_1^* = \lambda_1$, $\lambda_2^* = \lambda_2$, $m^* = m - t$, $n^* = n$, where $\theta = nr/b(= n\lambda_2/r)$.

Example 1.1. The incidence matrix of *SR*65: $v = b = 9$, $r = k = 6$, $\lambda_1 = 3$, $\lambda_2 = 4$, $m = n = 3$ may be partitioned in to 3×9 submatrices such that each column sum of the partitioned matrix is 2 as given below:

$$N = \left[\begin{array}{ccccccccc} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Removing a row of blocks of we obtain another SRGD design *SR*35: $v = 6$, $b = 9$, $r = 6$, $k = 4$, $\lambda_1 = 3$, $\lambda_2 = 4$, $m = 2$, $n = 3$.

1.2 μ-Resolvable design

A block design $D(v, b, r, k)$ whose b blocks can be divided into $t = r/\mu$ classes, each of size $\beta = v\mu/k$ and such that in each class of β blocks every element of D is replicated μ times, is called an μ-resolvable design. If $\mu = 1$, then the design is said to be resolvable.

Alternatively, if the incidence matrix N of a block design D(v, b, r, k) may be partitioned into submatrices as: $N = (N_1|N_2|\cdots|N_t)$ where each $N_i(1 \leq$

$i \leq t$) is a $v \times v\mu/k$ matrix such that each row sum of N_i is μ, then the design is μ-resolvabe.

1.3 Some combinatorial matrices

An $n \times n$ matrix $H = (H_{ij})$ with entries H_{ij} as ± 1 is called a *Hadamard matrix* if $HH' = H'H = nI_n$, where H' is the transpose of H and I_n is the identity matrix of order n. A Hadamard matrix is in normalized form if its first row and first column contain only +1s. A rectangular Hadamard matrix is an $m \times n$ $(m < n)$ matrix with elements 1, -1 such that $XX' = mI_m$.

A Hadamard matrix is *regular* if the sum of the elements in any row of the matrix is constant. It is known that the order of a regular Hadamard matrix is a perfect square $4t^2$, t a positive integer. The number of entries +1 in any row is a constant, either $2t^2 - t$ or $2t^2 + t$. In the first case, any two rows will have $t^2 - t$ positions wherein both have entry +1 whereas the second case has $t^2 + 1$ positions wherein both have entry +1.

A generalized Bhaskar Rao design GBRD$(v, b, r, k, \lambda; G)$ over a group G is a $v \times b$ array with entries from $G \cup \{0\}$ such that:

1. Each row has exactly r group element entries;
2. Each column has exactly k group element entries;
3. For each pair of distinct rows $(x_1, x_2, \ldots, x_b)$ and $(y_1, y_2, \ldots, y_b)$, the multi-set $\{x_i y_i^{-1}$:i=$1, 2, \ldots, b; x_i, y_i \neq 0\}$ contains each group element exactly $\lambda/|G|$ times.

When $|G| = 2$, such a design is a Bhaskar Rao design. A generalized Bhaskar Rao design GBRD$(v, b, r, k, \lambda; G)$ with $v = b$ and $r = k$ is also known as a *balanced generalized Weighing matrix* BGW$(v, k, \lambda; G)$. A *difference matrix* D$(k, \lambda g; G)$, is a GBRD$(v, \lambda g, \lambda g, k, \lambda g; G)$, i.e. difference matrices are precisely GBRD 's with non-zero entries. Further when $k = \lambda g$, the difference matrix is said to be a *generalised Hadamard matrix* over G of order λg and index λ, GH$(\lambda g; G)$, see de Launey [21]. If the diagonal entries of BGW$(v, k, \lambda; G)$ are zero and the inner product of any pair of distinct rows contains each element of G exactly λ times, then it is known as a *generalized conference matrix*, GC$(G; \lambda)$. The order of GC$(G; \lambda)$ is $\lambda g + 2$.

A *Conference matrix* of order n is an $n \times n$ matrix C with diagonal entries 0 and off-diagonal entries ± 1 such that $CC' = (n-1)I_n$. A conference matrix is *normalized* if all entries in its first row and first column are 1 (except the $(1,1)$-entry which is 0). The *core* of a normalized conference matrix C consists of all the rows and columns of C except the first row and column. For more details on combinatorial matrices we refer to Ionin and Kharghani [17], Abel et al. [1] and Tonchev [33].

1.4 Balanced incomplete block design

A balanced incomplete block design (BIBD) or a 2-(v, k, λ) design is an arrangement of v elements into b blocks, each of size k $(< v)$, such that every element occurs in exactly r blocks and any two distinct elements occur together in λ blocks.

It is well known that the existence of a Hadamard matrix of order $4t$ implies the existence of a BIBD or a Hadamard design with parameters: $v = b = 4t-1$, $r = k = 2t-1$, $\lambda - 1$, see Dey [9]. Such a design is *skew Hadamard* if $N + N' = (J - I)_{4t-1}$, where N is the incidence matrix of the 2-$(4t-1, 2t-1, t-1)$ design.

The main aim of the paper is to obtain solution of GD designs in the range of $r, k \leq 10$ available in Clatworthy [6] and elsewhere using matrix approaches. A comprehensive coverage on the constructions of GD designs may be found in Clatworthy [6], Dey and Balasubramanian [10], Dey [8, 9], Raghavarao [24], Raghavarao and Padgett [25], Saurabh et al. [28] and Saurabh and Sinha [29]. Kharaghani and Suda [20] introduced the concept of linked systems of symmetric GD designs. Several methods of constructions of SRGD and symmetric RGD designs by various authors are scattered throughout the literature, see Clatworthy [6] and elsewhere. Dey [7], Hedayat and Wallis [16], Bush [4], Kageyama and Tanaka [19], Gibbons and Mathon [15], Cheng [5], Sarvate and Seberry [26], Kadowaki and Kageyama [18] gave matrix approaches to their constructions. Apart from the works of these authors, some simple matrix approaches replace most of the earlier construction methods. These constructions cover all the SRGD and symmetric RGD designs found in Clatworthy [6] and elsewhere in the range of $r, k \leq 10$ except few. In the process μ-resolvable solutions of some SRGD designs are also obtained.

Notations:

I_n is the identity matrix of order n.

$J_{v\times b}$ is the $v \times b$ matrix all of whose entries are 1, $J_{v\times v}$ is denoted by J_v.

A' is the transpose of matrix A.

e_n is a $1 \times n$ matrix with entries 1.

$A \otimes B$ is the Kronecker product of two matrices A and B.

$0_{m\times n}$ is the zero matrix of order $m \times n$.

$\text{EA}(p^n) \approx C_p \times C_p \times \cdots \times C_p$ (n copies) denotes the elementary abelian group of order p^n and $C_p = \text{EA}(p)$ is a cyclic group of order p, where p is a prime.

*SR*X and *R*X numbers are from Clatworthy [6]. The design number *SR*X(a/b/c...) occurs between *SR*X and *SR*(X+1), see Freeman [12] and Dey [7].

2 Earlier constructions

Replacing 1 by I_2 and -1 by $(J-I)_2$ in a Hadamard matrix of order 2 we obtain a SRGD design *SR*1: $v = b = 4$, $r = k = 2$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = n = 2$. Further replacing 1 by I_2 and -1 by $(J-I)_2$ in a Hadamard matrix of order $4t$ we obtain:

Theorem 2.1 (Sinha [31], Kadowaki and Kageyama [18]).
The existence of a Hadamard matrix of order $4t$ *is equivalent to the existence of a symmetric* SRGD *design with parameters:* $v = b = 8t$, $r = k = 4t$, $\lambda_1 = 0$, $\lambda_2 = 2t$, $m = 4t$, $n = 2$.

Replacing the elements of a group G of order g by the corresponding $g \times g$ permutation matrices and each 0 entry by a $g \times g$ null matrix in a $\text{GBRD}(v, b, r, k, \lambda; G)$ we obtain:

Theorem 2.2 (Gibbons and Mathon [15]).
The existence of a $\text{GBRD}(v, b, r, k, \lambda; G)$ *over a group* G *implies the existence of a* GD *design with parameters:* $v^* = vg$, $b^* = bg$, $r^* = r$, $k^* = k$, $\lambda_1 = 0$, $\lambda_2 = \lambda/g$, $m = v$, $n = g$.

As a special case of Theorem 2.2, we have:

Theorem 2.3 (Sarvate and Seberry [26]).
The existence of a GBRD$(v, b, r, k, \lambda; G)$ *over an elementary abelian group* G *of order* g, EA(g) *implies the existence of a* GD *design with parameters:* $v^* = vg$, $b^* = bg$, $r^* = r$, $k^* = k$, $\lambda_1 = 0$, $\lambda_2 = \lambda/g$, $m = v$, $n = g$.

Theorem 2.4 (Raghavarao and Padgett [25]).
There exists a GD design with parameters: $v = b = 4s$, $r = k = s + 2$, $\lambda_1 = s - 2$, $\lambda_2 = 2$, $m = 4$, $n = s$; $s \geq 2$.

Theorem 2.5 (Raghavarao and Padgett [25]).
There exists a GD *design with parameters:* $v = b = 3n$, $r = k = n+1$, $\lambda_1 = n$, $\lambda_2 = 1$, $m = 3$, n.

Remark 2.6. The GD design in Theorem 2.4 is obtained by replacing 1 by I_n and -1 by $(J - I)_n$ in a regular Hadamard matrix of order 4 and the GD design in Theorem 2.5 is obtained by replacing 1 by I_n and -1 by J_n in the core of a conference matrix of order 4.

Theorem 2.7 (Bush [4], Kageyama and Tanaka [19], Corollary 4.1.1.).
If there exists a skew Hadamard design with parameters: $v' = b' = 4t - 1$, $r' = k' = 2t - 1$, $\lambda' = 1$, *then there is a symmetric regular* GD *design with parameters:* $v = b = 3(4t - 1)$, $r = k = 2t + 1$, $\lambda_1 = t - 1$, $\lambda_2 = 1$, $m = 3$, n.

Theorem 2.8 (Kageyama and Tanaka [19, Corollary 4.1.4]).
There exist for $n \geq 2$ *symmetric* RGD *designs with parameters:*

(i) $v = b = 7n$, $r = k = n + 2$, $\lambda_1 = n - 2$, $\lambda_2 = 1$, $m = 7$, n.

(ii) $v = b = 7n$, $r = k = 3n - 2$, $\lambda_1 = 3(n - 2)$, $\lambda_2 = n - 1$, $m = 7$, n.

3 The constructions

3.1 Construction theorems for SRGD designs

Theorem 3.1. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of* SRGD *designs with parameters:*

(i) $v = 2m$, $b = 4t$, $r = 2t$, $k = m$, $\lambda_1 = 0$, $\lambda_2 = t$, m, $n = 2$; (1)

(ii) $v = 4mt$, $b = 4t(4t - 1)$, $r = 2t(4t - 1)$, $k = 2mt$, $\lambda_1 = 2t(2t - 1)$, $\lambda_2 = t(4t - 1)$, m, $n = 4t$; (2)

where $1 < m < 4t$.

Proof. Let H^* be a rectangular Hadamard matrix obtained by deleting $4t-m-1$ rows of a Hadamard matrix of order $4t$ such that its first row contains only 1s. Let H be the $m \times 4t$ matrix obtained by deleting the first row of H^*. Then each row sum of H is zero and $J_{m\times 4t}H' = HJ'_{m\times 4t} = 0_m$.

(i) We claim that $N = \binom{(J_{m\times 4t})+H)/2}{(J_{m\times 4t})-H)/2} = \binom{N_1}{N_2}$ is the incidence matrix of the SRGD design with parameters (1). We have

$$\begin{aligned} N_1N_1' &= (J_{m\times 4t}+H)(J'_{m\times 4t}+H')/4 \\ &= (J_{m\times 4t}J'_{m\times 4t}+HH')/4 = 2tI_m + t(J-I)_m \end{aligned}$$

$$\begin{aligned} N_2N_2' &= (J_{m\times 4t}-H)(J'_{m\times 4t}-H')/4 \\ &= (J_{m\times 4t}J'_{m\times 4t}+HH')/4 = 2tI_m + t(J-I)_m \end{aligned}$$

$$\begin{aligned} N_1N_2' &= (J_{m\times 4t}+H)(J'_{m\times 4t}-H')/4 \\ &= (J_{m\times 4t}J'_{m\times 4t}-HH')/4 = t(J-I)_m \end{aligned}$$

Also each column sum of N is m. Hence N represents a SRGD design with Parameters (1).

(ii) Let H^{**} be the $4t \times (4t-1)$ matrix obtained by deleting the first column of a normalized Hadamard matrix of order $4t$. Let N be the $(0,1)$-matrix obtained by replacing 1 by $(J_{m\times 4t}+H)/2$ and -1 by $(J_{m\times 4t}-H)/2$ in H^{**}. Also each column sum of N is $2mt$. Then N represents a SRGD design with Parameters (2) which may be easily verified. □

Remark 3.2. Theorem 3.1(i) is the matrix construction of the Theorem 2.7 of Bush [4].

Example 3.3. Let $m=5$, $t=2$. Let H^* be a rectangular Hadamard matrix of order 6×8 whose first row contains only ones. Then a 5×8 rectangular Hadamard matrix H obtained by deleting first row of H^* is given as:

$$H = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

Now using Theorem 3.1(i), $N = \binom{(J_{5\times 8}+H)/2}{(J_{5\times 8}-H)/2}$ represents a SRGD design $SR52$: $v = 10$, $r = 4$, $k = 5$, $b = 8$, $\lambda_1 = 0$, $\lambda_2 = 2$, $m = 5$, $n = 2$ whose blocks are given as:

$$\begin{array}{cccc} (1,2,3,4,5); & (2,4,6,8,10); & (1,4,5,7,8); & (3,4,6,7,10); \\ (1,2,3,9,10); & (2,5,6,8,9); & (1,7,8,9,10); & (3,5,6,7,9). \end{array}$$

The 5×2 array is given as transpose of the array: $\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{array}$.

Example 3.4. Let $m = 3$, $t = 1$. Consider a normalized Hadamard matrix H^* of order 4:

$$H^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Then

$$H = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

and

$$H^{**} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Now replacing 1 by $(J_{3\times 4}+H)/2$ and -1 by $(J_{3\times 4}-H)/2$ in H^{**}, we obtain a $(0,1)$-matrix

$$N = \begin{pmatrix} (J_{3\times 4}+H)/2 & (J_{3\times 4}+H)/2 & (J_{3\times 4}+H)/2 \\ (J_{3\times 4}-H)/2 & (J_{3\times 4}+H)/2 & (J_{3\times 4}-H)/2 \\ (J_{3\times 4}+H)/2 & (J_{3\times 4}-H)/2 & (J_{3\times 4}-H)/2 \\ (J_{3\times 4}-H)/2 & (J_{3\times 4}-H)/2 & (J_{3\times 4}+H)/2 \end{pmatrix},$$

which represents a SRGD design $SR68$: $v = b = 12$, $r = k = 6$, $\lambda_1 = 2$, $\lambda_2 = 3$, $m = 3$, $n = 4$ [vide Theorem 3.1(ii)] whose blocks are given as:

$$\begin{array}{cccc} (1,2,3,7,8,9); & (2,4,6,8,10,12); & (1,5,6,7,11,12); & (3,4,5,9,10,11); \\ (1,2,3,4,5,6); & (2,5,7,9,10,12); & (1,4,8,9,11,12); & (3,6,7,8,10,11); \\ (1,2,3,10,11,12); & (2,4,6,7,9,11); & (1,5,6,8,9,10); & (3,4,5,7,8,12) \end{array}$$

The 3×4 array is given as: $\begin{array}{cccc} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{array}$.

van Lint and Wilson [34, p.229] used E_i-matrices $(1 \le i \le 3)$ in the construction of a BIBD with parameters: $v = 9$, $b = 12$, $r = 4$, $k = 3$, $\lambda = 1$, where E_i denotes a 3 by 3 matrix with 1 in column i and 0s elsewhere. Here we are defining E_i-matrices as an n by n matrix with 1s in i-th row and 0s elsewhere. A permutation matrix P is an by matrix with entries 0 and 1 such that each row and column of P contains 1 exactly once and is 0 elsewhere. Then

(i) $\sum_{i=1}^{n} E_i = J_n; \sum_{i=1}^{n} E_i E_i' = nI_n$

(ii) $E_i P = E_i$, $(1 \le i \le n)$.

Clearly $\alpha = \text{CIRC}.(0\ 1\ 0\ \cdots\ 0)$ is an n by n permutation matrix.

Theorem 3.5. *There exists an* SRGD *design with parameters:*

$$v = 3n,\ b = n^2,\ r = n,\ k = 3,\ \lambda_1 = 0,\ \lambda_2 = 1,\ m = 3,\ n. \tag{3}$$

Proof. Let E_i $(1 \le i \le n)$ denote an $n \times n$ matrix whose i-th row contains only +1s and is 0 elsewhere. Let $\alpha = \text{CIRC}.(0\ 1\ 0\ \cdots\ 0)$ denote a circulant matrix of order n with +1 at the second position of the first row and is 0 elsewhere. Then

$$N = \begin{pmatrix} E_1 & E_2 & \cdots & E_n \\ I_n & I_n & \cdots & I_n \\ I_n & \alpha & \cdots & \alpha^{n-1} \end{pmatrix}$$

is the incidence matrix of an SRGD design with Parameters (3). This may be easily verified. □

Theorem 3.6. *There exists a g-resolvable* SRGD *design with parameters:*

$$\begin{gathered} v = g(\lambda g + 1),\ b = \lambda g^2,\ r = \lambda g,\ k = \lambda g + 1, \\ \lambda_1 = 0,\ \lambda_1 = \lambda,\ m = \lambda g + 1,\ n = g, \end{gathered} \tag{4}$$

when g is a prime or prime power.

Proof. It is well known (see Kharaghani and Suda [20]) that the existence of a $\text{GH}(\lambda g; G)$ over $G = \text{EA}(g)$ implies the existence of an SRGD design with parameters:

$$\begin{gathered} v' = b' = \lambda g^2,\ r' = k' = \lambda g, \\ \lambda_1' = 0,\ \lambda_2' = \lambda,\ m' = \lambda g,\ n' = g. \end{gathered} \tag{5}$$

Let M be the incidence matrix of a GD design with Parameters (5). We construct a matrix as follows:

$$\begin{bmatrix} (E_1\ E_2\ \cdots\ E_g) & \cdots & (E_1\ E_2\ \cdots\ E_g) \\ & M & \end{bmatrix},$$

where E_i $(1 \le i \le g)$ is a $g \times g$ matrix whose i-th row contains only $+1$s and is 0 elsewhere; and $(E_1\ E_2\ \cdots\ E_g)$ is adjoined λ times in a row above M. Clearly N can be partitioned into submatrices each of size $v \times g^2$, such that each row sum of partitioned matrix is g. Hence we obtain a g-resolvable SRGD design with Parameters (4) . □

3.2 Non-isomorphic solutions of some SRGD designs

Example 3.7. Consider a GH$(6; C3)$ with entries from a cyclic group $C_3 = \{1, w, w^2\}$:

$$\text{GH}(6; C_3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w & w^2 & 1 \\ 1 & w^2 & w & w & 1 & w^2 \\ 1 & w^2 & w^2 & 1 & w & w \\ 1 & 1 & w & w^2 & w^2 & w \\ 1 & w & 1 & w^2 & w & w^2 \end{pmatrix}.$$

Then using GH$(6; C_3)$ in Theorem 3.6;

$$N = \left(\begin{array}{ccc|ccc} E_1 & E_2 & E_3 & E_1 & E_2 & E_3 \\ I_3 & I_3 & I_3 & I_3 & I_3 & I_3 \\ I_3 & \alpha & \alpha^2 & \alpha & \alpha^2 & I_3 \\ I_3 & \alpha^2 & \alpha & \alpha & I_3 & \alpha^2 \\ I_3 & \alpha^2 & \alpha^2 & I_3 & \alpha & \alpha \\ I_3 & I_3 & \alpha & \alpha^2 & \alpha^2 & \alpha \\ I_3 & \alpha & I_3 & \alpha^2 & \alpha & \alpha^2 \end{array}\right)$$

represents a 3-resolvable SRGD design *SR*84: $v = 21$, $b = 18$, $r = 6$, $k = 7$, $\lambda_1 = 0$, $\lambda_2 = 2$, $m = 7$, $n = 3$, where $\alpha =$ CIRC.$(0\ 1\ 0)$ is a circulant matrix of order 3. For the same design a non-resolvable solution is reported in Clatworthy [6]. The resolution classes are

RI: $[(1, 4, 7, 10, 13, 16, 19)$; $(1, 5, 8, 11, 14, 17, 20)$; $(1, 6, 9, 12, 15, 18, 21)$; $(2, 4, 9, 11, 14, 16, 21)$; $(2, 5, 7, 12, 15, 17, 19)$; $(2, 6, 8, 10, 13, 18, 20)$; $(3, 4, 8, 12, 14, 18, 19)$; $(3, 5, 9, 10, 15, 16, 20)$; $(3, 6, 7, 11, 13, 17, 21)]$

RII: $[(1,4,9,12,13,17,20);\ (1,5,7,10,14,18,21);\ (1,6,8,11,15,16,19);$
$(2,4,8,10,15,17,21);\ (2,5,9,11,13,18,19);\ (2,6,7,12,14,16,20);$
$(3,4,7,11,15,18,20);\ (3,5,8,12,13,16,21);\ (3,6,9,10,14,17,19)]$

The 7×3 array is given as transpose of the array: $\begin{matrix} 1 & 4 & 7 & 10 & 13 & 16 & 19 \\ 2 & 5 & 8 & 11 & 14 & 17 & 20 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 \end{matrix}$

Example 3.8. Using GH(8; EA(4)) in Theorem 3.6, we obtain a 4-resolvable solution of $SR103$. For the same design a non-resolvable solution is reported in Clatworthy [6].

Example 3.9. Using GH($5; C_5$) in Theorem 2.3, it can be observed that

$$N_1 = \begin{pmatrix} I_5 & a^2 & a^3 & a^3 & a^2 \\ a & a^4 & a & a^2 & a^2 \\ a^4 & a^3 & a & a^3 & a^4 \\ a^4 & a^4 & a^3 & a & a^3 \\ a & a^2 & a^2 & a & a^4 \end{pmatrix} \quad \text{and} \quad N_1 = \begin{pmatrix} I_5 & a & a^4 & a^4 & a \\ I_5 & a^4 & I_5 & a^3 & a^3 \\ I_5 & a^2 & a & a^2 & I_5 \\ I_5 & I_5 & a^2 & a & a^2 \\ I_5 & a^3 & a^3 & I_5 & a^4 \end{pmatrix}$$

both represent $SR60$. Juxtaposing N_1 and N_2 we obtain a quasidouble resolvable solution of $SR61$, for which only a duplicate solution of $SR60$ is reported.

3.3 Construction theorems for RGD designs

Theorem 3.10. *The existence of a skew Hadamard design with parameters:*

$$v' = b' = 4t-1,\ r' = k' = 2t-1,\ \lambda' = t-1$$

implies the existence of a group divisible design witt-h parameters:

$$v = 8t = b,\ r = 4t-1 = k,\ \lambda_1 = 0,\ \lambda_2 = 2t-1,\ m = 4t,\ n = 2. \quad (6)$$

Proof. Let N be the incidence matrix of a skew Hadamard design with parameters:

$$v' = b' = 4t-1,\ r' = k' = 2t-1,\ \lambda' = t-1.$$

Then $N + N' = (J-I)_{4t-1}$. Let

$$M_1 = \begin{pmatrix} 0 & e_{4t-1} \\ e'_{4t-1} & N \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 0_{1\times(4t-1)} \\ 0'_{1\times(4t-1)} & N' \end{pmatrix}.$$

Then we claim that

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}$$

represents a GD design with Parameters (6). We have

(i) $M_1M_1' + M_2M_2'$

$$= \begin{pmatrix} 4t-1 & 2t-1 & 2t-1 & \dots & 2t-1 \\ 2t-1 & 2t & t & \cdots & t \\ 2t-1 & t & 2t & \cdots & t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2t-1 & t & t & \cdots & 2t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 2t-1 & t-1 & \cdots & t-1 \\ 0 & t-1 & 2t-1 & \cdots & t-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & t-1 & t-1 & \cdots & 2t-1 \end{pmatrix}$$

$$= (4t-1)I_{4t} + (2t-1)(J-I)_{4t}.$$

(ii) Also $N + N' = (J-I)_{4t-1}$

$$\Rightarrow N^2 + (N')^2 + NN' + N'N = (J-I)_{4t-1}(J-I)'_{4t-1}$$
$$\Rightarrow N^2 + (N')^2 = (2t-1)(J-I)_{4t-1}$$

$$M_1M_2' + M_2M_1' = \begin{pmatrix} 0 & 2t-1 & 2t-1 & \cdots & 2t-1 \\ 2t-1 & & & & \\ 2t-1 & & & N^2 + (N')^2 & \\ \vdots & & & & \\ 2t-1 & & & & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2t-1 & 2t-1 & \cdots & 2t-1 \\ 2t-1 & 0 & 2t-1 & \cdots & 2t-1 \\ 2t-1 & 2t-1 & 0 & \cdots & 2t-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2t-1 & 2t-1 & 2t-1 & \cdots & 0 \end{pmatrix}$$

$$= (2t-1)(J-I)_{4t}.$$

Hence M represents the incidence matrix of a GD design with Parameters (6). □

Theorem 3.11. *When $4t+1$ is a prime or prime power, there exists a group divisible design with parameters:*

$$v = 4(2t+1) = b,\ r = k = 4t+1,\ \lambda_1 = 0, \quad \lambda_2 = 2t,\ m = 2(2t+1),\ n = 2. \tag{7}$$

Proof. When $4t+1$ is a prime or prime power, the initial blocks:

$$(x^0, x^2, x^4, \ldots, x^{4t-2}) \quad \text{and} \quad (x^1, x^3, x^5, \ldots, x^{4t-1})$$

generate a BIBD with parameters:

$$v = 4t+1,\ b = 2(4t+1),\ r = 4t,\ k = 2t,\ \lambda = 2t-1,$$

where x is a primitive element of the Galois field $\mathrm{GF}(4t+1)$. Let N_1 be the incidence matrix corresponding to the block design with initial block $(x^0, x^2, x^4, \ldots, x^{4t-2})$ and N_2 be the incidence matrix corresponding to the block design with initial block $(x^1, x^3, x^5, \ldots, x^{4t-1})$. Then the rows and columns of N_1 and N_2 can be permuted such that $N_1 + N_2 = (J-I)_{4t+1}$. Let

$$M_1 = \begin{pmatrix} 0 & e_{4t+1} \\ e'_{4t+1} & N_1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 0_{1\times 4t+1} \\ 0'_{1\times 4t+1} & N_1 \end{pmatrix}.$$

Then we claim that

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}$$

represents a GD design with Parameters (7). We have

(i) $M_1M_1' + M_2M_2'$

$$= \begin{pmatrix} 4t+1 & 2t \quad 2t \quad \cdots \quad 2t \\ 2t & \\ 2t & \\ \vdots & J_{4t+1} + N_1N_1' + N_2N_2' \\ 2t & \end{pmatrix} = \begin{pmatrix} 4t+1 & 2t & 2t & \ldots & 2t \\ 2t & 4t+1 & 2t & \cdots & 2t \\ 2t & 2t & 4t+1 & \cdots & 2t \\ \vdots & \vdots & \vdots & \vdots & \\ 2t & 2t & 2t & \cdots & 4t+1 \end{pmatrix}$$

$$= (4t+1)I_{4t+2} + 2t(J-I)_{4t+2}.$$

(ii) $N_1 + N_2 = (J-I)_{4t+1} \Rightarrow N_1N_2' + N_2N_1' = 2t(J-I)_{4t+1}$

$$M_1M_2' + M_2M_1' = \begin{pmatrix} 0 & 2t \quad 2t \quad \cdots \quad 2t \\ 2t & \\ 2t & \\ \vdots & N_1N_2' + N_2N_1' \\ 2t & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2t & 2t & \cdots & 2t \\ 2t & 0 & 2t & \cdots & 2t \\ 2t & 2t & 0 & \cdots & 2t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2t & 2t & 2t & \cdots & 0 \end{pmatrix}$$

$$= 2t(J-I)_{4t+2}.$$

Hence M represents the incidence matrix of a GD design with Parameters (7). ☐

Remark 3.12. The GD designs in Theorem 3.11 have been constructed using difference sets and a connection between partial difference sets and GD designs may be found in Ma [23] and Arasu et al. [2]. Further Theorems 3.10 and 3.11 yield patterned constructions for the RGD designs $R177$a and $R197$a respectively. For these designs trial and error solutions are reported in Dey [7].

Theorem 3.13. *When $s \geq 2$ there exists a* GD *design with parameters*

$$\begin{gathered} v = b = sn, \ r = k = (s-1)n + 1, \\ \lambda_1 = (s-1)n, \ \lambda_2 = (s-2)n + 2, \ m = s, \ n \end{gathered} \tag{8}$$

Proof. We claim that $N = I_s \otimes I_n + (J - I)_s \otimes J_n$ is the incidence matrix of a GD design with Parameters (8). We have

$$NN' = \begin{pmatrix} I_n + (s-1)J_n^2 & 2J_n + (s-2)J_n^2 & \cdots & 2J_n + (s-2)J_n^2 \\ 2J_n + (s-2)J_n^2 & I_n + (s-1)J_n^2 & \cdots & 2J_n + (s-2)J_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ 2J_n + (s-2)J_n^2 & 2J_n + (s-2)J_n^2 & \cdots & I_n + (s-1)J_n^2 \end{pmatrix}$$

$$= \begin{pmatrix} \{n(s-1)+1\}I_n + n(s-1)(J-I)_n & \cdots & \{n(s-2)+2\}J_n \\ \vdots & \ddots & \vdots \\ \{n(s-2)+2\}J_n & \cdots & \{n(s-1)+1\}I_n + n(s-1)(J-I)_n \end{pmatrix}$$

$$= \{n(s-1)+1\}(I_s \otimes I_n) + \{n(s-1)\}I_s \otimes (J-I)_n + \{n(s-2)+2\}(J_s - I_s) \otimes J_n.$$

Hence N represents a GD design with Parameters (8). ☐

Theorem 3.14. *The existence of a Conference matrix of order $t \geq 6$ and a* BIBD *with parameters: $v = 2k$, b, r, k, λ implies the existence of a* RGD *design with parameters*

$$\begin{gathered} v^* = (t-1)v, \ b^* = (t-1)b, \ r^* = r + b(t-2)/2, \ k^* = k + v(t-2)/2, \\ \lambda_1^* = \lambda + b(t-2)/2, \ \lambda_2^* = r(t-2)/2, \ m^* = t-1, \ n^* = v \end{gathered} \tag{9}$$

Proof. Let C^* be the core of a normalized Conference matrix C and N be the incidence matrix of a BIBD with parameters: $v = 2k$, b, r, k, λ. Then replacing 1 by $J_{v,b}$, 0 by N and -1 by $0_{v,b}$ in C^* we obtain a GD design with Parameters (9). ☐

Remark 3.15. Theorem 3.14 is the generalization of Theorem 2.2 of Bhagwandas and Parihar [3]. For $t = 6$ we obtain Theorem 2.2 of Bhagwandas and Parihar [3].

For $N = I_2$ in Theorem 3.14, we obtain:

Corollary 3.16. *There exists a* RGD *design with parameters:*

$$\begin{gathered} v = b = 2(t-1),\ r = k = t-1, \\ \lambda_1 = t-2,\ \lambda_2 = (t-2)/2,\ m = t-1,\ n = 2. \end{gathered} \tag{10}$$

Theorem 3.17. *The existence of a symmetric* 2-(v, k, λ) *design implies the existence of a* RGD *design with parameters:*

$$\begin{gathered} v^* = b^* = sv,\ r^* = k^* = (s-1)v + k, \\ \lambda_1 = (s-1)v + \lambda,\ \lambda_2 = 2r + (s-2)v,\ m = s,\ n. \end{gathered} \tag{11}$$

Proof. Let N be the incidence matrix of a symmetric 2-(v, k, λ) design. Then $M = I_s \otimes N + (J - I)_s \otimes J_v$ is the incidence matrix of a GD design with Parameters (11). □

Theorem 3.18. *There exists a symmetric* RGD *design with parameters:*

$$\begin{gathered} v^* = b^* = 3g^2,\ r^* = k^* = 2g, \\ \lambda_1 = g,\ \lambda_2 = 1,\ m = 3g,\ n = g, \end{gathered} \tag{12}$$

where $g = p^n$ is a prime power.

Proof. Let C be the $g \times (g-1)$ matrix obtained by deleting the first column of a normalised $\mathrm{GH}(g^2; \mathrm{EA}(g))$. Let M be a $(0,1)$-block matrix obtained by replacing group elements of C by the corresponding permutation matrices and let the rows of M be $R_1, R_2, \ldots, R_g$. Then

$$N = \begin{pmatrix} \mathrm{CIRC}.(0_g\ 0_g\ \cdots\ 0_g | E_1\ E_2\ \cdots\ E_g | E_1'\ R_1) \\ \mathrm{CIRC}.(0_g\ 0_g\ \cdots\ 0_g | E_1\ E_2\ \cdots\ E_g | E_2'\ R_1) \\ \vdots \\ \mathrm{CIRC}.(0_g\ 0_g\ \cdots\ 0_g | E_1\ E_2\ \cdots\ E_g | E_g'\ R_1) \end{pmatrix}$$

represents a RGD design with Parameters (12), where the E_i-matrices $(1 \leq i \leq g)$ are matrices whose i-th row contains only $+1$s and is 0 elsewhere. □

Example 3.19. For $g = 3$ we obtain $R170$: $v = b = 27$, $r = k = 6$, $\lambda_1 = 3$, $\lambda_2 = 1$, $m = 9$, $n = 3$, whose incidence matrix is given as:

$$N = \begin{pmatrix} \text{circ.}(0_3\ 0_3\ 0_3 | E_1\ E_2\ E_3 | E_1'\ I_3\ I_3) \\ \text{circ.}(0_3\ 0_3\ 0_3 | E_1\ E_2\ E_3 | E_2'\ \alpha^2\ \alpha) \\ \text{circ.}(0_3\ 0_3\ 0_3 | E_1\ E_2\ E_3 | E_3'\ \alpha\ \alpha^2) \end{pmatrix}$$

$$= \begin{pmatrix} 0_3 & 0_3 & 0_3 & E_1 & E_2 & E_3 & E_1' & I_3 & I_3 \\ E_1' & I_3 & I_3 & 0_3 & 0_3 & 0_3 & E_1 & E_2 & E_3 \\ E_1 & E_2 & E_3 & E_1' & I_3 & I_3 & 0_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & 0_3 & E_1 & E_2 & E_3 & E_2' & \alpha^2 & \alpha \\ E_2' & \alpha^2 & \alpha & 0_3 & 0_3 & 0_3 & E_1 & E_2 & E_3 \\ E_1 & E_2 & E_3 & E_2' & \alpha^2 & \alpha & 0_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & 0_3 & E_1 & E_2 & E_3 & E_3' & \alpha & \alpha^2 \\ E_3' & \alpha & \alpha^2 & 0_3 & 0_3 & 0_3 & E_1 & E_2 & E_3 \\ E_1 & E_2 & E_3 & E_3' & \alpha & \alpha^2 & 0_3 & 0_3 & 0_3 \end{pmatrix}$$

Example 3.20. For $g = 4$ we obtain $R190$: $v = b = 48$, $r = k = 8$, $\lambda_1 = 4$, $\lambda_2 = 1$, $m = 12$, $n = 4$, whose incidence matrix is given as:

$$N = \begin{pmatrix} \text{circ.}(0_4\ 0_4\ 0_4\ 0_4 | E_1\ E_2\ E_3\ E_4 | E_1'\ I_4\ I_4\ I_4) \\ \text{circ.}(0_4\ 0_4\ 0_4\ 0_4 | E_1\ E_2\ E_3\ E_4 | E_2'\ A\ B\ C) \\ \text{circ.}(0_4\ 0_4\ 0_4\ 0_4 | E_1\ E_2\ E_3\ E_4 | E_2'\ C\ A\ B) \\ \text{circ.}(0_4\ 0_4\ 0_4\ 0_4 | E_1\ E_2\ E_3\ E_4 | E_2'\ B\ C\ A) \end{pmatrix},$$

where $A = I_2 \otimes (J - I)_2$, $B = (J - I)_2 \otimes (J - I)_2$, $A = (J - I)_2 \otimes I_2$.

4 Tables of designs

This section contains Tables (1–3) of semi-regular and symmetric regular GD designs listed in Clatworthy [6] and elsewhere in the range of $r, k \leq 10$ constructed using the present theorems. The designs obtained by duplicating, deletion of groups and taking complement are not included in the Tables.

The generalised Hadamard matrices, $\text{GH}(\lambda g; G)$ used in Tables 1 and 2 may be found in de Launey [22]. H_n denotes a Hadamard matrix of order n. The SRGD design SR109a may be found in Ghosh and Divecha [14].

Table 1: Symmetrical semi-regular group divisible designs

No.	GD: $(v, k, \lambda_1, \lambda_2, m, n)$	Source
1	*SR*1: (4, 2, 0, 1, 2, 2)	Th. 2.1, H_2
2	*SR*23: (9, 3, 0, 1, 3, 3)	Th. 3.6(5); GH$(3; C_3)$
3	*SR*36: (8, 4, 0, 2, 4, 2)	Th. 2.1, H_4
4	*SR*44: (16, 4, 0, 1, 4, 4)	Th. 3.6(5); GH$(4; EA(4))$
5	*SR*60: (25, 5, 0, 1, 5, 5)	Th. 3.6(5); GH$(5; C_5)$
6	*SR*67: (12, 6, 0, 3, 6, 2)	Th. 2.8(i); $m = 6$, $t = 3$
7	*SR*68: (12, 6, 2, 3, 3, 4)	Th. 3.1(ii); $m = 3$, $t = 1$
8	*SR*72: (18, 6, 0, 2, 6, 3)	Th. 3.6(5); GH$(6; C_3)$
9	*SR*87: (49, 7, 0, 1, 7, 7)	Th. 3.6(5); GH$(7; C_7)$
10	*SR*92: (16, 8, 0, 4, 8, 2)	Th. 2.1, H_8
11	*SR*95: (32, 8, 0, 2, 8, 4)	Th. 3.6(5); GH$(8; EA(4))$
12	*SR*97: (64, 8, 0, 1, 8, 8)	Th. 3.6(5); GH$(8; EA(8))$
13	*SR*102: (27, 9, 0, 3, 9, 3)	Th. 3.6(5); GH$(9; C_3)$
14	*SR*105: (81, 9, 0, 1, 9, 9)	Th. 3.6(5); GH$(9; EA(9))$
15	*SR*108: (20, 10, 0, 5,10, 2)	Th. 3.1(i); $m = 10$, $t = 5$
16	*SR*109a: (50, 10, 0, 2,10, 5)	Th. 3.6(5); GH$(10; C_5)$

Table 2: Asymmetrical semi-regular group divisible designs

No.	GD: $(v, r, k, b, \lambda_1, \lambda_2, m, n)$	Source
17	SR30: (18, 6, 3, 36, 0, 1, 3, 6)	Th. 3.5, $n = 6$
18	SR34: (30, 10, 3, 100, 0, 1, 3, 10)	Th. 3.5, n= 10
19	SR38: (8, 6, 4, 12, 2, 3, 2, 4)	Th. 3.1 (ii); $m = 2$, $t = 1$
20	SR41: (12, 3, 4, 9, 0, 1, 4, 3)	Th. 3.6(4); GH$(3; C_3)$
21	SR51: (40, 10, 4, 100, 0, 1, 4, 10)	Unknown
22	SR58: (20, 4, 5, 16, 0, 1, 5, 4)	Th. 3.6(4); GH(4; EA(4))
23	SR66: (12, 4, 6, 8, 0, 2, 6, 2)	Th. 3.1 (i); $m = 6$, $t = 2$
24	SR71: (12, 10, 6, 20, 4, 5, 2, 6)	Dual of SR106
25	SR75: (30, 5, 6, 25, 0, 1, 6, 5)	Th. 3.6(4); GH$(5; C_5)$
26	SR80: (14, 4, 7, 8, 0, 2, 7, 2)	Th. 3.1 (i); $m = 7$, $t = 2$
27	SR84: (21, 6, 7, 18, 0, 2, 7, 3)	Th. 3.6(4); GH$(6; C_3)$
28	SR91: (16, 6, 8, 12, 0, 3, 8, 2)	Th. 2.8 (i); $m = 8$, $t = 3$
29	SR96: (56, 7, 8, 49, 0, 1, 8, 7)	Th. 3.6(4); GH$(7; C_7)$
30	SR103: (36, 8, 9, 32, 0, 2, 9, 4)	Th. 3.6(4); GH(8; EA(4))
31	SR104: (72, 8, 9, 64, 0, 1, 9, 8)	Th. 3.6(4); GH(8; EA(8))
32	SR106: (20, 6, 10, 12, 0, 3, 10, 2)	Th. 3.1 (i); $m = 10$, $t = 3$
33	SR107: (20, 8, 10, 16, 0, 4, 10, 2)	Th. 3.1 (i); $m = 10$, $t = 4$
34	SR109: (30, 9, 10, 27, 0, 3, 10, 3)	Th. 3.6(4); GH$(9; C_3)$
35	SR110: (90, 9, 10, 81, 0, 1, 10, 9)	Th. 3.6(4); GH(9; EA(9))

Remark 4.1. The incidence matrix N of the above series (except Theorem 3.1) of SRGD designs are partitioned into submatrices such that each partitioned matrix has column sum one. Hence removing a row of blocks of N we obtain another SRGD design. And continuing so on we obtain: $SR108 \to \cdots \to SR5$; $SR105 \to \cdots \to SR16$; $SR36 \to \cdots \to SR2$; $SR102 \to \cdots \to SR8$; $SR95 \to \cdots \to SR10$; $SR97 \to \cdots \to SR15$; $SR87 \to \cdots \to SR14$; $SR60 \to \cdots \to SR11$; $SR23 \to SR6$; $SR34 \to SR17$; $SR92 \to \cdots \to SR4$; $SR106 \to \cdots \to SR3$; $SR30 \to SR13$; $SR44 \to \cdots \to SR9$; $SR109a \to SR103a \to \cdots \to SR61$.
The parameters of SR103a, SR95a and SR86a are $(45, 10, 9, 50, 0, 2, 9, 5)$, $(40, 10, 8, 50, 0, 2, 8, 5)$, and $(35, 10, 7, 50, 0, 2, 7, 5)$ respectively. $SR$35 is the complement of $SR$6.

Table 3: Symmetric regular group divisible designs

No.	GD: $(v, k, \lambda_1, \lambda_2, m, n)$	Source
1	$R$42: (6, 3, 2, 1, 3, 2)	Th. 2.4; $n = 2$
2	$R$54: (8, 3, 0, 1, 4, 2)	Th. 3.10, $t = 1$
3	$R$94: (6, 4, 3, 2, 2, 3)	Th. 3.13; $s = 2, n = 3$
4	$R$104: (9, 4, 3, 1, 3, 3)	Th. 2.5; $n = 3$
5	$R$109: (12, 4, 2, 1, 6, 2)	Th. 3.18; $g = 2$
6	$R$112: (14, 4, 0, 1, 7, 2)	Th. 2.2; BGW$(7, 4, 2; C_2)$
7	$R$114: (15, 4, 0, 1, 5, 3)	Th. 2.2; BGW$(5, 4, 3; C_3)$
8	$R$133: (8, 5, 4, 2, 2, 4)	Th. 3.13; $s = 2, n = 4$
9	$R$139: (10, 5, 4, 2, 5, 2)	Corollary 3.16; $t = 6$
10	$R$143: (12, 5, 4, 1, 3, 4)	Th. 2.5; $n = 4$
11	$R$144: (12, 5, 0, 2, 6, 2)	Th. 3.11; $t = 1$
12	$R$145: (12, 5, 1, 2, 4, 3)	Th. 2.4; $s = 3$
13	$R$166: (10, 6, 5, 2, 2, 5)	Th. 3.5; $s = 2, n = 5$
14	$R$168: (15, 6, 5, 1, 3, 5)	Th. 2.5; $n = 5$
15	$R$170: (27, 6, 3, 1, 9, 3)	Th. 3.18; $g = 3$
16	$R$171: (28, 6, 2, 1, 7, 4)	Th. 2.8 (i); $n = 4$
17	$R$172: (9, 7, 6, 5, 3, 3)	Th. 3.13; $s = n = 3$
18	$R$173: (12, 7, 6, 2, 2, 6)	Th. 3.13; $s = 2, n = 6$
19	$R$177: (14, 7, 6, 3, 7, 2)	Corollary 3.16; $t = 8$
20	R177a: (16, 7, 0, 3, 8, 2)	Th. 3.10, $t = 2$
21	R177b: (16, 7, 2, 3, 4, 4)	Unknown
22	$R$178: (18, 7, 6, 1, 3, 6)	Th. 2.4; $n = 6$
23	$R$179: (20, 7, 3, 2, 4, 5)	Th. 2.5; $s = 5$
24	$R$180: (20, 7, 6, 2, 10, 2)	Unknown
25	R180a: (21, 7, 3, 2, 7, 3)	Th. 2.8 (ii); $n = 3$
26	R180b: (24, 7, 0, 2, 8, 3)	Th. 2.2; GC$(C_3; 2)$
27	$R$182: (33, 7, 2, 1, 3, 11)	Th. 2.7; $t = 3$
28	R182a: (35, 7, 3, 1, 7, 5)	Th. 2.8 (i); $n = 5$

29	R182b: (45, 7, 0, 1, 15, 3)	Th. 2.2; BGW$(15, 7, 3; C_3)$
30	$R$183: (48, 7, 0, 1, 8, 6)	Th. 2.2; GC$(C_6; 1)$
31	$R$187: (14, 8, 7, 2, 2, 7)	Th. 3.13; $s = 2$, n=7
32	$R$188: (21, 8, 7, 1, 3, 7)	Th. 2.5; $n = 7$
33	$R$189: (24, 8, 4, 2, 4, 6)	Th. 2.4; $s = 6$
34	R189a: (42, 8, 4, 1, 7, 6)	Th. 2.8 (i); $n = 6$
35	$R$190: (48, 8, 4, 1, 12, 4)	Th. 3.18; $g = 4$
36	$R$191: (63, 8, 0, 1, 9, 7)	Th. 2.2; GC$(C_7; 1)$
37	$R$193: (12, 9, 8, 6, 3, 4)	Th. 3.13; $s = 3$, n=4
38	$R$195: (16, 9, 8, 2, 2, 8)	Th. 3.13; $s = 2$, n=8
39	$R$196: (18, 9, 6, 4, 6, 3)	Unknown
40	$R$197: (18, 9, 8, 4, 9, 2)	Corollary 3.16; $t = 10$
41	R197a: (20, 9, 0, 4, 10, 2)	Th. 12, $t = 2$
42	R197b: (20, 9, 3, 4, 4, 5)	Unknown
43	$R$198: (24, 9, 8, 1, 3, 8)	Th. 2.5; $n = 8$
44	R198b: (24, 9, 6, 3, 12, 2)	[11, Theorem 2.4];
45	$R$199: (26, 9, 0, 3, 13, 2)	Th. 2.2; BGW$(13, 9, 6; C_2)$
46	$R$200: (28, 9, 5, 2, 4, 7)	Th. 2.4; $s = 7$
47	R200a: (38, 9, 0, 2, 19, 2)	Th. 2.2; BGW$(19, 9, 4; C_2)$
48	R200b: (39, 9, 0, 2, 13, 3)	[29]
49	R200c: (40, 9, 0, 2, 10, 4)	Th. 2.2; BGW$(10, 9, 8; C_4)$
50	R200d: (45, 9, 3, 1, 3, 15)	Th. 2.7; $t = 4$
51	R200e: (49, 9, 5, 1, 7, 7)	Th. 2.8 (i); $n = 7$
52	$R$201: (78, 9, 0, 1, 13, 6)	Th. 2.2; BGW$(13, 9, 6; S_3)$
53	$R$202: (80, 9, 0, 1, 10, 8)	Th. 2.2; GC$(Q_8; 1)$
54	$R$203: (12, 10, 9, 8, 4, 3)	Th. 3.13; $s = 4$, n=3
55	$R$204: (14, 10, 8, 6, 2, 7)	Th. 3.17; $s = 2$ and 2–$(7, 3, 1)$ design
56	$R$206: (18, 10, 9, 2, 2, 9)	Th. 3.13; $s = 2$, n=9
57	R206a: (21, 10, 9, 4, 7, 3)	[28]
58	R206b: (21, 10, 8, 3, 3, 7)	[3, Theorem 3.1];
59	$R$207: (27, 10, 9, 1, 3, 9)	Th. 2.5; $n = 9$
60	R207a: (28, 10, 6, 3, 7, 4)	Th. 2.8 (ii); $n = 4$
61	$R$208: (32, 10, 6, 2, 4, 8)	Th. 2.4; $s = 8$
62	R208a: (56, 10, 6, 1, 7, 8)	Th. 2.8 (i); $n = 8$
63	R208b: (49, 10, 1, 2, 7, 7)	Unknown, [27]
64	$R$209: (75, 10, 5, 1, 15, 5)	Th. 3.18; $g = 5$

Q_8 is a Quaternion group of order 8 and S_3 is the symmetric group of degree 3 and order 6. The balanced generalized Weighing matrices and generalized Conference matrices used in the Table 3 may be found in Ionin and Kharghani [17] and the RGD designs in Table 3 may be found in Clatworthy [6] and Sinha [32].

5 Conclusion

Some of the series obtained above may be new as these are not found in Raghavarao [24], Dey [8, 9], Raghavarao and Padgett [25]. This paper unifies and generalizes some earlier constructions of GD designs. The paper also provides a short survey on the methods of constructions of GD designs by matrix approaches. Tables 1, 2 and 3 of SRGD and symmetric RGD designs are presented above along with their methods of construction. One SRGD and five symmetric RGD designs in the range of $r, k \leq 10$ could not be obtained by the above constructions.

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