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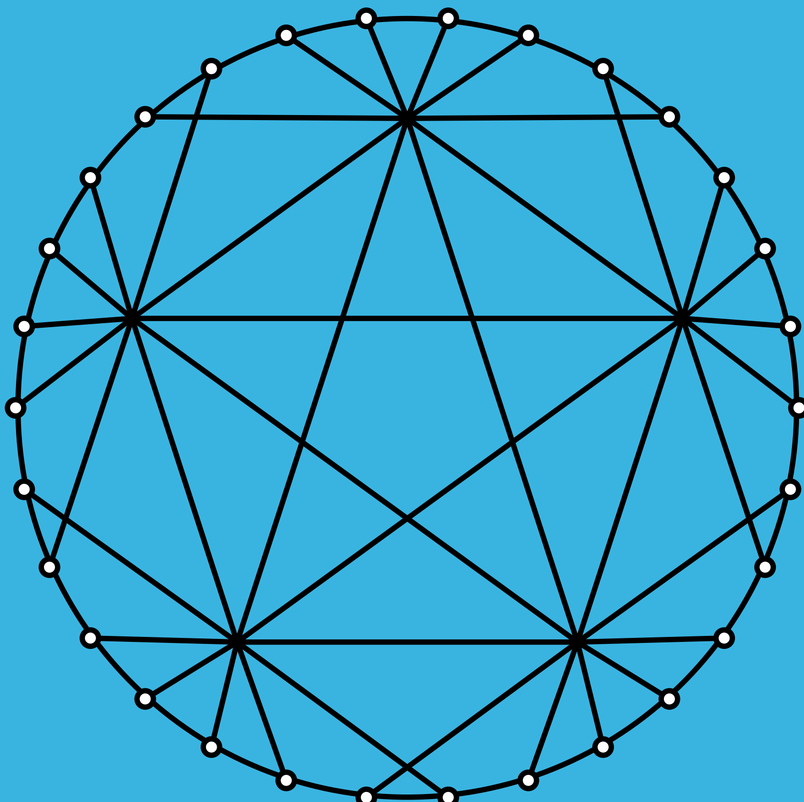

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Vertex-magic trees with n central vertices and kn leaves have bounded order for each k exceeding the square root of 3

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Abstract. It is well-known that trees with n central vertices and more than $2n$ leaves do not possess vertex-magic total labelings. Furthermore, this restriction is tight in the sense that there are examples of vertex-magic total labelings of trees with n central vertices and $2n$ leaves. In this paper we prove that any such vertex-magic graph must satisfy $n \leq 3$. This answers a long standing open question posed by Wallis. We do this by providing an upper bound on n for any vertex-magic graph with kn degree 1 vertices and n other vertices, provided that $k > \sqrt{3}$. We then provide alternative open questions with a related conjecture, in the same spirit as Wallis' original question, and begin the process of exploring them. In particular, for each $n \geq 1$, we provide an elegant vertex-magic total labeling of a tree with n central vertices and $n + 1$ leaves.

1 Introduction

A *total labeling* of a graph G with vertex set V and edge set E is a bijective map λ from $V \cup E$ onto $\{1, 2, \ldots, |V| + |E|\}$. The weight $wt(v)$ of the vertex v is given by $wt(v) = \lambda(v) + \sum \lambda(e)$ where the sum is over all edges e incident with v. The total labeling λ is called *vertex-magic* if there is a constant h such that $wt(v) = h$ for every vertex $v \in V$. In this case, λ is called a *vertex-magic total labeling (VMTL)* and h is its *magic constant*. A graph that has a VMTL is called a *vertex-magic graph* and we will, more simply, refer to such a graph as *magic*. If a graph does not have a VMTL, we will call it *non-magic*. VMTLs were formally introduced in [5] where

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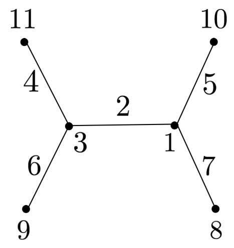

Figure 1: A VMTL with $h = 15$ for a tree with $n = 2$ and four leaves.

it was already shown that a tree with n central vertices and more than $2n$ leaves must be non-magic. Furthermore, they provide (see Figure 1) a VMTL for a tree with two central vertices and four leaves, showing that their bound is tight.

This paper was inspired by a problem which originally appears in the first edition of the book [7] by W.D. Wallis (2001), throughout which research problems are generously sprinkled. The following is the one of interest:

Research Problem 3.5 (from [7]) For each positive integer n, find a tree with n central vertices and $2n$ leaves, which is vertex-magic.

The problem is repeated in the second edition [6], by A.M. Marr and W.D. Wallis (2013), with the same wording. This suggests an anticipation that such graphs *exist*. One of the goals of this paper is to prove they do not, unless $n \leq 3$. Indeed, this follows from Theorem 2.1, as the special case with $k = 2$, where k is defined as the ratio of the number of leaves to the number of central vertices. Theorem 2.1 places an upper bound on n in terms of k, whenever $k > \sqrt{3}$.

In Section 3, we provide alternatives to Wallis' *Research Problem 3.5* from [7]. In particular, we wonder if $k < \sqrt{3}$ is enough to guarantee that there is a vertex-magic tree with n central vertices and kn leaves. We provide as a special example a VMTL for a tree with n central vertices and $n + 1$ leaves, for all $n \geq 1$. While this example does not feature k being close to $\sqrt{3}$, it is unusually simple for a general VMTL construction.

For extra context, note that necessary restrictions for certain trees and forests to have a VMTL are provided in [2]. Yet, interestingly, they show that the disjoint union of s 3−vertex paths (i.e. the so-called *galaxy* $sK_{1,2}$) has a VMTL for every s.

However, the presence of degree 1 vertices does provide a formidable obstruction to the existence of VMTLs. In his 2001 book, Wallis proves the following:

Theorem 3.15 (from [7]): Let G be any graph of order v. If G has e edges, then a $G - sun\ G^*$ is not vertex-magic total when

$$e > \frac{-1 + \sqrt{1 + 8v^2}}{2}$$

Note that, by definition, a $G - sun$ is obtained from a graph G by adjoining a new vertex of degree 1 to each vertex of G. The term *sun graph*, without reference to an originating graph G, presumes that G is a cycle.

It is also important to mention that the special case where $n = 4$ of Wallis' *Research Problem 3.5* was decided via a computer search by Jağer and Arnold in [3]. In that paper, the strategy was to first identify the 60 relevant trees with four central vertices and eight leaves, using NAUTY. Then, an integer programming algorithm verified that none of them had a VMTL. It is perhaps fortunate that we were unaware of this interesting result at the time that our work was being done. The end result is that our paper is self-contained and that it did not benefit from computer assistance.

Our interest in *Research Problem 3.5* started with the general question of attempting to determine the role of the degree sequence itself, in deciding whether or not a graph is vertex-magic. Some progress was made in [1] where, among other results, the following was shown:

Theorem 8 (from [1]): Let G be a graph with $2n$ vertices and $2n$ edges, with $n \geq 5$. If G has $t = \left\lfloor \frac{n}{3} \right\rfloor$ components isomorphic to $K_{1,3}$ then G is not vertex-magic.

The strategy for the proof of our main theorem (in the next section) is analogous to the one used for Theorem 8 in [1]. In that paper, the goal was to provide degree sequences that are ambiguous in the sense that there are both magic and non-magic graphs sharing the same degree sequence. The sun graph was shown to be magic, and Theorem 8 was used to construct non-magic graphs *without a component isomorphic to* $K_{1,1}$, yet still having the same degree sequence as the sun. (Any graph with a component of $K_{1,1}$ is trivially seen to be non-magic).

It is a noteworthy heuristic observation that graphs within a family of graphs with similar degree sequences seem to have more VMTLs as the order increases. Furthermore, the number of VMTLs also may grow very quickly with the order of the graphs. For a more precise illustration of this phenomenon, J. S. Kimberley and J. A. MacDougall ([4]) provide tables for the numbers of a particular kind of VMTL, the so-called *strong* ones for odd order 2−regular graphs, and this number grows stunningly quickly with the order of the graph.

The next section provides an example that goes against this trend, as the degree 1 vertices present too much of an obstacle for larger orders.

2 The main theorem

In this section we prove the following:

Theorem 2.1. *Let G be a graph with l leaves and n other vertices. Assume G has $n+l-1$ edges and set $k = l/n$. If G is vertex-magic and $k > \sqrt{3}$ then*

$$n \leq \frac{k+1}{k^2-3}$$

Remark: Any tree with n central vertices and l leaves satisfies the hypothesis of the theorem. Trees having $l = 2n$ leaves (and therefore $k = 2$), corresponding to $n = 2$ and $n = 3$, along with their VMTLs are shown in Figures 1 and 2 respectively.

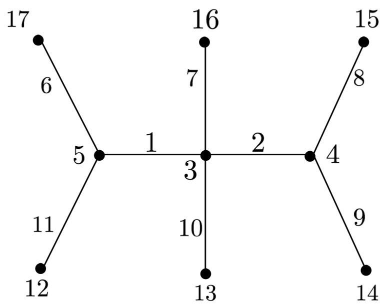

Figure 2: A VMTL with $h = 23$ for a tree with $n = 3$ and six leaves.

Proof. Assume G is a graph as in the hypothesis of the theorem, with VMTL λ having a magic constant of h. Let v_i denote a leaf with adjacent edge e_i, for $i = 1, 2, \ldots l$.

Let S_v and S_e denote, respectively, the sum of all vertex labels and the sum of all edge labels. Since there are $n + l$ vertices and $n + l - 1$ edges, we have:

$$\begin{aligned} S_v + S_e &= 1 + 2 + \cdots + (2n + 2l - 1) \\ &= (n + l)(2n + 2l - 1) \end{aligned}$$

Summing over the weights of all $n + l$ vertices will result in each edge label being counted twice. Hence:

$$S_v + 2S_e = (n + l)h$$

Combining these two equations results in

$$h = 2n + 2l - 1 + \frac{S_e}{n + l} \tag{1}$$

Claim: $h \leq 2l + 4n - 1$

Proof of Claim. We focus on the degree 1 vertices, and for each one, the corresponding two labels that must sum to h. We can therefore choose a_i so that:

$$\{\lambda(v_i), \lambda(e_i)\} = \{2n + 2l - a_i, h + a_i - 2n - 2l\} \text{ for } i = 1, 2, \ldots l \tag{2}$$

Notice that since the maximum label is $2n + 2l - 1$, it follows that $a_i > 0$ and furthermore we may choose the notation so that

$$2n + 2l - a_i > h + a_i - 2n - 2l \text{ for } i = 1, 2, \ldots l \tag{3}$$

Therefore,

$$h < 4n + 4l - 2a_i \text{ for each } i = 1, 2, \ldots l$$

Since, $a_i \neq a_j$ whenever $i \neq j$, it follows that there is some choice of index i^* so that $a_{i^*} \geq l$. Therefore, $h < 4n + 4l - 2l = 4n + 2l$. Since h, n and l are all integral, we see that $h \leq 4n + 2l - 1$, which proves the claim. □

Next we provide a lower bound for h. We do this by providing a lower bound for S_e and then using equation (1).

For the leaf edges, we use equation (2) and inequality (3) to see that

$$\sum_{i=1}^{l} \lambda(e_i) \geq \sum_{i=1}^{l} (h + a_i - 2n - 2l) \tag{4}$$

The usefulness of inequality (4) is the key insight required for the proof. The rest is straightforward. We use (1) to replace h in the above expression with $2n + 2l - 1 + \frac{S_e}{n+l}$. Hence:

$$\begin{aligned}
\sum_{i=1}^{l} \lambda(e_i) &\geq \sum_{i=1}^{l} (2n + 2l - 1 + \frac{S_e}{n+l} + a_i - 2n - 2l) \\
&= \frac{lS_e}{n+l} + \sum_{i=1}^{l} (a_i - 1) \\
&\geq \frac{lS_e}{n+l} + \sum_{i=1}^{l} (i - 1) \\
&= \frac{lS_e}{n+l} + \frac{l(l-1)}{2}
\end{aligned}$$

Since there are $n + l - 1$ edges in total, there are $n - 1$ other edges to consider. The sum of their labels is (trivially) at least

$$1 + 2 + \ldots (n - 1) = n \left(\frac{n-1}{2} \right)$$

Whence,

$$S_e \geq \frac{lS_e}{n+l} + l \left(\frac{l-1}{2} \right) + n \left(\frac{n-1}{2} \right)$$

i.e.

$$\frac{nS_e}{n+l} \geq l \left(\frac{l-1}{2} \right) + n \left(\frac{n-1}{2} \right)$$

and so

$$\frac{S_e}{n+l} \geq k \left(\frac{l-1}{2} \right) + \frac{n-1}{2}$$

This is the desired lower bound for S_e. Substituting this back into (1) yields:

$$h \geq 2n + 2l - 1 + k \left(\frac{l-1}{2} \right) + \frac{n-1}{2}$$

From our earlier claim, $h \leq 2l + 4n - 1$. By comparing these two bounds for h, we get:

$$2n + 2l - 1 + k \left(\frac{l-1}{2} \right) + \frac{n-1}{2} \leq 2l + 4n - 1$$

After slightly rearranging and canceling, we get:

$$-2n + \frac{kl}{2} + \frac{n}{2} \leq \frac{k}{2} + \frac{1}{2}$$

Since $l = kn$, we get

$$\left(\frac{k^2-3}{2}\right) n \leq \frac{k+1}{2}$$

Since, by assumption, $k > \sqrt{3}$, we can divide throught by $(k^2-3)/2$ to get

$$n \leq \frac{k+1}{k^2-3}$$

as required. $\square$

Taking $k = 2$ gives us the following:

Corollary 2.2. *Let G be a tree with n central vertices and $2n$ leaves. If G is vertex-magic then $n \leq 3$.*

3 A vertex-magic tree with n central vertices and $n+1$ leaves

The goal of this section is to provide alternatives to Wallis' original *Research Problem 3.5* (discussed in the introduction) and then begin the process of finding partial answers for one of them.

Problem 3.1. *For which pairs of integers (n, l) is there a vertex-magic tree with n central vertices and l leaves?*

We know from Theorem 2.1 that $l/n > \sqrt{3}$ provides a limitation. We do not know exactly what happens if $l/n < \sqrt{3}$ or even if $\sqrt{3}$ is the correct constant to consider. Therefore, we ask the following:

Problem 3.2. *If possible, find the largest constant κ such that $l/n \leq \kappa$ guarantees that there is a tree with n central vertices and l leaves.*

Theorem 2.1 suggests a candidate for κ, namely $\sqrt{3}$. Furthermore, Theorem 2.1 implies that, for each choice of $k > \sqrt{3}$, there are at most finitely many trees with n central vertices and kn leaves. Note that this observation still allows for the possibility that the total over all such k is infinite. However, it is conceivable that $\kappa < \sqrt{3}$. We propose the following:

Conjecture 3.3. *There are only finitely many pairs of positive integers* (n, l) *with* $l/n > \sqrt{3}$ *such that there is a vertex-magic tree with* n *central vertices and* l *leaves.*

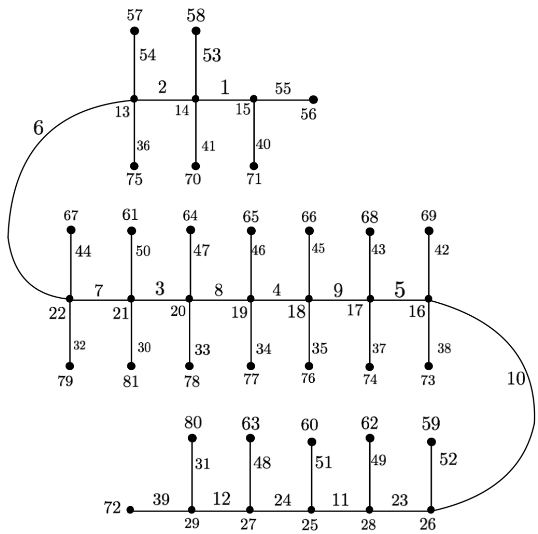

Figure 3: A VMTL with $h = 111$ for a tree with 15 central vertices and 26 leaves.

If Conjecture 3.3 is *incorrect*, then the labeling in Figure 3 provides an infinitesimal piece of evidence against the conjecture, as $l/n = 26/15 > \sqrt{3}$ for the magic tree shown there. In any case, it provides part of an answer for Problem 3.1.

It may be quite difficult to find enough explicit general constructions to solve Problem 3.2. We expect that solutions of these problems will require existence proofs.

We conclude by providing a VMTL for a tree with n central vertices and $n + 1$ leaves (see Figure 4). The original motivation for looking at this example was to begin the process of understanding Problems 3.1 and 3.2. We include it due to its surprising simplicity and elegance, as well as the fact that it generalizes a well-known VMTL for a path with three vertices, by taking $n = 1$.

Theorem 3.4. *Let* T_n *be the tree with* n *central vertices* v_i, $i = 1, 2, \cdots n$ *and* $n+1$ *leaves* w_i, $i = 1, 2, \cdots n + 1$ *with edges* $v_i v_{i+1}$, $1 \le i \le n - 1$ *and*

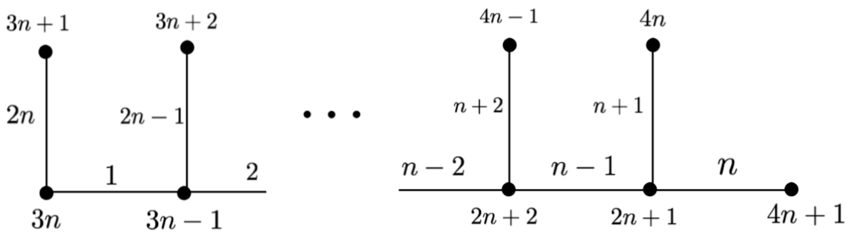

Figure 4: A VMTL for a tree with n central vertices and $n+1$ leaves.

v_iw_i, $1 \leq i \leq n$ and v_nw_{n+1}. Then, for each $n \geq 1$ there is a vertex-magic total labeling λ of T_n with magic constant $h = 5n+1$.

Proof. Starting with the leaves, set $\lambda(w_i) = 3n+i$ for $1 \leq i \leq n+1$ and $\lambda(v_iw_i) = 2n+1-i$ for $1 \leq i \leq n$ with $\lambda(v_nw_{n+1}) = n$. Evidently the weight of every leaf is $5n+1$ as claimed in the statement of the theorem.

For the central vertices, set $\lambda(v_i) = 3n+1-i$ for $1 \leq i \leq n$ and if $i < n$ set $\lambda(v_iv_{i+1}) = i$. It follows that for $i > 1$, $\lambda(v_{i-1}v_i) = i-1$. Thus the weight $wt(v_i)$ of these vertices, for $1 < i < n$, satisfies:

$$\begin{aligned} wt(v_i) &= \lambda(v_i) + \lambda(v_{i-1}v_i) + \lambda(v_iv_{i+1}) + \lambda(v_iw_i) \\ &= (3n+1-i) + (i-1) + i + (2n+1-i) \\ &= 5n+1 \end{aligned}$$

For the remaining two vertices, note that

$$\begin{aligned} wt(v_1) &= \lambda(v_1) + \lambda(v_1v_2) + \lambda(v_1w_1) \\ &= 3n+1+2n \\ &= 5n+1 \end{aligned}$$

Also,

$$\begin{aligned} wt(v_n) &= \lambda(v_n) + \lambda(v_{n-1}v_n) + \lambda(v_nw_{n+1}) + \lambda(v_nw_n) \\ &= (2n+1) + (n-1) + n + (n+1) \\ &= 5n+1 \end{aligned}$$

It remains to show that λ is bijective. For the convenience of the reader, we provide the following table, providing the use of each integer $1, 2, \cdots 4n+1$.

Labels	Corresponding object(s)
$1, 2, \cdots, n-1$	central edges
n	the edge $v_n w_{n+1}$
$n+1, n+2, \cdots, 2n$	the leaf-adjacent edges $v_i w_i$
$2n+1, 2n+2, \cdots, 3n$	central vertices
$3n+1, 3n+2, \cdots, 4n, 4n+1$	leaves

□

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