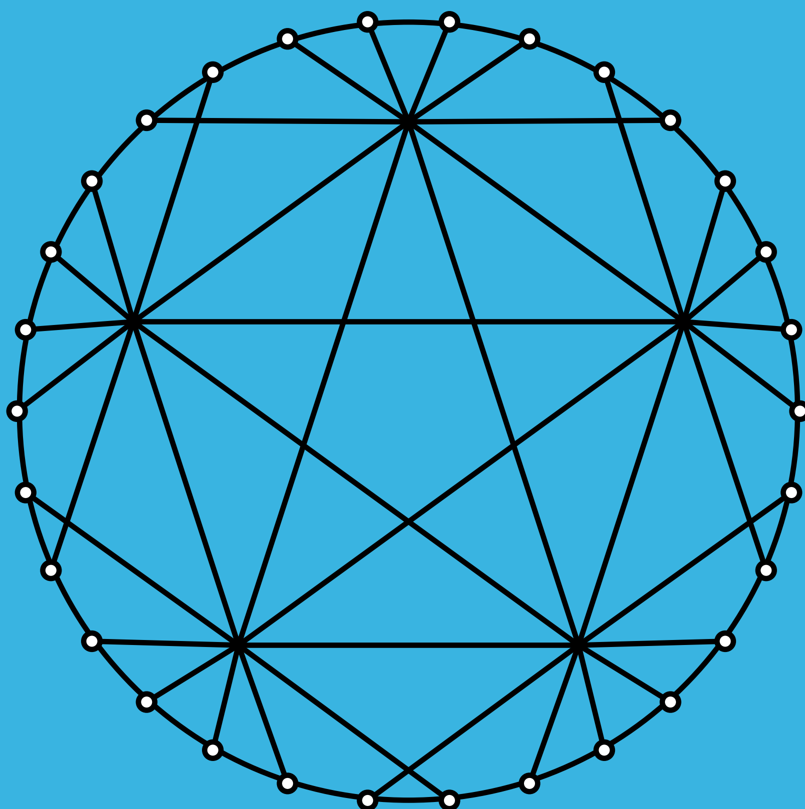
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 98
June 2023



Duluth, Minnesota, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **98** (2023), 140–147

The extended graphical and bigraphical generalized Steiner systems

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Abstract. A *set system* $(X, \mathscr{B})$ consists of a v-element set X of *points* and a collection $\mathscr{B}$ of subsets of X called *blocks*. A set system $(X, \mathscr{B})$ is a (proper) t-$(v, \mathscr{K}, \lambda)$ design, provided

1. every t–element set of points is contained in precisely λ blocks,
2. $\mathscr{K}$ is a set of integers, each of which is $> t$ and $< v$, and
3. if $B \in \mathscr{B}$, then $|B| \in \mathscr{K}$.

The parameters t and λ are called respectively the *strength* and *index* of the design. When the index is 1, they are called *generalized Steiner systems*. With respect to a given point $x \in X$, the blocks containing x (with x discarded) are called the *derived design* and the blocks that do not contain x are called the *residual design*.

If $X = E(K_n)$, then the blocks are subgraphs of K_n. We say that the set system is *graphical*, if whenever B is a block then all subgraphs of K_n isomorphic to B are also blocks. A t-$(v, \mathscr{K}, \lambda)$ design is an *extended graphical design* if respect to some fixed point x, both the residual and derived designs are graphical. In this note all extended graphical generalized Steiner systems are determined. In a similar fashion *bigraphical designs* are defined for $X = E(K_{m,n})$ and all extended bigraphical generalized Steiner systems are also determined. This note concludes with several open problems that arise naturally from this investigation.

1 Exordium

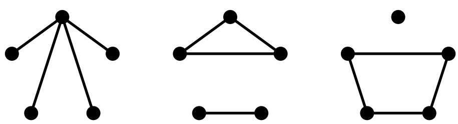
A 2-$(v, \mathscr{K}, \lambda)$ design is a *pairwise balanced design*, see [4, Part IV]. Among the first examples of pairwise balanced designs are those constructed by

AMS (MOS) Subject Classifications: 05B05,05C99
Key words and phrases: graphical, bigraphical, t-wise balanced, Steiner systems

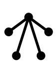
Received: 19 July 2022
Accepted: 17 March 2022

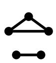
Kirkman [10] in 1847, which he used to solve the schoolgirls problem on $v = 5\cdot 3^{m+1}$ points, [4, page 15]. Kramer [12] referred to t-$(v, \mathscr{K}, \lambda)$ designs as *t-wise balanced designs* or *tBDs.* However Hanani [8] in 1963 first introduced such designs as *C-designs*, but called them pairwise balanced designs when $t = 2$. For the case $t = 3$ Brouwer [2] used the term *triplewise balanced designs.* When the index $\lambda = 1$, the adjective *Steiner* is affixed even though they are only loosely related to Steiner's original problem, see [5]. Indeed t-$(v, \mathscr{K}, 1)$ designs were called *Hanani generalized Steiner systems* in [17] and *generalized Steiner systems* in [1, 18]. Note that generalized Steiner systems, as introduced in [7], are H-designs (see [16]) with certain coding properties and they should not be confused with the t-$(v, \mathscr{K}, 1)$ designs that are discussed here. Regardless of the terminology it is not difficult to see that t-$(v, \mathscr{K}, 1)$ designs have deep connections to geometries and codes. Furthermore they have proven to be invaluable in the construction of desired combinatorial configurations.

A set system $(X, \mathscr{B})$ is *graphical* if X is the set of $v = \binom{n}{2}$ labelled edges of the complete graph K_n with vertex set $\{1, 2, \ldots, n\}$ and it has the natural action of the symmetric group S_n on the edges as an automorphism group. Thus if $B \in \mathscr{B}$, then all subgraphs isomorphic to B are also in $\mathscr{B}$. Hence a graphical set system can be presented as a list of unlabeled graphs. For example



graphically represents a 3-$(10, 4, 1)$ design. The actual blocks are:

 $= \big\{ \{12,13,14,15\},\{21,23,24,25\},\{31,32,34,35\},\{41,42,43,45\},\{51,52,53,54\} \big\}$

 $= \left\{ \begin{array}{l} \{12,23,31,45\},\{12,24,41,35\},\{14,43,31,25\},\{42,23,34,15\},\{12,25,51,34\} \\ \{13,35,51,24\},\{32,25,53,14\},\{14,45,51,23\},\{24,45,52,13\},\{34,45,53,12\} \end{array} \right\}$

and

 $= \left\{ \begin{array}{l} \{12,23,34,41\},\{13,34,42,21\},\{14,42,23,31\},\{12,23,35,51\},\{13,35,52,21\} \\ \{15,52,23,31\},\{12,24,45,51\},\{14,45,52,21\},\{15,52,24,41\},\{13,34,45,51\} \\ \{14,45,53,31\},\{15,53,34,41\},\{23,34,45,52\},\{24,45,53,32\},\{25,53,34,42\} \end{array} \right\}$

where $ab = ba = \{a, b\}$. A complete list of graphical t-$(v, K, 1)$ designs can be found in [3] and is provided for reference in Table 2. The presentation of a design by a list of unlabeled graphs is I think quite beautiful. It gives a visual interpretation for the blocks of the design.

If $(X, \mathscr{B})$ is a set system and $x \in X$, then the *derived* set system with respect to x is $(X \setminus \{x\}, \mathscr{B}_x)$, where

$$\mathscr{B}_x = \{B \setminus \{x\} : x \in B \in \mathscr{B}\}$$

and the *residual* set system with respect to x is $(X \setminus \{x\}, \mathscr{B}^x)$, where

$$\mathscr{B}^x = \{B \in \mathscr{B} : x \notin B\}.$$

A t-$(v, \mathscr{K}, \lambda)$ design $(X, \mathscr{B})$ is an *extended graphical design*, if there is a point x such that both $\mathscr{B}_x$ and $\mathscr{B}^x$ are graphical. This article was motivated by my discovery that the 3-$(16, \{4, 6\}, 1)$ design studied by Assmus and Sardi [1] is the extended graphical design listed as D_4 in Table 1. There is, by the way, a unique 3-$(16, \{4, 6\}, 1)$ design. It consists of the best biplane on 16 points and its 60 ovals. Having made this discovery (see [6]) I became interested in finding all extended graphical generalized Steiner systems. This article is the results of my efforts.

2 Proventus

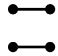
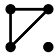
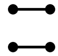
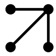
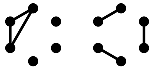
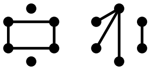
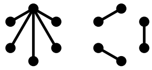
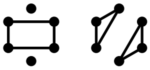
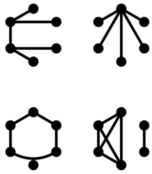
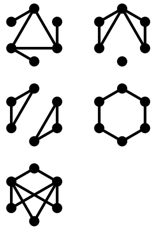
It is not difficult to determine the extended graphical t-$(v, \mathscr{K}, 1)$ designs $(X \cup \{\infty\}, \mathscr{B})$, when the derived set system $(X, \mathscr{B}_\infty)$ is a known graphical $(t-1)$-$(v-1, \mathscr{K}', 1)$ design. Consider $\mathscr{B}_\infty$ as a list of unlabeled graphs on $v-1$ points and let $\mathscr{R}$ be the t-edge unlabeled graphs on $v-1$ points that are not a subgraph of any graph in $\mathscr{B}_\infty$. Let $\mathscr{C}$ be the set of unlabeled graphs on $v-1$ points that have more than t edges. (Incidentally [13, Corollary 1.3] shows that subgraphs with more than $\frac{1}{2}\binom{v-1}{2}$ edges need not be considered.) Define the matrix $A : \mathscr{R} \times \mathscr{C} \to \mathbb{Z}_{\geq 0}$ by

$$A[R, C] = |\{c \in C : r \text{ is a subgraph of } c\}|,$$

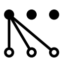
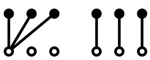
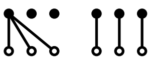
where r is any fixed labeled subgraph in R. Then there will be a solution $U : \mathscr{C} \to \{0, 1\}$ to the matrix equation $AU = J$, where $J[r] = 1$, for all $r \in \mathscr{R}$, if and only if $\mathscr{B}^\infty = \{C \in \mathscr{C} : U[C] = 1\}$ is such that $(X, \mathscr{B}_\infty \cup \mathscr{B}^\infty)$ is an extended graphical t-$(v, \mathscr{K}, 1)$ design. Columns $C \in \mathscr{C}$ of A that contain an entry exceeding 1 can of course be removed. Although the matrix equations $AU = J$ that need to be considered in this note are perhaps small enough that they could possibly be constructed and solved by hand, I employed computer algorithms. I used the algorithms

Table 1: Derived and residual systems

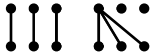
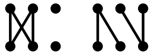
Extended graphical generalized Steiner systems

	Parameters	X	$\mathscr{B}_\infty$	$\mathscr{B}^\infty$
D_1	2-(7, 3, 1)	$E(K_4) \cup \{\infty\}$		
D_2	2-(7, 3, 1)	$E(K_4) \cup \{\infty\}$		
D_3	3-(16, 4, 1)	$E(K_6) \cup \{\infty\}$		
D_4	3-(16, {4, 6}, 1)	$E(K_6) \cup \{\infty\}$		
D_5	5-(16, {6, 8}, 1)	$E(K_6) \cup \{\infty\}$		

Extended BW-bigraphical generalized Steiner systems

	Parameters	X	$\mathscr{B}_\infty$	$\mathscr{B}^\infty$
D_6	2-(10, {3, 4}, 1)	$E(K_{33}) \cup \{\infty\}$		
D_7	2-(10, {3, 4}, 1)	$E(K_{33}) \cup \{\infty\}$		

Extended bigraphical generalized Steiner systems

	Parameters	X	$\mathscr{B}_\infty$	$\mathscr{B}^\infty$
D_8	3-(10, 4, 1)	$E(K_{33}) \cup \{\infty\}$		

described in [14, Chapter 6] to construct the matrices and I used the dancing links data structure described in [11] to solve them. The dancing link implementation I employed can be downloaded from:

http://pottonen.kapsi.fi/libexact.html

Theorem 1. *The only extended graphical* t-$(v, \mathscr{K}, 1)$ *designs are* D_1, D_2, *…,* D_5*, which are listed in Table 1.*

Proof. Let $(X \cup \{\infty\}, \mathscr{B})$ be an extended graphical t-$(v, \mathscr{K}, 1)$ design. Then $(X, \mathscr{B}_\infty)$ is a graphical $(t-1)$-$(v-1, \mathscr{K}', 1)$ design. There is only one on K_4, one on K_5 and three on K_6. It is computationally easy, by the method described above, to determine how a $(t-1)$-$(v-1, \mathscr{K}', 1)$ design $(X, \mathscr{B}_\infty)$ can be completed to an extended graphical t-$(v, \mathscr{K}, 1)$ design. □

The automorphism group $\mathrm{AUT}(K_{m,n})$ of the undirected complete bipartite graph $K_{m,n}$ with vertex set $\{1', 2', \ldots, m', 1, 2, \ldots, n\}$ is the cross product $S_m \times S_n$, if $m \neq n$ and is the wreath product $S_n \wr S_2$, if $m = n$. A set system $(X, \mathscr{B})$ is *bigraphical* if X is the set of $v = mn$ labelled edges of $K_{m,n}$ and the natural action of $\mathrm{AUT}(K_{m,n})$ on the edges of $K_{m,n}$ is an automorphism group. Thus if $B \in \mathscr{B}$, then all subgraphs of $K_{m,n}$, isomorphic to B, are also in $\mathscr{B}$.

If the vertices of $K_{n,n}$ are 2-colored, so that one independent set is colored black and the other is colored white, then the automorphism group G is $S_n \times S_n$. ($S_n \times S_n$ is subgroup of index 2 in $\mathrm{AUT}(K_{n,n})$.) A set system $(X, \mathscr{B})$ is *BW-bigraphical* if X is the set of $v = n^2$ labelled edges of the 2-colored $K_{n,n}$ and the natural action of G is an automorphism group. Thus if $B \in \mathscr{B}$, then all 2-colored subgraphs of $K_{m,n}$ isomorphic to B are also in $\mathscr{B}$.

A complete list of bigraphical t-$(v, \mathscr{K}, 1)$ designs can be found in [9] and is provided in Table 3.

The proofs of Theorems 2 and 3 are similar to the proof of Theorem 1.

Theorem 2. *The only extended bigraphical* t-$(v, \mathscr{K}, 1)$ *design is* D_8*, which is listed in Table 1.*

Theorem 3. *The only extended BW-bigraphical* t-$(v, \mathscr{K}, 1)$ *designs are* D_6 *and* D_7*, which are listed in Table 1.*

Table 2: Graphical generalized Steiner systems (see [3]).

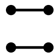
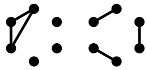
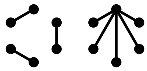
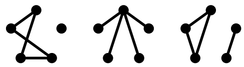
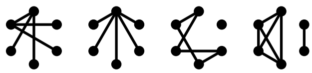
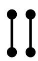
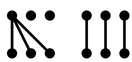
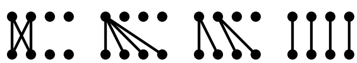
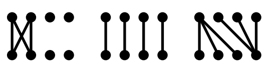
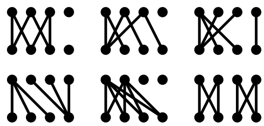
Parameters	X	Graphical representation
1-$(6, 2, 1)$	$E(K_4)$	
2-$(15, 3, 1)$	$E(K_6)$	
2-$(15, \{3, 5\}, 1)$	$E(K_6)$	
3-$(10, 4, 1)$	$E(K_5)$	
4-$(15, \{5, 7\}, 1)$	$E(K_6)$	

Table 3: Bigraphical generalized Steiner systems (see [9]).

Parameters	X	Graphical representation
1-$(n^2, n, 1)$	$E(K_{n,n})$	$K_{1,n}$
1-$(4, 2, 1)$	E$(K_{2,2})$	
2-$(9, 3, 1)$	E$(K_{3,3})$	
3-$(16, 4, 1)$	E$(K_{4,4})$	
3-$(16, \{4, 6\}, 1)$	E$(K_{4,4})$	
5-$(16, \{6, 8\}, 1)$	E$(K_{4,4})$	

Here are a few problems to consider.

1. What are the extended graphical and bigraphical designs with index 2? A complete list of graphical designs with index 2 can be found in [3] and the bigraphical designs with index 2 are in [15].

2. Do there exist doubly extended graphical and bigraphical designs? How should multi-extended graphical and bigraphical designs be defined?

3. Are there other actions of the symmetric group that yield interesting designs? Do they have extensions?

I close by thanking the reviewer for the inspiration to add more history and detail to my note. I also thank Ortrud Oellermann and Doug Stinson who read a preprint of this note and provided me with useful comments.

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