On local antimagic chromatic number of a corona product graph

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Abstract. In this paper, we provide a correct proof for the lower bounds of the local antimagic chromatic number of the corona product of friendship and fan graphs with null graph respectively as in [On local antimagic vertex coloring of corona products related to friendship and fan graph, Indon. J. Combin., 5(2) (2021) 110–121]. Consequently, we obtained a sharp lower bound that gives the exact local antimagic chromatic number of the corona product of friendship and null graph.

1 Introduction

Let $G = (V, E)$ be a finite, undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $|V(G)| = p$ and $|E(G)| = q$ respectively. The friendship graph $f_n$ ($n \geq 2$) is a graph which consists of $n$ triangles with a common vertex. The fan graph $F_n$ ($n \geq 2$) is obtained by joining a new vertex to every vertex of a path $P_n$. The corona product of two graphs $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ along with $|V(G)|$ copies of $H$, and join the $i$-th vertex of $G$ to every vertex of the $i$-th copy of $H$, where $1 \leq i \leq |V(G)|$. For integers $a < b$, let $[a,b] = \{a, a+1, \ldots, b\}$. For graph-theoretic terminology, we refer to Chartrand and Lesniak [4].

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Key words and phrases: Local antimagic chromatic number, fan graph, friendship graph.

AMS (MOS) Subject Classifications: 05C78,05C15.
Hartsfield and Ringel [7] introduced the concept of antimagic labeling of a graph. For a graph $G$, let $f : E(G) \to \{1, 2, \ldots, q\}$ be a bijection. For each vertex $u \in V(G)$, the weight $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to $u$. If $w(u) \neq w(v)$ for any two distinct vertices $u$ and $v \in V(G)$, then $f$ is called an antimagic labeling of $G$. Hartsfield and Ringel conjectured that every connected graph with at least three vertices admits antimagic labeling [7]. Interested readers can refer to [5, 6].

Arumugam et al. in [1], and independently, Bensmail et al. in [3], posed a new definition as a relaxation of the notion of antimagic labeling. They called a bijection $f : E \to \{1, 2, \ldots, |E|\}$ a local antimagic labeling of $G$ if for any two adjacent vertices $u$ and $v$ in $V(G)$, the condition $w(u) \neq w(v)$ holds. Based on this notion, Arumugam et al. then introduced a new graph coloring parameter. Let $f$ be a local antimagic labeling of a connected graph $G$. The assignment of $w(u)$ to $u$ for each vertex $u \in V(G)$ induces naturally a proper vertex coloring of $G$ which is called a local antimagic vertex coloring of $G$. The local antimagic chromatic number, denoted $\chi_{la}(G)$, is the minimum number of colors taken over all local antimagic colorings of $G$ [1].

Arumugam et al. [2] obtained the local antimagic chromatic number for the graph $G \circ O_m$, where $G$ is a path, cycle or complete graph and $O_m$ is the null graph of order $m \geq 1$.

**Theorem 1.1** (Arumugam et al. [2]). Let $m \geq 2$, then

$$\chi_{la}(C_3 \circ O_m) = 3m + 3,$$

except $\chi_{la}(C_3 \circ O_1) = 5$.

**Theorem 1.2** (Arumugam et al. [2]). For $n \geq 2$, $\chi_{la}(K_n \circ K_1) = 2n - 1$.

In [8], the authors studied $\chi_{la}(f_n \circ O_m)$ and $\chi_{la}(F_n \circ O_m)$ for $n \geq 2$ and $m \geq 1$. We note that there are inconsistencies in the notations of $f_n$ and $F_n$ used. They proved that $\chi_{la}(f_n \circ O_m) \leq m(2n + 1) + 3$ and $\chi_{la}(F_n) \leq m(n + 1) + 3$ by providing a correct local antimagic labeling respectively. However, there are gaps in proving that $\chi_{la}(f_n \circ O_m) \geq m(2n + 1) + 3$ and $\chi_{la}(F_n) \geq m(n + 1) + 3$. Motivated by this, we shall first provide correct arguments to the proofs of the lower bounds. Consequently, we showed that $\chi_{la}(f_n \circ O_m) = m(2n + 1) + 2$ for $n \geq 2, m = 1$. Interested readers may refer to [9–12] for local antimagic chromatic number of graphs with pendant edges.
2 Lower bounds of $\chi_{la}(f_n \circ O_m)$ and $\chi_{la}(F_n \circ O_m)$

Lemma 2.1. For $n \geq 2, m \geq 1$, $\chi_{la}(f_n \circ O_m) \geq m(2n+1) + 3$ except $\chi_{la}(f_n \circ O_1) \geq m(2n+1) + 2$.

Proof. Let $G = f_n \circ O_m$ with $V(G) = \{x, u_i, v_i, x_j, u_j^i, v_j^i | 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(G) = \{xx_j, xu_i, xv_i, u_i v_i, u_i u_j^i, v_i v_j^i | 1 \leq i \leq n, 1 \leq j \leq m\}$. Clearly, $|E(G)| = q = m(2n+1) + 3n$.

Suppose $f : E(G) \to [1, q]$ is a local antimagic labeling of $G$. Clearly, all the $m(2n+1)$ pendant vertices must have distinct induced vertex colors that are at most $q$. Moreover, $w(x) \geq 1+2+\cdots+(2n+m) = (2n+m)(2n+m+1)/2 = s$. Now, $2s - 2q = (2n + m + 1)^2 + (2n + m + 1) - 6n - 2m(2n + 1) = 4n^2 + m^2 + m + 1 > 0$. Thus, $w(x) > q$. Therefore, $\chi_{la}(G) \geq m(2n+1) + 1$.

Without loss of generality, we consider the following 3 cases.

Case 1. $f(u_1 v_1) = q$. In this case, $w(u_1) \neq w(v_1) \neq w(x) > q$ so that $\chi_{la}(G) \geq m(2n+1) + 3$.

Case 2. $f(x u_1) = q$ or $f(u_1 u_j^1) = q$. In this case, $w(u_1) \neq w(x) > q$ so that $\chi_{la}(G) \geq m(2n+1) + 2$. Suppose equality holds. Clearly, for each $i \in [1, n]$, at most one of $u_i, v_i$ has induced vertex color $q$. So, there are at most $n$ vertices in $\{u_i, v_i\}$ with induced vertex color $q$. The sum of these $n$ induced vertex colors is at least $1 + 2 + \cdots + n(m+1) = n(m+2)[n(m+2)+1]$ and at most $nq = n[3n+m(n+1)]$. Since $n \geq 2$, it is easy to check that $n(m+2)[n(m+2)+1] - n[3n + m(n+1)] = 2n^2(m+1) + n + \frac{1}{2}mn(mn+1) - [3n^2 + mn(2n+1)] > 0$ if and only if $m > 1$. Consequently, $\chi_{la}(G) \geq m(n+1) + 2$ if $m = 1$, and $\chi_{la}(G) \geq m(n+1) + 3$ if $m \geq 2$.

Case 3. $f(xx_1) = q$. In this case, $w(x_1) = q$ and $w(u_j^i), w(v_j^i), w(x_j) < q$ ($x_j \neq x_1$) so that $\chi_{la}(G) \geq m(2n+1) + 1$. Suppose $w(v_i) < w(u_i) \leq q$ for $1 \leq i \leq n$, then $\sum_{i=1}^{n}[w(u_i) + w(v_i)]$ is at most $n(2q-1)$ and at least $1 + 2 + \cdots + n(2m+3) = n(2m+3)[n(2m+3)+1]/2$. Now,

\[
\begin{align*}
n(2m+3)[n(2m+3)+1] - 2n(2q-1) & = n(2m+3)[n(2m+3)+1] - 2n[2m(2n+1) + 6n - 1] \\
& = 4m^2n^2 + 4mn^2 - 2mn - 3n^2 + 5n > 0.
\end{align*}
\]
Thus, we may assume $w(u_1) > q$. Since $w(u_1) \neq w(x)$, we have $\chi_{la}(G) \geq m(2n + 1) + 2$. Suppose equality holds. By an argument similar to that in Case 2, we have $\chi_{la}(G) \geq m(2n+1)+2$ if $m = 1$ and $\chi_{la}(G) \geq m(2n+1)+3$ if $m \geq 2$.

Note that $F_2 \circ O_m = C_3 \circ O_m$, we next consider $F_n \circ O_m, n \geq 3, m \geq 1$.

**Lemma 2.2.** For $n \geq 3, m \geq 1$, $\chi_{la}(F_n \circ O_m) \geq m(n + 1) + 3$.

**Proof.** Let $G = F_n \circ O_m$ with $V(G) = \{x, x_j, v_i, v^i_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(G) = \{xx_j, xv_i, v_i v^i_j | 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. Clearly, $|E(G)| = m(n + 1) + 2n - 1 = q$.

Let $f$ be a local antimagic labeling of $G$ that induces $\chi_{la}(G)$ distinct vertex colors. Clearly, all the $m(n+1)$ pendant vertices must have distinct induced vertex colors that are at most $m$ colors. Clearly, all the $f$ vertex colors that are at most $m$ colors. Clearly, all the $f$ vertex colors that are at most $m$ colors. Clearly, all the $f$ vertex colors that are at most $m$ colors.

**Case 1.** $f(v_1 v_2) = q$ or $f(v_2 v_3) = q$ if $n \geq 4$. In this case, $w(x) \neq w(v_1) \neq w(v_2) > q$. Thus, $\chi_{la}(G) \geq m(n + 1) + 3$.

**Case 2.** $f(xv_1) = q$ (or $f(xv_2) = q$). In this case, $w(x) \neq w(v_1) > q$ (or $w(x) \neq w(v_2) > q$). Thus, $\chi_{la}(G) \geq m(n + 1) + 2$. Suppose equality holds. Note that if $w(v_i) > q$ for $3 \leq i \leq n$, then $w(v_i) = w(v_1)$. Moreover, $w(v_i) \neq w(v_{i+1})$ for $1 \leq i \leq n-1$. Suppose there are $r \geq 1$ vertices in $\{v_i | 1 \leq i \leq n\}$ with induced vertex color larger than $q$, then there are $n-r \geq 1$ vertices in $\{v_i | 1 \leq i \leq n\}$ with induced vertex color at most $q$. These $n-r$ vertices are incident to a total of $(m+2)n-1-r(m+1) = (m+1)(n-r) + n + 1$ edges. Therefore, their edge labels sum under $f$ is at most $(n-r)q$. However, the sum is at least $S = 1 + 2 + \cdots + [(m+1)(n-r) + n - 1] = \frac{1}{2}[(m+1)(n-r) + n - 1][(m+1)(n-r) + n]$. Note that $n-r \geq n/2$. Thus, $-r \geq -n/2$ and $2S - 2(n-r)q \geq \frac{n}{2}\left[\frac{m^2 n}{2} + \frac{n}{2} - 3\right] > 0$ except for $n = 3, m = 1$. This contradicts $S \leq (n-r)q$ for all $(n,m) \neq (3,1)$.
The second inequality is obtained as follows:

\[ 2S - 2(n - r)q = [(m + 1)(n - r) + n]^2 - [(m + 1)(n - r) + n] \]
\[ - 2(n - r)[m(n + 1) + 2n] \]
\[ = (m + 1)^2(n - r)^2 + (2n - 1)(m + 1)(n - r) \]
\[ + n^2 - n - 2(n - r)(mn + m + 2n) \]
\[ = (n - r)[m^2(n - r) + 2m(n - r) - 3m - n - r - 1] + n^2 - n \]
\[ \geq (n - r)[m^2(n - r) + 2m(n - r) - 3n - \frac{3n}{2} - 1] + n^2 - n \]
\[ \geq \frac{n}{2} \left[ \left( \frac{m^2 + 2m}{2} \right) - 3m - \frac{3n}{2} - 1 \right] + n^2 - n \]
\[ \geq \frac{n}{2} \left[ \frac{m^2n}{2} - \frac{3n}{2} - 1 + 2n - 2 \right] \]
\[ = \frac{n}{2} \left[ \frac{m^2n}{2} + \frac{n}{2} - 3 \right] > 0 \text{ except when } (n, m) = (3, 1) \]

Now, consider \( G = F_3 \circ O_1 \) that has \( q = 9 \). If \( G \) admits a local antimagic labeling that induces 6 distinct vertex colors, then \( w(v_1) = w(v_3) \leq 9 \). Since \( v_1 \) and \( v_3 \) are incident to 6 different edges, their total label sum is at least 21 so that \( w(v_1) = w(v_3) \geq 11 \), a contradiction. Therefore, \( \chi_{la}(G) \geq m(n + 1) + 3 \).

**Case 3.** \( f(v_1v_2^1) = q \) (or \( f(v_2v_2^2) = q \)). In this case, \( w(v_1) \neq w(x) > q \) (or \( w(v_2) \neq w(x) > q \)). Thus, \( \chi_{la}(G) \geq m(n + 1) + 2 \). Suppose equality holds. By an argument similar to Case 2, we have the same contradiction. \( \square \)

### 3 \( \chi_{la}(f_n \circ O_1) \)

In [8], the authors obtained local antimagic labelings that correctly show that \( \chi_{la}(f_n \circ O_m) \leq m(2n + 1) + 3 \) and \( \chi_{la}(F_n \circ O_m) \leq m(n + 1) + 3 \). By Lemma 2.1, we shall next show that \( \chi_{la}(f_n \circ O_1) = 2n + 3 \).

**Theorem 3.1.** For \( n \geq 2 \), \( \chi_{la}(f_n \circ O_1) = 2n + 3 \).

**Proof.** Let \( G = f_n \circ O_1 \) with \( V(G) \) and \( E(G) \) as defined in the proof of Lemma 2.1. Suffice to define a bijection \( f : E(G) \to [1, 5n + 1] \) that induces \( 2n + 3 \) distinct induced vertex colors. We shall use labeling matrices to describe the labeling of all the edges of \( f_n \circ O_1 \).
Suppose $n$ is odd. We first define $f(x_1) = 5n+1$. We now arrange integers in $[2n+1, 5n]$ as a $3 \times n$ matrix as follows:

1. In row 1, assign $4n + (i + 1)/2$ to column $i$ if $i = 1, 3, 5, \ldots, n$; assign $(9n + 1)/2 + i/2$ if $i = 2, 4, 6, \ldots, n - 1$. We have used integers in $[4n + 1, 5n]$.

2. In row 2, assign $(7n + 1)/2 + (i - 1)/2$ to column $i$ if $i = 1, 3, 5, \ldots, n$; assign $3n + i/2$ if $i = 2, 4, 6, \ldots, n - 1$. We have used integers in $[3n + 1, 4n]$.

3. In row 3, assign $3n + 1 - i$ to column $1 \leq i \leq n$. We have used integers in $[2n+1, 3n]$.

The resulting matrix is given in Table 1.

Table 1: Assignment of integers in $[2n + 2, 5n + 1]$

<table>
<thead>
<tr>
<th>4n + 1</th>
<th>$\frac{9n+3}{2}$</th>
<th>4n + 2</th>
<th>$\frac{9n+5}{2}$</th>
<th>$\cdots$</th>
<th>5n - 1</th>
<th>$\frac{9n-1}{2}$</th>
<th>5n</th>
<th>$\frac{9n+1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{7n+1}{2}$</td>
<td>3n + 1</td>
<td>$\frac{7n+3}{2}$</td>
<td>3n + 2</td>
<td>$\cdots$</td>
<td>$\frac{7n-3}{2}$</td>
<td>4n - 1</td>
<td>$\frac{7n-1}{2}$</td>
<td>4n</td>
</tr>
<tr>
<td>3n</td>
<td>3n - 1</td>
<td>3n - 2</td>
<td>3n - 3</td>
<td>$\cdots$</td>
<td>2n + 4</td>
<td>2n + 3</td>
<td>2n + 2</td>
<td>2n + 1</td>
</tr>
</tbody>
</table>

We next arrange integers in $[1, 3n]$ as a $3 \times n$ matrix as follows:

1. In row 1, assign $3n + 1 - i$ to column $1 \leq i \leq n$. We have used integers in $[2n + 1, 3n]$.

2. In row 2, assign $(3n + 1)/2 + (i - 1)/2$ to column $i$ if $i = 1, 3, 5, \ldots, n$; assign $n + i/2$ to column $i$ if $i = 2, 4, 6, \ldots, n - 1$. We have used integers in $[n + 1, 2n]$.

3. In row 3, assign $(i + 1)/2$ to column $i$ if $i = 1, 3, 5, \ldots, n$; assign $(n + 1)/2 + i/2$ to column $i$ if $i = 2, 4, 6, \ldots, n - 1$. We have used integers in $[1, n]$.

The resulting matrix is given in Table 2.

Table 2: Assignment of integers in $[2n + 2, 5n + 1]$

<table>
<thead>
<tr>
<th>3n</th>
<th>3n - 1</th>
<th>3n - 2</th>
<th>3n - 3</th>
<th>$\cdots$</th>
<th>2n + 4</th>
<th>2n + 3</th>
<th>2n + 2</th>
<th>2n + 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3n+1}{2}$</td>
<td>n + 1</td>
<td>$\frac{3n+3}{2}$</td>
<td>n + 2</td>
<td>$\cdots$</td>
<td>$\frac{3n-3}{2}$</td>
<td>2n - 1</td>
<td>$\frac{3n-1}{2}$</td>
<td>2n</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{n+3}{2}$</td>
<td>2</td>
<td>$\frac{n+5}{2}$</td>
<td>$\cdots$</td>
<td>n - 1</td>
<td>$\frac{n-1}{2}$</td>
<td>n</td>
<td>$\frac{n+1}{2}$</td>
</tr>
</tbody>
</table>
For $1 \leq k \leq 3$, $1 \leq i \leq n$, let $a_{k,i}$ be the $(k,i)$-entry of Table 1, and $b_{k,i}$ be the $(k,i)$-entry of Table 2. Note that $b_{1,i} = a_{3,i}$. Define $f(u_iu_1^i) = a_{1,i}$, $f(xu_i) = a_{2,i}$, $f(u_iv_i) = a_{3,i}$, $f(xv_i) = b_{2,i}$ and $f(v_iv_1^i) = b_{3,i}$. It is obvious that $f$ is a bijective function.

Now, column sum of each column of Table 1 is $(21n + 3)/2$. Thus, $w(u_i) = (21n + 3)/2$ and $w(u_1^i) \in [4n + 1, 5n]$ for $1 \leq i \leq n$. Similarly, the column sum of each column of Table 2 is $(9n + 3)/2$. Thus, $w(v_i) = (9n + 3)/2$ and $w(v_1^i) \in [1, n]$ for $1 \leq i \leq n$. Moreover, $w(x) = (n + 1) + \cdots + (2n) + (3n + 1) + \cdots + 4n + (5n + 1) = (n + 1)(5n + 1)$. Clearly, $w(x) \neq w(u_i) \neq w(u_1^i) \neq w(x_1) = 5n + 1$ for $1 \leq i \leq n$. Note that $4n + 1 \leq w(v_i) = (9n + 3)/2 \leq 5n + 1$ is odd for $n \geq 3$. Therefore, $f$ is a local antimagic labeling that induces $2n + 3$ distinct vertex colors. Consequently, $\chi_{la}(f_n \circ O_1) = 2n + 3$ for odd $n \geq 3$.

We now consider even $n \geq 2$. Figures 1 and 2 show that $\chi_{la}(f_2 \circ O_1) = 7$ and $\chi_{la}(f_4 \circ O_1) = 11$.

Figure 1: $\chi_{la}(f_2 \circ O_1) = 7$ with induced vertex colors in $\{7, 5, 9, 10, 11, 20, 28\}$

Figure 2: $\chi_{la}(f_4 \circ O_1) = 11$ with induced vertex colors in $\{5, 6, 7, 9, 10, 16, 17, 18, 21, 46, 85\}$
Consider $n \geq 6$. We first define $f(xx_1) = 3n + 3$, $f(u_nv_n) = 1$, $f(u_nu_n^1) = 2n + 2$, $f(xu_n) = 2n$, $f(v_nv_1^n) = 2n + 3$ and $f(xx_n) = 2n + 1$. We now have $w(x_1) = 3n + 3$, $w(u_n) = 4n + 3$, $w(u_1^n) = 2n + 2$, $w(v_n) = 4n + 5$ and $w(v_1^n) = 2n + 3$. We now consider the remaining integers in $[2, 2n - 1] \cup [2n + 4, 3n + 2] \cup [3n + 4, 5n + 1]$. We now arrange integers in $[2n + 4, 3n + 2] \cup [3n + 4, 5n + 1]$ as a $3 \times (n-1)$ matrix as follows:

1. In row 1, assign $4n + 3 + (i - 1)/2$ to column $i$ if $i = 1, 3, 5, \ldots, n - 1$; assign $9n/2 + 2 + i/2$ if $i = 2, 4, 6, \ldots, n - 2$. We have used integers in $[4n + 3, 5n + 1]$.

2. In row 2, assign $7n/2 + 3 + (i - 1)/2$ to column $i$ if $i = 1, 3, 5, \ldots, n - 1$; assign $3n + 3 + i/2$ if $i = 2, 4, 6, \ldots, n - 2$. We have used integers in $[3n + 4, 4n + 2]$.

3. In row 3, assign $3n + 3 - i$ to column $1 \leq i \leq n - 1$. We have used integers in $[2n + 4, 3n + 2]$.

The resulting matrix is given in Table 3.

Table 3: Assignment of integers in $[2n + 4, 3n + 2] \cup [3n + 4, 5n + 1]$

<table>
<thead>
<tr>
<th>$4n + 3$</th>
<th>$\frac{9n}{2} + 3$</th>
<th>$4n + 4$</th>
<th>$\frac{9n}{2} + 4$</th>
<th>$\cdots$</th>
<th>$5n$</th>
<th>$\frac{9n}{2} + 1$</th>
<th>$5n + 1$</th>
<th>$\frac{9n}{2} + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{7n}{2} + 3$</td>
<td>$3n + 4$</td>
<td>$\frac{7n}{2} + 4$</td>
<td>$3n + 5$</td>
<td>$\cdots$</td>
<td>$\frac{7n}{2} + 1$</td>
<td>$4n + 1$</td>
<td>$\frac{7n}{2} + 2$</td>
<td>$4n + 2$</td>
</tr>
<tr>
<td>$3n + 2$</td>
<td>$3n + 1$</td>
<td>$3n$</td>
<td>$3n - 1$</td>
<td>$\cdots$</td>
<td>$2n + 7$</td>
<td>$2n + 6$</td>
<td>$2n + 5$</td>
<td>$2n + 4$</td>
</tr>
</tbody>
</table>

We next arrange integers in $[2, 2n - 1] \cup [2n + 4, 3n + 2]$ as a $3 \times n$ matrix as follows:

1. In row 1, assign $3n + 3 - i$ to column $1 \leq i \leq n - 1$. We have used integers in $[2n + 4, 3n + 2]$.

2. In row 2, assign $3n/2 + (i - 1)/2$ to column $i$ if $i = 1, 3, 5, \ldots, n - 1$; assign $n + i/2$ to column $i$ if $i = 2, 4, 6, \ldots, n - 2$. We have used integers in $[n + 1, 2n - 1]$.

3. In row 3, assign $(i + 3)/2$ to column $i$ if $i = 1, 3, 5, \ldots, n - 1$; assign $n/2 + 1 + i/2$ to column $i$ if $i = 2, 4, 6, \ldots, n - 2$. We have used integers in $[2, n]$.

The resulting matrix is given in Table 4.
Table 4: Assignment of integers in \([2, 2n - 1] \cup [2n + 4, 3n + 2]\)

<table>
<thead>
<tr>
<th>3n + 2</th>
<th>3n + 1</th>
<th>3n</th>
<th>3n - 1</th>
<th>⋯</th>
<th>2n + 7</th>
<th>2n + 6</th>
<th>2n + 5</th>
<th>2n + 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{3n}{2})</td>
<td>(n + 1)</td>
<td>(\frac{3n}{2} + 1)</td>
<td>(n + 2)</td>
<td>⋯</td>
<td>(\frac{3n}{2} - 2)</td>
<td>(2n - 2)</td>
<td>(\frac{3n}{2} - 1)</td>
<td>(2n - 1)</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{n}{2} + 2)</td>
<td>3</td>
<td>(\frac{n}{2} + 3)</td>
<td>⋯</td>
<td>(n - 1)</td>
<td>(\frac{n}{2})</td>
<td>(n)</td>
<td>(\frac{n}{2} + 1)</td>
</tr>
</tbody>
</table>

For \(1 \leq k \leq 3\), \(1 \leq i \leq n - 1\), let \(c_{k,i}\) be the \((k, i)\)-entry of Table 3, and \(d_{k,i}\) be the \((k, i)\)-entry of Table 4. Note that \(d_{1,i} = c_{3,i}\). Define \(f(xu_i) = c_{1,i}\), \(f(xv_i) = c_{3,i}\), \(f(xv_i) = d_{2,i}\) and \(f(v_i v_1^i) = d_{3,i}\). It is obvious that \(f\) is a bijective function.

Now, column sum of each column of Table 3 is \(21n/2 + 8\). Thus, \(w(u_i) = 21n/2 + 8\) and \(w(u_1^i) \in [4n + 3, 5n + 1]\) for \(1 \leq i \leq n - 1\). Similarly, the column sum of each column of Table 4 is \(9n/2 + 4\). Thus, \(w(v_i) = 9n/2 + 4\) and \(w(v_1) \in [2, n]\) for \(1 \leq i \leq n - 1\). Moreover, \(w(x) = [2n + (2n + 1) + (3n + 3)] + (3n + 4) + \cdots + (4n + 2) + (n + 1) + \cdots + (2n - 1) = (7n + 4) + (n - 1)(5n + 3) = 5n^2 + 5n + 1\). Clearly, for \(1 \leq i \leq n - 1\), \(w(x) \neq w(u_i^1) \neq w(u_i) \neq w(v_i^1) \neq w(u_1^1) \neq w(v_1) \neq w(x_1)\). Note that \(4n + 3 \leq w(u_n) = 4n + 3 \neq w(v_n) = 4n + 5 \leq 5n + 1\) for even \(n \geq 6\). Therefore, \(f\) is a local antimagic labeling that induces \(2n + 3\) distinct vertex colors. Consequently, \(\chi_{la}(f_n \circ O_1) = 2n + 3\) for even \(n \geq 6\).

**Example 3.1.** Figures 3 and 4 below give the labelings of \(f_3 \circ O_1\) and \(f_6 \circ O_1\) according to the proof in Theorem 3.1.

![Figure 3: \(\chi_{la}(f_3 \circ O_1) = 9\) with induced vertex colors in \([1, 3] \cup [13, 16] \cup \{33, 64\}\)](image-url)
Figure 4: $\chi_{la}(f_6 \circ O_1) = 15$ with induced vertex colors in $[1,6] \cup [27,31] \cup \{21,71,211\}$

References


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