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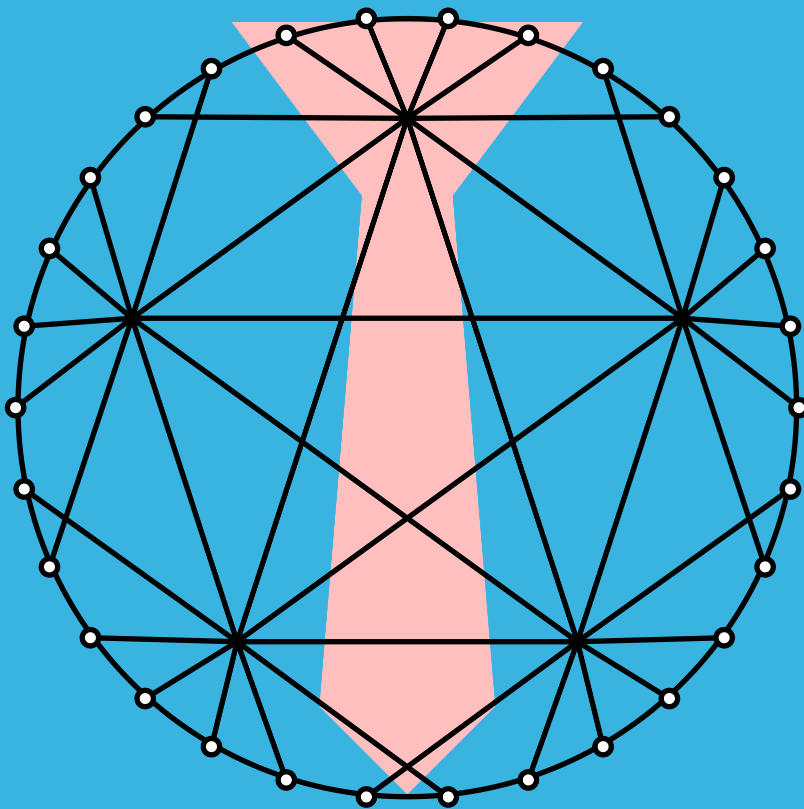

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Total, equitable and neighbor-sum distinguishing total colorings of some classes of circulant graphs

Shantharam Prajnanaswaroopa*, Jayabalan Geetha, Kanagasabapathi Somasundaram

Abstract. Total Coloring of a graph is a variety of proper colorings in which no two adjacent vertices, edges incident on the same vertex, or an edge and its incident vertices receive the same color. The total chromatic number is the minimum number of colors required in any total coloring of a graph. The neighbor-sum distinguishing and equitable total chromatic numbers are generalizations of the total chromatic number. The computation of all three numbers is shown to be NP-hard. The circulant graphs are regular graphs with varying applications within and outside graph theory. They are the easiest examples of regular graphs that come to mind. They can be seen as the Cayley graphs on cyclic groups. In this paper, we have obtained better bounds for the total chromatic and equitable and neighbor-sum distinguishing total chromatic numbers of some classes of the circulant graphs.

1 Introduction

Let $G = (V(G), E(G))$ be a graph with the sets of vertices $V(G)$ and edges $E(G)$, respectively. All the graphs considered here are finite, simple, connected, and undirected. The total coloring of a simple graph is the coloring of the vertices and the edges such that any adjacent vertices or edges and their incident vertices do not receive the same color. The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

*Corresponding author: s_prajnanaswaroopa@cb.amrita.edu
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The Total Coloring Conjecture (TCC) (Behzad [1] and Vizing [11] independently proposed) is a famous open problem in graph theory that states that any simple graph G with maximum degree $\Delta(G)$ can be totally colored with at most $\Delta(G) + 2$ colors. The progress on this conjecture had been low in the initial years, but several advancements are in progress [4]. The graphs that can be totally colored in $\Delta(G) + 1$ colors are said to be type I, and the graphs with the total chromatic number $\Delta(G) + 2$ are said to be type II.

Equitable total coloring is a total coloring of a graph such that any two color classes of the total coloring differ by at most one element. The minimum colors required in such a coloring of the graph is called the equitable total chromatic number, denoted by $\chi''_{=}(G)$. Fu [3] conjectured that every graph G has an equitable total k-coloring for each $k \geq \max\{\chi''(G), \Delta(G) + 2\}$. Later, Wang [12] proposed a weaker conjecture that $\chi''_{=}(G) \leq \Delta(G) + 2$.

A proper total k coloring of G is called the neighbor-sum distinguishing total coloring if $\sum_c(u) \neq \sum_c(v)$ for each edge uv, where $\sum_c(u)$ is the sum of the color of the vertex u and the colors of edges incident with u. The minimum number of colors required is called the neighbor-sum distinguishing total chromatic number, denoted by $\chi''_{\Sigma}(G)$. Pilśniak and Woźniak [8] conjectured that $\Delta(G) + 2 \leq \chi''_{\Sigma}(G) \leq \Delta(G) + 3$ for any graph G.

In an edge coloring of a graph, a rainbow matching is a set of independent edges such that each edge in the independent set has a different color. For example, we can color the edges of any even cycle C_{2n} such that it has a rainbow perfect matching (an n-coloring of the edges with each perfect matching receiving n colors is a rainbow coloring).

A Cayley graph, $\mathrm{Cay}(G, S)$, for a group H with symmetric generating set $S \subset H$ is a simple graph with vertices as all the elements of the group; and edges between every two elements of the form g and gs, where $g \in H, s \in S$. Note that a symmetric subset implies $s \in S \implies s^{-1} \in S$. The set S is also supposed not to have the identity element of the group. A circulant graph is a Cayley graph on the group $H = \mathbb{Z}_n$, the cyclic group on n vertices. A power of cycle C_n^k is a circulant graph on $\mathbb{Z}_n$ with generating set $\{1, 2, \ldots, k, n-k, \ldots, n-2, n-1\}$.

In other words, given a sequence of positive integers $1 \leq d_1 < d_2 < ... < d_l \leq \lfloor \frac{n}{2} \rfloor$, the circulant graph $G = C_n(d_1, d_2, ..., d_l)$ has the vertex set $V = Z_n = \{0, 1, 2, ..., n-1\}$, two vertices x and y being adjacent if and only if $x = (y \pm d_i) \mod n$ for some $i, 1 \leq i \leq l$ and a graph is a power of cycle, denoted C_n^k, n and k are integers, $1 \leq k < \lfloor \frac{n}{2} \rfloor$, if $V(C_n^k) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(C_n^k) = E^1 \cup E^2 \cup ... \cup E^k$, where $E^i = \{e_0^i, e_1^i, ..., e_{n-1}^i\}$ and $e_j^i = (v_j, v_{(j+i) \mod n})$, $0 \leq j \leq n-1$ and $1 \leq i \leq k$.

Campos and de Mello [2] verified the TCC for the power of cycle C_n^k, if n is even and $2 < k < \frac{n}{2}$. Also, they showed that one could obtain a $\Delta(C_n^k) + 2$-total coloring for these graphs in polynomial time. They also proved that C_n^k with $n \cong 0 \mod (\Delta(C_n^k)+1)$ are type I, and they proposed the following conjecture.

Conjecture 1.1. Let $G = C_n^k$, with $2 \leq k < \lfloor \frac{n}{2} \rfloor$. Then,

$$\chi''(G) = \begin{cases} \Delta(G) + 2, & \text{if } k > \frac{n}{3} - 1 \text{ and } n \text{ is odd} \\ \Delta(G) + 1, & \text{otherwise.} \end{cases}$$

Geetha et al. and Prajnanaswaroopa et al. ([5] and [9]) proved Campos and de Mello's conjecture for some classes of powers of cycles. Also, in [5], they verify the TCC for the complement of powers of cycles.

A Latin square is an $n \times n$ array consisting of n entries of numbers (or symbols), with each row and column containing only one instance of each element. This means the rows and columns are permutations of one single n vector with different entries. A Latin square is said to be commutative if it is symmetric. A Latin square containing numbers is considered idempotent if each diagonal element contains a number equal to its row (column) number. In addition, if the rows of the Latin square are just cyclic permutations (one shift of the elements to the right) of the previous row, then the Latin square is said to be circulant (anti-circulant, if the cyclic permutations are left shifts), the matrix (corresponds to the Latin square) can be generated from a single row vector. The Latin square

1	k+2	2	k+3	...	2k+1	k+1
k+2	2	k+3	3	...	k+1	1
...	...	...	...	...	...	...
...	...	...	...	...	...	...
k+1	1	k+2	2	...	k	2k+1

is anti-circulant, commutative and idempotent. The entries of the square are as follows:

$$L = (l_{ij}) = \begin{cases} m, & \text{if } i + j = 2m \\ k + 1 + m, & \text{if } i + j = 2m + 1. \end{cases}$$

From the above, it can be easily seen that the Latin square corresponding to the matrix L is commutative, idempotent, and also anti-circulant. The following lemma is due to Stong [10] (Corollary 2.3.1).

Lemma 1.1. *If G is an even order abelian group with generating set S, then the graph $Cay(G, S)$ is 1-factorizable.*

2 Results on powers of cycles

The following theorem is a slight generalization of the one found in Theorem 2.6 of ([9]). We state it here, along with the proof of the previous version, for better comprehension.

Theorem 2.1. *Let $G = C_n^k$ be a power of cycle graph and $(k+i)|n$ for some $1 \leq i \leq k+1$. Then G satisfies TCC. In particular, G is type I if n is even.*

Proof. Case 1. n is even.

Let $(k+i) = 2m+1$. Then the given condition $n \equiv 0 \bmod (k+i)$, implies $n \equiv 0 \bmod (2m+1)$. We know that there exists a commutative idempotent Latin square of order $2m+1$ that we denote by C'.

Let e_{ij} denotes the $(ij)^{th}$ entry in C'. We define an upper triangular Tableau D of order $2m-k$ by

$$D = (d_{ij}) = \begin{cases} e_{ij}, \quad \text{if } \ i = 1, 2, ..., 2m-k, \\ j = i+k+1, i+k+2, ..., 2m+1 \\ \text{empty, otherwise.} \end{cases}$$

Tableau D is shown by Table 1.

$e_{1,k+2}$	$e_{1,k+3}$	$\cdots$	$\cdots$	$e_{1,2m+1}$
	$e_{2,k+3}$	$e_{2,k+4}$	$\cdots$	$e_{2,2m+1}$
		$\cdots$	$\cdots$	$e_{3,2m+1}$
			$\cdots$	$e_{4,2m+1}$
				$e_{2m-k,2m+1}$

Table 1: Tableau D.

Now, we construct three new Tableaux A, B, and C of orders k, k, and $2m+1$, respectively, using some portions of C', D, and additional colors. The Tableau A is upper triangular whose entries in the main diagonal are $2k+1$, and the subsequent sub-diagonals are filled respectively with $2k, 2k-1$ down to $2m+2$. Similarly, Tableau B is lower triangular with main diagonal entries being $2m+2$, and subsequent sub-diagonals increase values up to $2k+1$. The constructions of A, B and C based on m, $\frac{k}{2} \leq m \leq k$, are as follows:

(i) If $m = \frac{k}{2}$, then A and B are shown in 2. In this case, Tableau $C = C'$ and Tableau D are empty.

$2m+2$				
$2m+3$	$2m+2$			
$2m+4$	$2m+3$	$2m+2$		
$\dots$	$\dots$	$\dots$	$2m+2$	
$2k+1$	$\dots$	$\dots$	$\dots$	$2m+2$

$2k+1$	$2k$	$2k-1$	$\dots$	$2m+2$
	$2k+1$	$2k$	$\dots$	$2m+3$
		$2k+1$	$2k$	$\dots$
			$2k+1$	$2k$
				$2k+1$

Table 2: Left-Tableau B and Right-Tableau A when $m = \frac{k}{2}$.

(ii) If $\frac{k}{2} < m < k$, there will be $2m-k$ unfilled upper sub-diagonals in Tableau A. We fill these unfilled upper sub-diagonals of A using the entries from D as shown in Table 3. Similarly, there will be $2m-k$ unfilled lower sub-diagonals in Tableau B, and these sub-diagonals are filled using D^T. The Tableau C is obtained by deleting D and D^T from C'.

$2k+1$	$2k$	$\dots$	$2m+2$	$e_{1,k+2}$	$\dots$	$e_{1,2m}$	$e_{1,2m+1}$
	$2k+1$	$2k$	$\dots$	$2m+2$	$\dots$		$e_{2,2m+1}$
		$2k+1$	$2k$	$\dots$	$2m+2$		$\vdots$
			$2k+1$	$2k$	$\dots$	$2m+2$	$e_{k+2,2m+1}$
				$2k+1$	$2k$		$2m+2$
					$2k+1$	$2k$	$\vdots$
						$2k+1$	$2k$
							$2k+1$

Table 3: Tableau A in the case $\frac{k}{2} < m < k$.

(iii) If $m = k$, then $A = D$ and $B = D^T$. Again the Tableau C is obtained by deleting D and D^T from C'.

The three Tableaux are arranged as given below to form the $n \times n$ total color matrix.

We place l copies of Tableau C (of order $2m+1$) along the main diagonal as depicted in Table 4, where $n = l(k+i)$. Each cell represents a sub-matrix of order $2m+1$. In the cell $(i, i+1)$, $0 < i < l$, one copy of B (i odd) or A^T (i even) is placed bottom-left justified. Similarly, in the cell $(i+1, i)$ one copy of B^T (i odd) or A (i even) is placed top-right justified. One copy each of A and A^T are respectively placed at cells $(1, l)$ and $(l, 1)$.

In the arrangement shown, the rows of Tableau B start from the $(2m-k+2)$-th row and $(2m+2)$-th column. This ensures that the Tableaux B and A are differently placed so that none of the numbers in the color matrix coincide. Similarly, the last row of the last B^T is at the $(n-2m+k-2)$-th row. This ensures that the colors in the last B^T and A^T do not coincide. From the above observations, the entries in rows (columns) of Tableau A placed at cell $(1, l)$ and Tableau B at $(1, 2)$ (at $(l-1, l)$), respectively do not coincide. By symmetry, the colors in B^T at $(2, 1)$ (at $(l, l-1)$) and A^T at $(l, 1)$ also do not coincide. Hence, we obtain a proper total coloring of C_n^k, a type-I total coloring.

C	B				A
B^T	C	A^T			
	A	C	B		
		B^T	C	A^T	
			A	C	B
A^T				B^T	C

Table 4: Color matrix structure.

The edges of C_n^k are a disjoint union of C_n^m and the edges of the circulant graph on $\mathbb{Z}_n$ with generating set $\{m+1, m+2, \dots, k, n-k, \dots, n-m-2, n-m-1\}$. By Lemma 1.1 and the conditions given on the set $\{m+1, m+2, \dots, k\}$, the circulant graph is 1-factorizable. This ensures that the remaining edges of the graph after removing C_n^m will be edge colorable exactly with $2(k-m)$ colors used in Tableaux A and B and have no clashes in the colors. Thus, the above procedure gives us a type-I total coloring of C_n^k.

Case 2. n is odd.

We first give a $(k+i)$-partial total coloring of the graph that corresponds to the total coloring of the subgraph $C_n^{\lfloor \frac{k+i}{2} \rfloor}$ by using the similar concept of the above case. That is, we first totally color the graph induced by the first $k+i$ vertices as in a commutative idempotent Latin square of order $k+i$. We repeat the same pattern for the remaining vertices, taking $k+i$ at a time (as in the above case) and then extend it to the complete total coloring by coloring the remaining edges, which can always be done in $2k-(k+i)+2 = k-i+2$ colors by using the Vizing's theorem. This is possible as the vertices are given a $k+i$ coloring, ensuring the edge colors do not clash with the already given vertex colors. Hence, the total number of colors used is $2k+2$, which verifies TCC for such graphs. $\square$

Example 2.1. For example, we see that the graph C_{21}^6 can be totally colored by first partially giving the part of the graph C_{21}^3 a type I total coloring and then extending it to the whole graph by giving the remaining edges a class 2 edge coloring, which is guaranteed by Vizing's theorem. The first part of the color matrix can be seen in Table 5. The color matrix is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	5	2	6															3	7	4
1	5	2	6	3	7															4	1
2	2	6	3	7	4	1															5
3	6	3	7	4	1	5	2														
4		7	4	1	5	2	6	3													
5			1	5	2	6	3	7	4												
6				2	6	3	7	4	1	5											
7					3	7	4	1	5	2	6										
8						4	1	5	2	6	3	7									
9							5	2	6	3	7	4	1								
10								6	3	7	4	1	5	2							
11									7	4	1	5	2	6	3						
12										1	5	2	6	3	7	4					
13											2	6	3	7	4	1	5				
14												3	7	4	1	5	2	6			
15													4	1	5	2	6	3	7		
16														5	2	6	3	7	4	1	
17															6	3	7	4	1	5	2
18	3															7	4	1	5	2	6
19	7	4															1	5	2	6	3
20	4	1	5															2	6	3	7

Table 5: Partial total coloring of C_{21}^6.

then extended by giving a suitable 7 edge coloring of the remaining edges (which are the edges of the Cayley graph on $\mathbb{Z}_{21}$ with generating set as $\{4, 5, 6, 15, 16, 17\}$), which is guaranteed by Vizing's theorem. Hence, the total colors required for this graph is $7+7=14$.

There are several classes of graphs that satisfy the condition of Theorem 2.1. For example, if n is highly composite, we have several classes of such graphs.

The canonical total coloring of a complete graph of odd order n is coloring the vertices and edges of the complete graph in a particular fashion, which we describe here. First, we observe that the complete graph can be written as the circulant graph with generating set consisting of all elements of the cyclic group of order n except 0, that is $\{1, 2, \ldots, n-1\}$. Now, the total color matrix is filled as follows:

(i) The vertices are colored $1 - -2 - \ldots - n$, corresponding to filling the diagonal in the pattern $1 - -2 - \ldots - n$.

(ii) For the first row of the color matrix, the colors corresponding to the generator s_i is $\frac{s_i}{2} + 1$ if s_i is even and $\lceil \frac{n}{2} \rceil + \lceil \frac{s_i}{2} \rceil$ if s_i is odd.

(iii) The remaining rows of the total color matrix are filled by the circulant pattern, that is, rotating the first row to the left by one color cyclically.

Note that the above canonical coloring implies that if we have the color x in the first row of the total color matrix at position s_i, we have the color $x - s_i \pmod{n}$ at position $n - s_i$ in the first row, with the number 0 corresponding to the color n.

The canonical total coloring of a complete graph of even order n is just derived from the canonical total coloring of the complete graph of odd order by deleting the last row and column of the total color matrix of the complete graph of order $n-1$, constructed as described above. Tables 6 and 7 show the canonical total coloring of the complete graphs K_9 and K_8, respectively.

	0	1	2	3	4	5	6	7	8
0	1	6	2	7	3	8	4	9	5
1	6	2	7	3	8	4	9	5	1
2	2	7	3	8	4	9	5	1	6
3	7	3	8	4	9	5	1	6	2
4	3	8	4	9	5	1	6	2	7
5	8	4	9	5	1	6	2	7	3
6	4	9	5	1	6	2	7	3	8
7	9	5	1	6	2	7	3	8	4
8	5	1	6	2	7	3	8	4	9

Table 6: Canonical total coloring of K_9.

	0	1	2	3	4	5	6	7
0	1	6	2	7	3	8	4	9
1	6	2	7	3	8	4	9	5
2	2	7	3	8	4	9	5	1
3	7	3	8	4	9	5	1	6
4	3	8	4	9	5	1	6	2
5	8	4	9	5	1	6	2	7
6	4	9	5	1	6	2	7	3
7	9	5	1	6	2	7	3	8

Table 7: Canonical total coloring of K_8.

Theorem 2.2. *The equitable total chromatic number of power of cycle graph, $G = C_n^k$ with $(2k+1)|n$, n is even, is equal to its total chromatic number. Further, $\chi''_{\Sigma}(G) \leq 2k+3 = \Delta(G)+3$.*

Proof. Let $G = C_n^k$ and $n = 2m(2k+1)$. From case 1 of the previous theorem, the graph C_n^k is type I, and it is easy to see that the equitable total chromatic number of C_n^k is equal to its total chromatic number (we remove the vertices in each color class, the remaining graphs have a perfect matching).

Secondly, we see that the edges of the hamiltonian cycle formed by the generator 1 in the totally colored graph have two rainbow-perfect matchings. We then give two new colors to the two rainbow perfect matchings, which will then give us a different sum of the colors incident at each vertex for each adjacent pair of vertices (as the row/column sum for any consecutive $2k+1$ rows/ columns will be different in the total color matrix). Hence, the number of colors used in such a coloring is $2k+1+2 = 2k+3$. However, it may be possible to reduce further the number of total colors required by 1, a possibility that we cannot rule out. Hence, the neighbor-sum distinguishing total chromatic number, $\chi''_{\Sigma}(G) \leq 2k+3$. □

Example 2.2. Let us consider the total coloring of the graph C_{18}^4. In this case, we have $2 \cdot 4 + 1 = 9|18$, and the graph is type I. We could write its total color matrix as in Table 8.

We modify the two rainbow perfect matchings formed by the generating elements $\{1, 17\}$ with two new colors as explained in the theorem to give a coloring such that the neighbor-sum distinguishing total chromatic number

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	6	2	7	3										8	4	9	5
1	6	2	7	3	8	4										9	5	1
2	2	7	3	8	4	9	5										1	6
3	7	3	8	4	9	5	1	6										2
4	3	8	4	9	5	1	6	2	7									
5		4	9	5	1	6	2	7	3	8								
6			5	1	6	2	7	3	8	4	9							
7				6	2	7	3	8	4	9	5	1						
8					7	3	8	4	9	5	1	6	2					
9						8	4	9	5	1	6	2	7	3				
10							9	5	1	6	2	7	3	8	4			
11								1	6	2	7	3	8	4	9	5		
12									2	7	3	8	4	9	5	1	6	
13										3	8	4	9	5	1	6	2	7
14	8										4	9	5	1	6	2	7	3
15	4	9										5	1	6	2	7	3	8
16	9	5	1										6	2	7	3	8	4
17	5	1	6	2										7	3	8	4	9

Table 8: Equitable total coloring of C_{18}^4.

is 11 from the total chromatic number 9. Hence, Table 8 could be modified to Table 9.

3 Results on circulant graphs

Very few results on TCC for circulant graphs have been known. Some results on the same can be found in [5], [6], and [7].

In this section, we obtain an upper bound for the total chromatic numbers (equitable and neighbor-sum distinguishing total chromatic numbers in a few cases) for certain classes of circulant graphs.

Theorem 3.1. *Let G be a circulant graph with even order n, $\Delta(G) \geq \frac{n}{2}$, and the generating set S satisfying the following properties:*

(i) *It does not have the involution element $\frac{n}{2}$.*

(ii) *The set S has a subset S_1 of the form $\{1, 2, 3, \ldots, \frac{n}{4}, n - \frac{n}{4}, \ldots, n-1\}$ (which is a generating set of the power of cycle $C_n^{\frac{n}{4}}$).*

(iii) *The complement of the subset S_1 in S generates the whole group.*

Then G satisfies TCC.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	10	2	7	3										8	4	9	11
1	10	2	11	3	8	4										9	5	1
2	2	11	3	10	4	9	5										1	6
3	7	3	10	4	11	5	1	6										2
4	3	8	4	11	5	10	6	2	7									
5		4	9	5	10	6	11	7	3	8								
6			5	1	6	11	7	10	8	4	9							
7				6	2	7	10	8	11	9	5	1						
8					7	3	8	11	9	10	1	6	2					
9						8	4	9	10	1	11	2	7	3				
10							9	5	1	11	2	10	3	8	4			
11								1	6	2	10	3	11	4	9	5		
12									2	7	3	11	4	10	5	1	6	
13										3	8	4	10	5	11	6	2	7
14	8										4	9	5	11	6	10	7	3
15	4	9										5	1	6	10	7	11	8
16	9	5	1										6	2	7	11	8	10
17	11	1	6	2										7	3	8	10	9

Table 9: Neighbor-sum distinguishing total coloring of C_{18}^4.

Proof. We bifurcate the total color matrix into two parts; the first one consists of the partial total coloring of the graph induced by the generating set $\{1, 2, 3, \ldots, \frac{n}{4}, \ldots, n-1\}$. The second part consists of edge coloring of the remaining edges. The bifurcation of the graph is done using the generating set S_1, the graph corresponding to which is a power of a cycle. The second graph is induced by $\overline{S_1}$.

Here, the first part of the total color matrix requires only $\frac{n}{2} + 2$ colors by using the result of Campos and de Mello [2]. From Lemma 1.1, the graph induced by the remaining edges forms a Cayley graph and is 1-factorizable. Therefore, we can color the remaining edges with $\Delta(G) - \frac{n}{2}$ colors. This gives us the complete total coloring of the original circulant graph using $\Delta(G) - \frac{n}{2} + \frac{n}{2} + 2 = \Delta(G) + 2$ colors, thus satisfying TCC. $\square$

Example 3.1. Let us consider the total coloring of the Cayley graph on $\mathbb{Z}_{20}$ with generating set $\{1, 2, 3, 4, 5, 7, 8, 12, 13, 15, 16, 17, 18, 19\}$. Here, we have a 14-regular graph. The first part consists of the graph corresponding to the generating subset $\{1, 2, 3, 4, 5, 15, 16, 17, 18, 19\}$. We can follow the process given in [2] to give a total 12 coloring of the subgraph induced by this subset. Since the subset's complement is $\{7, 8, 12, 13\}$ generates the whole group, we could color the remaining edges with 4 extra colors. This implies that we could give a total coloring of G in $12 + 4 = 16$ colors which are equal to $\Delta(G) + 2$ colors, or in other words, G satisfies TCC.

Theorem 3.2. *If a circulant graph G of even order n has the generating set $S = \{s_i\}$ with $|S| = \frac{n}{2} - 2$ such that none of $s_i, -s_j$ are congruent modulo $\frac{n}{2}$, $s_i, s_j \in S$. Then $\chi''(G) \leq \frac{n}{2} + 1 = \Delta(G) + 3$.*

Proof. We begin by noticing that we can partition the vertices into $\frac{n}{2}$ independent sets as $\{0, \frac{n}{2}\}, \{1, \frac{n}{2}+1\}, \ldots, \{\frac{n}{2}-1, n-1\}$ as $\frac{n}{2} \notin S$. We observe that the induced subgraphs formed by the vertices $\{0, 1, \ldots, \frac{n}{2} - 1\}$ and $\{\frac{n}{2}, \ldots, n-1\}$ are isomorphic. Now, we totally color the induced subgraphs using the total coloring of a complete graph of order $\frac{n}{2}$ (we give a similar total coloring to the edges between two vertices of the subgraph as in the complete graph). In order to color the edges joining the two subgraphs, we observe that the edges missed from the complete graph while coloring each subgraph is later covered in the joining edges between the subgraphs. This is so possible because if we have an edge between two vertices, say g_i and g_j. We will not have an edge between g_j and $g_i + \frac{n}{2} \pmod{\frac{n}{2}}$ or vice-versa; as this would imply that both $g_j - g_i \pmod{\frac{n}{2}}$ and $g_i + \frac{n}{2} - g_j = \frac{n}{2} - (g_j - g_i) \pmod{\frac{n}{2}}$ belong to the generating set, which is in contradiction to the assumption that we have none of $s_i \not\equiv -s_j \pmod{\frac{n}{2}}$. Therefore, we could give the color of the edge between g_i and g_j, which is missed from the complete graph, to the edge between g_i and $g_j + \frac{n}{2} \pmod{\frac{n}{2}}$ in G. Hence, we could color G using the same colors as the complete graph. Since we require $\frac{n}{2}+1$ colors to totally color a complete graph of order $\frac{n}{2}$, we are done. □

Example 3.2. Consider the circulant graph on $\mathbb{Z}_{24}$ with generating set $\{1, 3, 4, 5, 10, 14, 19, 20, 21, 23\}$. We see that it satisfies the conditions of the theorem and hence can be totally colored with 13 colors. The total color matrix, in this case, can be written as in Table 10.

Theorem 3.3. *If a circulant graph G of even order n has a generating set S such that the following properties hold:*

(i) *It has a subset M of S such that $|M| = \frac{n}{2} - 2$ and $m_i \not\equiv -m_j \pmod{\frac{n}{2}}$, $m_i, m_j \in M$.*

(ii) *The complement of M in S generates the whole group.*

Then, the graph G satisfies $\chi''(G) \leq \Delta(G) + 3$.

Proof. The previous theorem shows that the total chromatic number of the subgraph induced by the generating set M is bounded above by $\frac{n}{2}+1$. Now, the complement of M in S generates the whole group. From Lemma 1.1, the graph induced by the remaining edges forms a Cayley graph, and it is 1-factorizable. Therefore, we can color the remaining edges with $\Delta(G) -$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	1	8		9	3	10					6				2					11	5	12		7
1	8	2	9		10	4	11					7				3					12	6	13	
2		9	3	10		11	5	12					2				4					13	7	1
3	9		10	4	11		12	6	13					3				5					1	8
4	3	10		11	5	12		13	7	1					4				6					2
5	10	4	11		12	6	13		1	8	2					5				7				
6		11	5	12		13	7	1		2	9	3					6				8			
7			12	6	13		1	8	2		3	10	11					7				9		
8				13	7	1		2	9	3		4	5	12					8				10	
9					1	8	2		3	10	4		12	6	13					9				11
10	6					2	9	3		4	11	5		13	7	1					10			
11		7					3	10	4		5	12	13		1	8	2					11		
12			2					11	5	12		13	1	8		9	3	10					6	
13				3					12	6	13		8	2	9		10	4	11					7
14	2				4					13	7	1		9	3	10		11	5	12				
15		3				5					1	8	9		10	4	11		12	6	13			
16			4				6					2	3	10		11	5	12		13	7	1		
17				5				7					10	4	11		12	6	13		1	8	2	
18					6				8					11	5	12		13	7	1		2	9	3
19	11					7				9					12	6	13		1	8	2		3	10
20	5	12					8				10					13	7	1		2	9	3		4
21	12	6	13					9				11					1	8	2		3	10	4	
22		13	7	1					10				6					2	9	3		4	11	5
23	7		1	8	2					11				7					3	10	4		5	12

Table 10: Total coloring of the circulant graph in Example 3.2.

$(\frac{n}{2}-2) = \Delta(G) - \frac{n}{2} + 2$ colors. Thus, the number of colors required for the total coloring of G is at most $\frac{n}{2} + 1 + \Delta(G) - \frac{n}{2} + 2 = \Delta(G) + 3$ colors. $\square$

Example 3.3. Consider the circulant graph on $\mathbb{Z}_{24}$ with generating set

$$\{1, 2, 3, 4, 5, 7, 10, 14, 17, 19, 20, 21, 22, 23\}.$$

We see that this graph satisfies the conditions of the theorem. Here we have $M = \{1, 3, 4, 5, 10, 14, 19, 20, 21, 23\}$. Theorem 2.2 shows that the graph induced by the set M requires 13 colors. The complement of M in S, $\{2, 7, 17, 22\}$ has cardinality of 4, whence the extra colors required are just 4. Hence, the graph can be totally colored with 17 colors.

Theorem 3.4. *Let G be a circulant graph with $n = 2m$ with m odd and $\Delta(G) > m$. If the generating sets S and $S_1 \subset S$, $|S_1| = m - 1$, have the following properties:*

(i) $s_i \not\equiv -s_j \pmod{m}$ *where* $s_i, s_j \in S_1$.

(ii) S_1 *has a group generator in it.*

(iii) *Complement of* S_1 *in* S *generates the whole group.*

Then $\chi''_{=}(G) = \Delta(G) + 1$ *and* $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$.

Proof. We first totally color the subgraph induced by the generating subset S_1. For this, we first color the vertices with the canonical colors

$1, 2, 3, \ldots, m$, which implies that we have two vertices in each color class. The diagonal of the total color matrix is filled in the pattern $1 - -2 - \ldots - (m-1) - m - 1 - -2 - \ldots - (m-1) - m$ (As the diagonals represent the colors corresponding to vertices, and we have given the vertices a m-coloring with the vertices i, $i+m$ getting the same color). This is possible as $m \notin S_1 (m \equiv -m \pmod{m})$. Now, to fill the sub-diagonals, we use the same pattern for the diagonals, with the starting colors being different. The starting colors (entries in the first row) are chosen as follows:

> For $s_i < m$, the entries corresponding to s_i in the first row are chosen by using the index of s_i ($s_i \pmod{m}$) in the canonical total coloring of a complete graph of order m. As for the sub-diagonals starting from $s_i > m$, we use the colors used for inverses of $s_i \pmod{m}$ in the canonical total coloring of the complete graph.

This ensures that there will be no clashes between the edge colors (as $s_i \not\equiv -s_j \pmod{m}$). This also ensures that there are no clashes between the vertex and edge colors because, if there are clashes between vertex and edge colors for $s_i > \frac{n}{2}$, there will be clashes at the colors between the vertices and sub-diagonals starting at $s_i < m$, which is not possible as $s_i \not\equiv s_j \pmod{m}$. Thus, we can give a type I total coloring for this subgraph using m colors. Now, the complement of S_1 in S generates the whole group, and by Lemma 1.1, the remaining edges of G are one factorizable. Therefore, G is type I.

Since, in the above total coloring, we have the equitable coloring of vertices and the graph is type I, each total color class having vertices has a perfect matching of the remaining graph (the graph with the two vertices removed). The total color classes having only edges have just one element less than the total color classes with vertices. As such, the graph has an equitable total coloring using $\Delta(G) + 1$ colors as desired. Thus, $\chi''_{=}(G) = \chi''(G) = \Delta(G) + 1$.

In order to prove that $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$, we first give a $\Delta(G) + 1$ total coloring of G as above and then take a Hamiltonian cycle, (since the generating set of the graph has a group generator, by property ii) which has two rainbow perfect matchings. We give two extra colors to its edges. This ensures that the sum of the colors of neighbors is different. Hence, $\chi''_{\Sigma}(G) \leq (\Delta(G) + 1) + 2 = \Delta(G) + 3$. □

Example 3.4. We take the graph G as the circulant graph on $\mathbb{Z}_{18}$ with generating set $S = \{1, 2, 4, 6, 7, 8, 10, 11, 12, 14, 16, 17\}$. Here, S_1 is $\{1, 2, 4, 6,$

$12, 14, 16, 17\}$. We first give a 9-total coloring of the subgraph induced by S_1. To give a total 9 coloring, we use the total color matrix of the canonical coloring of a complete graph of order 9. This gives us the starting colors for the sub-diagonals starting at $1, 2, 4, 6$ to be $6, 2, 3, 4$. Correspondingly, the sub-diagonals start at $12, 14, 16, 17$ to be $7, 8, 9, 5$. The part of the graph induced by $\{7, 8, 12, 13\}$, which is also a circulant graph, can be given a 4 coloring (from Corollary 2.3.1 of [10]). This gives us a 13 total coloring of the graph, which is also equitable. Now, we take the rainbow Hamiltonian cycle induced by the set $\{1, 17\}$ and give it a 2-coloring using two extra colors (replacing the original coloring), which is possible as the cycle induced is Hamiltonian (and hence even). This gives us a neighbor-sum distinguishing total coloring using 15 colors, as the colors replaced in each row and column has a different sum owing to the rainbow nature of the prior coloring. The total color matrix of the subgraph corresponding to S_1 is given in Table 11.

The final total color matrix can be obtained using the Lemma 1.1 and 4 extra colors to give us an equitable 13-coloring of the final graph.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	6	2		3		4						7		8		9	5
1	6	2	7	3		4		5						8		9		1
2	2	7	3	8	4		5		6						9		1	
3		3	8	4	9	5		6		7						1		2
4	3		4	9	5	1	6		7		8						2	
5		4		5	1	6	2	7		8		9						3
6	4		5		6	2	7	3	8		9		1					
7		5		6		7	3	8	4	9		1		2				
8			6		7		8	4	9	5	1		2		3			
9				7		8		9	5	1	6	2		3		4		
10					8		9		1	6	2	7	3		4		5	
11						9		1		2	7	3	8	4		5		6
12	7						1		2		3	8	4	9	5		6	
13		8						2		3		4	9	5	1	6		7
14	8		9						3		4		5	1	6	2	7	
15		9		1						4		5		6	2	7	3	8
16	9		1		2						5		6		7	3	8	4
17	5	1		2		3						6		7		8	4	9

Table 11: Partial cquitable cotal coloring of the circulant graph in Example 3.4.

To give a neighbor-sum distinguishing total coloring using 15 colors, we modify the prior total color matrix by replacing the colors of the Rainbow Hamiltonian cycle induced by the set $\{1, 17\}$ as in Table 12.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	14	2		3		4				6				8		9	15
1	14	2	15	3		4		5				7				9		1
2	2	15	3	14	4		5		6				8				1	
3		3	14	4	15	5		6		7				9				2
4	3		4	15	5	14	6		7		8				1			
5		4		5	14	6	15	7		8		9				2		
6	4		5		6	15	7	14	8		9		1				3	
7		5		6		7	14	8	15	9		1		2				4
8			6		7		8	15	9	14	1		2		3			
9				7		8		9	14	1	15	2		3		4		
10	6				8		9		1	15	2	14	3		4		5	
11		7				9		1		2	14	3	15	4		5		6
12			8				1		2		3	15	4	14	5		6	
13				9				2		3		4	14	5	15	6		7
14	8				1				3		4		5	15	6	14	7	
15		9				2				4		5		6	14	7	15	8
16	9		1				3				5		6		7	15	8	14
17	15	1		2				4				6		7		8	14	9

Table 12: Partial total coloring of the circulant craph in Example 3.4 that is neighbor-sum distinguishing.

The final total color matrix can be obtained using Lemma 1.1 and 4 extra colors to give us a total coloring with $\chi''_{\Sigma}(G) = 15$.

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S. Prajnanaswaroopa
Department of Mathematics, Amrita School of Physical Sciences,
Coimbatore Amrita Vishwa Vidyapeetham, India
sntrm4@rediffmail.com,s_prajnanaswaroopa@cb.amrita.edu

J. Geetha
Department of Mathematics, Amrita School of Physical Sciences,
Coimbatore Amrita Vishwa Vidyapeetham, India
j_geetha@cb.amrita.edu

K. Somasundaram
Department of Mathematics, Amrita School of Physical Sciences,
Coimbatore Amrita Vishwa Vidyapeetham, India
s_sundaram@cb.amrita.edu