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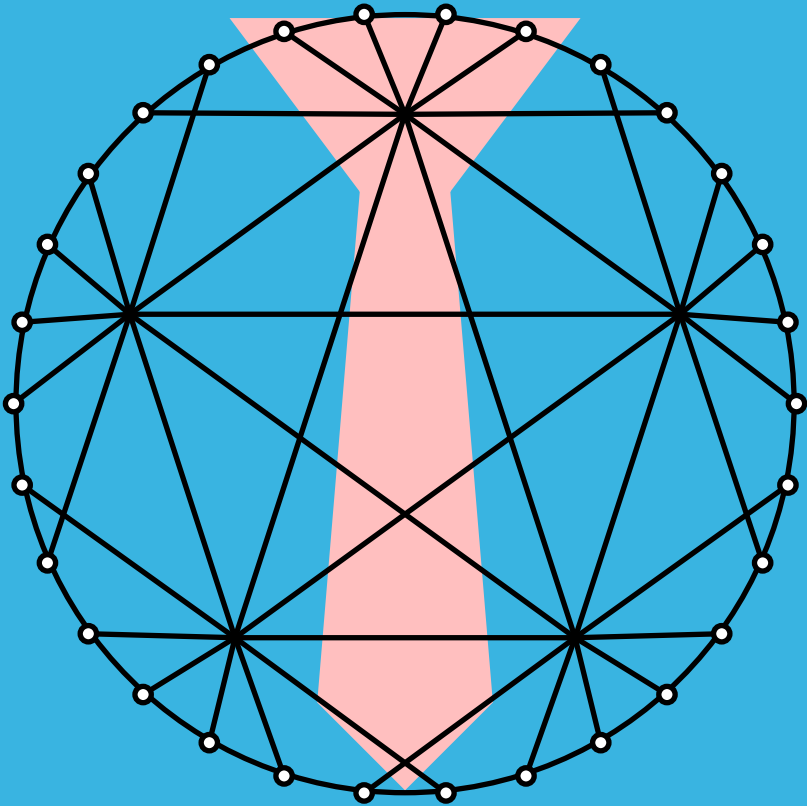
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Graph embeddings with no hamiltonian extensions

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Abstract. We show that extending an embedding of a graph Γ in a surface to an embedding of a hamiltonian supergraph can be blocked by certain planar subgraphs but, for some subdivisions of Γ, hamiltonian extensions must exist.

1 Introduction

The objects studied in this paper are 2-cell embeddings of graphs in (closed) surfaces. We ask: *When can such an embedding be extended to an embedding of a hamiltonian graph, containing the original graph as a subgraph?* The embedding is into the same surface so that the supergraph is obtained as a subdivision of some of the regions of the original embedding but the edges of the original graph are not subdivided. See Figure 1.
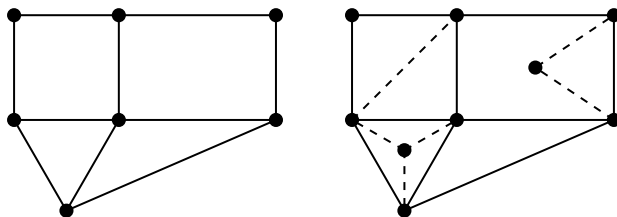

Figure 1: An extension of an embedding

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This problem is a variant of the differently specified question asked in [8], "*When is a graph, embeddable on a surface S, a subgraph of a hamiltonian graph which is also embeddable on S?*" McKenzie and Overbay showed [8] that the bipartite complete graphs, with genus $\gamma \leq 1$ which are *not* hamiltonian, are subgraphs of genus-γ graphs that *are* hamiltonian.

The formulation here emphasizes the embedding itself, rather than the possibility of being embedded. The idea of extending graph invariants to graph embeddings goes back (at least) to [3, 4, 5, 6, 9].

Merely being non-hamiltonian isn't enough to prevent a hamiltonian extension. For instance, the Petersen graph has an embedding in the torus, and one can add three edges to the embedding to make the enlarged graph hamiltonian where each added edge occurs within a region of the original embedding. *Which embeddings ensure that no such hamiltonian extension can be found?*

We obtain a large family of non-hamiltonian-extendable embeddings using an idea of Klee (see Malkevitch [7]) and conjecture that there are no other such non-hamiltonian-extendable embeddings. However, if the edges of the original graph can be subdivided before trying to extend it, then we show that every graph embedding has such a *topological* hamiltonian extension.

The paper proceeds as follows: Section 2 has definitions; in Section 3 we build non-hamiltonian-extendable graph embeddings. Section 4 proves that weakening the condition of extendability to allow subdivision of edges of the original graph makes it possible to always find a Hamiltonian extension.

2   Definitions

A *2-cell embedding* i of a finite graph Γ in a surface S is a continuous embedding $i : \Gamma \to S$ such that $S \setminus i(\Gamma)$ is a disjoint union of open 2-disks, the *regions* (of i). If G is some graph which contains Γ as a subgraph and $j : G \to S$ is a 2-cell embedding, then we say that j *extends* i if $j|_\Gamma = i$.

We call a 2-cell embedding i of Γ in S *hamiltonian extendable* if i can be extended to an embedding of a Hamiltonian supergraph G in S. Otherwise, i is *non-Hamiltonian-extendable*.

A path or cycle is *oriented* if its edges are assigned a consistent direction. If P is an oriented path, let P^o denote P *minus its terminal point*.

A subdivision of an edge is a path whose endpoints agree with the endpoints of the edge. A *subdivsion* of a graph is a graph obtained by subdividing some or all of the edges. Two graphs are homeomorphic iff they have isomorphic subdivisions.

An embedding $i : \Gamma \to S$ will be said to have a *topological extension* if there exists a subdivision Γ' of Γ and an extension $j : G \to S$ of i', where i' is the embedding $\Gamma' \to S$ induced by i.

A 2-cell embedding $i : \Gamma \to S$ is of *Klee type* if the number r of regions exceeds the number p of vertices; i is of *local* Klee type if there exists a cycle C contained in Γ such that (i) C separates Γ, (ii) $i(C)$ separates S (into *inside* and *outside*), and (iii) if r_C is the number of regions of $i(\Gamma)$ inside C and p_C is the number of vertices of Γ inside or on C, then $r_C \geq p_C$. See Figure 2. Labeling inside/outside is arbitrary and both parts of $S \setminus C$ could be nonplanar.

3 Graph embeddings of Klee type

Extending a 2-cell embedding of Klee or local Klee type to include points in the interiors of too many regions must produce a non-hamiltonian-extendable graph. We conjecture that these obstacles are the only way to produce such non-hamiltonian-extendable graphs.

Theorem 1.

(a) *Let $i : \Gamma \to S$ be an embedding of Klee type with $r > p$. Then, for any extension $j : G \to S$, G is not hamiltonian provided G contains vertices $w_1, \ldots, w_s$ inside distinct regions of i, $R_1, \ldots, R_s$, for $r \geq s \geq p + 1$.*

(b) *Let $i : \Gamma \to S$ be an embedding of local Klee type with $r_C \geq p_C$. Then, for any extension $j : G \to S$, G is not hamiltonian provided G contains vertices $w_1, \ldots, w_s$ inside distinct regions of i, $R_1, \ldots, R_s$, inside $i(C)$ for $r_C \geq s \geq p_C$.*

Proof. We argue by contradiction. Suppose there is an extension $j : G \to S$ of i and let Z be any oriented cycle contained in G which includes all s points. By construction, between any two consecutive (with respect to Z) points, say, w_k, w_{k+1} ($k = 1, \ldots, s$, addition mod s), there is a unique

vertex v_k in the boundary of the region R_k of i containing w_k such that v_k is in Z and the subpath P_k of Z from w_k to v_k contains no other point in $W := \{w_1, \dots, w_s\}$ and no other point in the boundary of R_k.

In case (a), Z contains at least s points in $V\Gamma$, which contradicts the assumption $s \geq p + 1$. In case (b), if $r_C > p_C$, then as in (a), no such cycle Z can exist, while if $r_C = p_C$, the only possibility is that Z includes all vertices on C (and some inside it), so Z can't include the vertices of Γ outside C. $\qquad\square$

Using the genus formula [10, 2] for cubes, $\gamma(Q_d) = 1 + (d - 4)2^{d-3}$, easy calculation shows that for the d-cube, the number of regions in the genus embedding is $r := r(d) := d\,2^{d-2} > 2^d = p$ for $d \geq 5$. Indeed, by Euler's formula,
$$2^d - d2^{d-1} + r(d) = 2 - 2(d - 4)2^{d-3} - 2.$$

Solving for $r(d)$ gives the result. So cubes of dimension ≥ 5 are of Klee type. Using the construction in case (a) above, one obtains for the 5-cube, by adding one new vertex w in the middle of s of the square faces, $33 \leq s \leq 40$, and using any of the 11 ways to connect each w_k to ≥ 2 of the 4 vertices on the boundary of the face which contains it, the number of distinct 2-connected non-hamiltonian graphs with embedding in S_5 extending that of the 5-cube is
$$N = \sum_{k=33}^{40} \binom{40}{k} 11^k \approx 1.45 \times 10^{43}.$$

The *stellation* of a triangular region puts one new vertex into the interior and joins it to all three corners. Iterating this operation on the resulting three triangles gives a local Klee type graph embedding with $C = K_3$, where $r_C = 9$ and $p_C = 7$. Hence, stellating all 9 of the regions produces a non-hamiltonian-embeddable graph, no matter where it occurs in some potentially large graph embedding. Here the inside region is what was inside the triangle. See Figure 2.
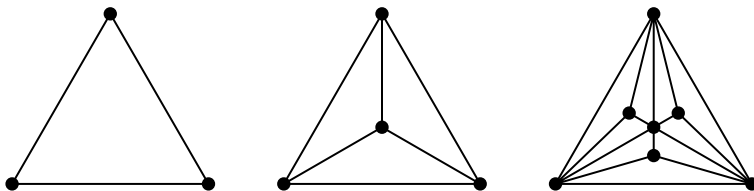

Figure 2: Local Klee type embedding on right; triangle C on left

4   Topological extensions

The planar case of Theorem 2 below is (implicitly) in [11, p. 32].

Theorem 2. *Any embedding $\Gamma \subset S$ has a hamiltonian topological extension.*

Proof. Let $i : \Gamma \to S$ be an embedding. Consider the $p = |V\Gamma|$ points $i(v) \in S$ for $v \in V\Gamma$. As S is a closed surface, it cannot be disconnected by the removal of any path (or any other contractible subset). Hence, for any enumeration of the points $i(v)$, say $i(v_1), \ldots, i(v_p)$, there is a topological path P_1 in S from v_1 to v_2, then a path P_2 in $S \setminus P_1^o$ from v_2 to v_3, and so on, until one chooses a path P_p in $S \setminus \bigcup_{k=1}^{p-1} P_k^o$ from $i(v_p)$ to $i(v_1)$. The union of the paths $P_1, \ldots, P_p$ is a non-self-intersecting closed curve $\mathcal{C}$ with $i(V\Gamma) \subset \mathcal{C} \subset S$. Since S is triangulable, there is an arbitrarily small perturbation $\mathcal{C}'$ of $\mathcal{C} \setminus i(V\Gamma)$ so that

$$\mathcal{C}' \cap i(\Gamma) = i(V\Gamma) \cup Y,$$

where Y is a finite set of points at which $\mathcal{C}'$ crosses interiors of edges of Γ.

Take the points in Y as subdivision vertices for the edges of Γ, and let Γ' be the resulting subdivision of Γ. Define a graph G as the union of Γ' and the new edges which result by subdividing $\mathcal{C}'$ using both the vertices of Γ' and the subdivision points. The resulting copy of G in S extends the embedding of Γ' and G has the subdivided $\mathcal{C}'$ as hamiltonian cycle. $\square$

We ask: What is the least number of subdivision points needed?

An alternate means to find a hamiltonian embedding extending a subdivision of some given embedding might be achievable using the "mesh surface" methods in Akleman et al. [1].

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