BULLETIN OF THE ICA
Volume **102** (2024), 71–92

On some extensions of mutually orthogonal graph squares

R. El-Shanawany* and M. E. Abdel-Aal

Abstract. A decomposition $\mathcal{G}$ =$\{G_0, G_1, \ldots, G_{n-1}\}$ of a graph $K_{n,n}$ is a partition of the edge set of $K_{n,n}$ into edge disjoint subgraphs $G_0, \ldots, G_{n-1}$ (called *pages*) in which all G_i, $i \in \{0, 1, \ldots, n-1\}$ are isomorphic to a specific graph G, and $\mathcal{G}$ is called a decomposition of $K_{n,n}$ by G. A family of decompositions $\{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{k-1}\}$ of a complete bipartite graph $K_{n,n}$ is a collection of k mutually orthogonal graph squares (MOGS) if $\mathcal{G}_i$ and $\mathcal{G}_j$ are orthogonal for all $i, j \in \{0, 1, \ldots, k-1\}$ and $i \neq j$. For any subgraph G of $K_{n,n}$ with n edges, $N(n, G)$ represents the greatest number k in the largest feasible set $\{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{k-1}\}$ of MOGS of $K_{n,n}$ by G. In this paper, we present several novel results pertaining to mutually orthogonal graph squares of the complete bipartite graph. Our focus lies in exploring starter functions of MOGS, as well as utilizing the technique of Kronecker product of MOGS to construct new mutually orthogonal sets of disjoint union stars.

	Nomenclature
$L(x, y)$	Entry in row x and column y of the square matrix L
mG	m disjoint copies of G
$G \cup H$	Disjoint union of G and H

1 Introduction

An edge decomposition $\mathcal{G} = \{G_0, G_1, \ldots, G_{n-1}\}$ of a graph $K_{n,n}$ is a partition of the edge set of $K_{n,n}$ into edge disjoint subgraphs $G_0, G_1, \ldots, G_{n-1}$ (called *pages*) in which all G_i, $i \in \{0, 1, \ldots, n-1\}$ are isomorphic to a particular graph G and $\mathcal{G}$ is called a decomposition of $K_{n,n}$ by G. For the

*Corresponding author: ramadan_elshanawany380@yahoo.com

Key words and phrases: Latin squares; Mutually orthogonal graph squares; Edge decomposition
AMS (MOS) Subject Classifications: 05C70, 05B30A

Received: 17 July 2023
Accepted: 11 March 2024

collection $\mathcal{G}$, we have $|E(G_i) \cap E(G_j)| = 0$ for all $i, j \in \{0, 1, \ldots, n-1\}$, $i \neq j$ and $\bigcup_{i=0}^{n-1} E(G_i) = E(K_{n,n})$. Two edge decompositions $\mathcal{G}$ and $\mathcal{F}$ of $K_{n,n}$ by G are orthogonal when $|E(G_i)| = |E(F_j)| = n$ and $|E(G_i) \cap E(F_j)| = 1$ for all $i, j \in \{0, 1, \ldots, n-1\}$. A family of decompositions $\{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{k-1}\}$ of a complete bipartite graph $K_{n,n}$ by G is considered a set of k mutually orthogonal edge decompositions if $\mathcal{G}_i$ and $\mathcal{G}_j$ are orthogonal for all $i, j \in \{0, 1, \ldots, k-1\}$ and $i \neq j$.

Decompositions of complete bipartite graphs have a variety of uses in experiment design as well as graph code generation. The authors of [13] offered authentication codes and the authors of [6], [7] used MOGS to construct graph orthogonal arrays and build a large number of authentication codes where there is a large number of graphs that can be used for decompositions of complete bipartite graph. Hence, if an opponent knows the used code generated by a certain graph, then we can use another graph for the construction of an authentication code.

A Latin square of side n is $n \times n$ matrix with entries from a set A with n different elements, where each element of A appears once in every row and every column. A collection of k-orthogonal Latin squares of order n constitutes a group of k Latin squares each pair of which is orthogonal. It is common practice to represent $N(n) = \max\{k : \exists\ k\text{-MOLS}\}$. A Latin square of side n is identical to an edge decomposition of $K_{n,n}$ by $nK_2 \simeq nK_{1,1}$. Two edge decompositions $\mathcal{G}$ and $\mathcal{F}$of $K_{n,n}$ by nK_2 are orthogonal if and only if the corresponding Latin squares of side n are orthogonal thus $N(n, nK_2) = N(n)$. The computation of $N(n)$ is one of the most complicated problems in combinatorial designs see the survey articles by Colbourn and Dinitz in [3], [4]. Many studies have been published about the mutually orthogonal Latin squares (MOLS) problem see [2, 5, 16] and [17].

For any subgraph G of $K_{n,n}$ with n edges, $N(n, G)$ denotes the maximum number k in a largest possible set $\{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{k-1}\}$ of mutually orthogonal graph squares (MOGS) of $K_{n,n}$ by G. Mutually orthogonal graph squares have deep links to core topics such as finite algebra, encryption, finite geometry and the design of experiments. El-Shanawany proved the relationship $N(n, G) \leq n$ for $n \geq 2$ in [8]. He also conjectured that if n is a prime number, then $N(n, p_{n+1}) = n$. This supposition has been proven using two approaches, see [9], [15]. MacNeish [14] has proved that if $N(m, mK_2) = k_1$and $N(n, nK_2) = k_2$ and $\min\{k_1, k_2\} = k$, then there are k-MOLS of order mn. El-Shanawany has shown in [10] that if

$N(m, mK_2) = k$ and $N(n, G) = k$, then $N(mn, mG) \geq k$. Numerous authors investigated (MOGS) of $K_{n,n}$ by G, where $G \neq nK_2$ (see the survey articles [9], [10], [12] and [15]). To describe the many constructions for sets of MOGS, we must first address basic concepts for graph squares (see also [8], [9]). The formal definitions of a G-square over additive group $\mathbb{Z}_m$ are given below.

Definition 1.1 (El-Shanawany, [10])**.** Let G be a subgrap h of $K_{m,m}$ with m edges. A square matrix L of order m is called a *G-square* if every element in $\mathbb{Z}_m$ occurs precisely m times and the graphs G_i, $i \in \mathbb{Z}_m$ with $E(G_i) = \{(x, y) : L(x, y) = i, \, x, y \in \mathbb{Z}_m\}$ are isomorphic to the graph G.

For an edge decomposition $\mathcal{G}_i$ we may associate bijectively an $m \times m$-square with entries belonging to $\mathbb{Z}_m$ indicated by $L_i = L_i(x, y)$, $0 \leq i \leq k-1$, $x, y \in \mathbb{Z}_m$ with

$$L_i(x, y) = \gamma \Leftrightarrow (x, y) \in E(G_{i\gamma}), \; \gamma \in \mathbb{Z}_m. \tag{1}$$

In a similar way to Definition 1.1, we define:

Definition 1.2. Let i, j be distinct positive integers. Two square matrices L_i and L_j of order m are said to be *orthogonal* if for any ordered pair (a, b), there is precisely one position (x, y) for $L_i(x, y) = a$, and $L_j(x, y) = b$. That is, the two graph squares have the property that, when superimposed, every ordered pair appears precisely once.

Now we mention some constructions that we can use.

Definition 1.3 (El-Shanawany, [8])**.** Let F be a subgraph of $K_{m,m}$ with m edges and G be a subgraph of $K_{n,n}$ with n edges. Then the *composite graph* $F[G]$ is a subgraph of $K_{mn,mn}$ with mn edges defined by

$$E(F[G]) = \{(x, y) : (x, y) = (na+c, nb+d), \, (a, b) \in E(F), \, (c, d) \in E(G)\}.$$

Note that acording of Defination 1.1 each $F[G]$-square represents edge decomposition of $K_{mn,mn}$ by $F[G]$.

El-Shanawany *et al.*, presented the following result in [11].

Proposition 1.4 (El-Shanawany and El-Mesady, [11])**.** *If there are k-MOGS of order m of the graph G and k-MOGS of order n of the graph H, then there are k-MOGS of order mn of the graph $H[G]$.*

Following that, if $N(m, G_1) = k_1$ and $N(n, G_2) = k_2$ and $\min\{k_1, k_2\} = k$, we can generate k-MOGS by Proposition 1.4 of order mn, where $k < mn$. Hence, Proposition 1.4 is a generalization to MacNeish and El-Shanawany's previous results. In this study, we are concerned with a branch of combinatorial design theory that deals with mutually orthogonal graph squares MOGS. We compute several generalisations of the well-known $N(n, G) = k \geq 3$, where G represents disjoint union of smaller subgraphs of $K_{n,n}$ each being a star. In addition, we provide some results based on the Kronecker product in Proposition 1.4.

All graphs in this study are finite, simple and undirected, the elements of $\mathbb{Z}_q \times \mathbb{Z}_2$ utilised to label the vertices of $K_{q,q}$ where $(v, j) \in \mathbb{Z}_q \times \mathbb{Z}_2$ will be written v_j which refers to the corresponding vertex and the edge $\{x_i, y_j\} \in E(K_{q,q})$ if and only if $x, y \in \mathbb{Z}_q$ and $i, j \in \mathbb{Z}_2$ such that $i \neq j$. If there is no possibility of ambiguity, we shall write (x, y) instead of $\{x_0, y_1\}$ for the edge between the vertices x_0 and y_1 refer to Figure 1.1. The results in [15],[10] and [9] motivated us to consider MOGS for the disjoint union of certain complete bipartite graphs. In [9] El-Shanawany provides the formal basic definitions of subgraph of $K_{n,n}$ produced by a function over $\mathbb{Z}_n$ as the following.

Definition 1.5 (El-Shanawany, [9])**.** Let G be a subgraph of $K_{n,n}$ and $f : \mathbb{Z}_n \to \mathbb{Z}_n$. Then G is called an *f-starter* if

$$E(G) = \bigcup_{x \in \mathbb{Z}_n} \big(x, f(x)\big),$$

and is denoted by G_f.

Definition 1.6 (El-Shanawany, [9])**.** Assume G_f is an f-starter graph and let $\eta \in \mathbb{Z}_n$. Then the graph

$$G_f + \eta = \big\{(x, f(x) + \eta) : (x, f(x)) \in E(G_f)\big\}$$

is called the *(η, f)-translate* of G_f.

In [9] the following theorem has been proven by El-Shanawany.

Theorem 1.7 (El-Shanawany, [9])**.** *The union of all translates of G_f yields an edge decomposition of $K_{n,n}$. (i.e. $\bigcup_{\eta \in \mathbb{Z}_n} E(G_f + \eta) = E(K_{n,n})$).*

The following example demonstrates a direct application of Definition 1.1 and Equation (1).

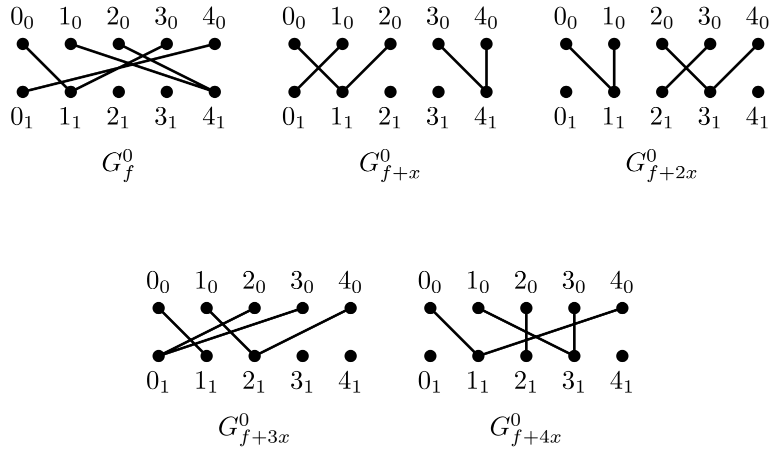

Figure 1.1: Mutually orthogonal graphs $2K_{1,2} \cup K_2$ with respect to $\mathbb{Z}_5$.

Example 1.8. The subgraph $G \simeq 2K_{1,2} \cup K_2$ of $K_{5,5}$ is an f-starter graph G_f induced by the function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ defined by $f(x) = x^2 + (2 + i)x + 1$, for all $i, x \in \mathbb{Z}_5$ as illustrated in Figure 1.1. Applying Definition 1.1 with $n = 5$ and for all $0 \leq s \leq 4$ there exist five mutually orthogonal decompositions of $K_{5,5}$ by $2K_{1,2} \cup K_2$ and we have $L_s(x, y) = y - f(x) - sx$, $x, y \in \mathbb{Z}_5$ and $0 \leq s \leq 4$. That means there are 5-MOGS of $K_{5,5}$ as follows:

$$L_0 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Note that every figure in Figure 1.1 and its translations represents edge decompositions of $K_{5,5}$ by $(2K_{1,2} \cup K_2)$. That is equivalent L_i squares $i = 0, 1, 2, 3, 4$, and in each square every row contains one x, where $x \in \mathbb{Z}_5$, and there are exactly two columns have two x-entry (in other words, 2 vertices x_1 have degree two) and one column has one x-entry (that is, one vertex x_1 has degree one) and two columns have no x-entry (2 vertices x_1 have degree zero) and all vertices x_0 have degree one.

The remainder of this paper is divided as follows: Section 2 is dedicated to construct MOGS based on starters function. Section 3 introduced numerous additional large MOGS constructions depending on the Kronecker product in Proposition 1.4. The fourth section is dedicated to the conclusion and future work.

2 Main results

We employ the starter function technique in this section to offer some novel direct constructions for $N(n, G) = k \geq 3$, where G represents a disjoint union of certain complete bipartite graphs. That is, in the following constructions we obtain more flexibility results than in construction of El-shanawany *et al.*, [12].

Let q be an odd prime number and $f : \mathbb{Z}_q \to \mathbb{Z}_q$ then G_f is a starter function of $K_{q,q}$ with q edges and $N(n, G_f)$ denotes the maximum number k in a largest possible set $\{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{k-1}\}$ of MOGS of $K_{q,q}$ by G_f. For all $x, y \in \mathbb{Z}_q$, let $L_s(x, y) = j$, where $y = f(x) + sx + j$ and $x, j \in \mathbb{Z}_q$ this implies that $j = y - f(x) - sx$ and we have

$$L_s(x, y) = y - f(x) - sx, \tag{2}$$

for all $0 \leq s \leq k - 1$. That is, there are k-MOGS of $K_{q,q}$. Our goal is to demonstrate the orthogonality of graph squares. It is simply demonstrated that for all different $0 \leq s, r \leq k - 1$ the pair(L_s, L_r) is orthogonal under the condition:

$$\big(L_s(x, y), L_r(x, y)\big) = (y - f(x) - sx, y - f(x) - rx)\, \forall x, y \in \mathbb{Z}_q. \tag{3}$$

The following result was shown in [8], but here we retrieve it by new technique.

Theorem 2.1. *Let p be an odd prime number. Then*

$$N\left(p, \left(\frac{p-1}{2}\right)K_{1,2} \cup K_{1,1}\right) = p.$$

Proof. Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a function defined by

$$f(x) = x^2 + 2x + 1, \tag{4}$$

is a starter function of the subgraph $(\frac{p-1}{2})K_{1,2} \cup K_{1,1}$of $K_{p,p}$. Assume $L_s(x, y) = j$, with $y = f(x) + sx + j$ such that $s, j \in \mathbb{Z}_p$. Hence, $j = y - f(x) - sx$. Now, we shall construct p-mutually orthogonal $(\frac{p-1}{2})K_{1,2} \cup K_{1,1}$ squares of order p as follows $L_s(x, y) = y - f(x) - sx$ for all $0 \leq s \leq p-1$. It is easy to check that the squares (L_s, L_r) are orthogonal under the condition $(L_s(x, y), L_r(x, y)) = (y - f(x) - sx, y - f(x) - rx)\, \forall x, y \in \mathbb{Z}_p$, for all different $0 \leq s, r \leq p-1$. Hence the squares $L_s, 0 \leq s \leq p-1$ are mutually orthogonal. It remains to prove the isomorphism of $G_f^s, 0 \leq s \leq p-1$ to the subgraph $((\frac{p-1}{2})K_{1,2} \cup K_{1,1})$ of $K_{p,p}$. Now, we show that G_f^s formed from the i-entries in L_s induce a copy of $(\frac{p-1}{2})K_{1,2} \cup K_{1,1}$ for all $s, i \in \mathbb{Z}_p$. It is clear that every row contains one i there are exactly $(\frac{p-1}{2})$ columns have two i-entry (in other words, $(\frac{p-1}{2})$ vertices $x_1$have degree two), one column has one i-entry (that is, one vertex $x_1$has degree one) and $(\frac{p-1}{2})$ columns have no i-entry (in other words, $(\frac{p-1}{2})$ vertices $x_1$have degree zero). Hence we get that every graph $G^i_{f+sx}, 0 \leq s \leq p-1$ is isomorphic to the subgraph $(\frac{p-1}{2})K_{1,2} \cup K_{1,1}$ of $K_{p,p}$. □

As a direct construction of Theorem 2.1 for $p = 5$ then

$$N(5, 2K_{1,2} \cup K_{1,1}) \geq 5,$$

(see Example 1.8) there exist 5-MOGS of $K_{5,5}$ by $2K_{1,2} \cup K_2$ with respect to $\mathbb{Z}_5$, which are represented by the graphs in Figure 1.1 where $G^0_{f+sx} \cong (2K_{1,2} \cup K_{1,1})$ is the graph corresponding to the entry 0 in the square $L_s, 0 \leq s \leq 4$.

In the upcoming theorem, we delve into a broader scope of MOGS in comparison to the Theorem 2.1.

Theorem 2.2. *Let p, q be odd prime integers such that $p \neq q$. Then $N(pq, G) \geq k = \min\{p, q\}$, where*

$$G \cong \left(\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)K_{1,4} \cup \left(\frac{p+q-2}{2}\right)K_{1,2} \cup K_{1,1}\right).$$

Proof. Using the technique presented in Theorem 2.1, the result follows by applying Equations (2), (3) and (4) for all distinct $0 \leq s, r \leq k-1$ and $x, y \in \mathbb{Z}_{pq}$. Now we show that the subgraph G_f^0 produced from the 0-entries of L_0 is isomorphic to $(\frac{p-1}{2})(\frac{q-1}{2})K_{1,4} \cup (\frac{p+q-2}{2})K_{1,2} \cup K_{1,1}$. A similar strategy may indeed be presented for the other subgraphs G_{f+sx}^i which produced from the i-entries of $L_s, 0 \leq s \leq k-1, i \in \mathbb{Z}_{pq}$. It is clear that every row contains one zero, there are exactly $(\frac{p-1}{2})(\frac{q-1}{2})$ columns have four 0-entry, there are exactly $(\frac{p+q-2}{2})$ columns have two zeros, one column has one zero and $(\frac{3pq-p-q-1}{2})$ columns have no zero. Hence we get that every graph G_{f+sx}^i, $0 \leq s \leq k-1, i \in \mathbb{Z}_{pq}$ is isomorphic to the subgraph $(\frac{p-1}{2})(\frac{q-1}{2})K_{1,4} \cup (\frac{p+q-2}{2})K_{1,2} \cup K_{1,1}$ of $K_{pq,pq}$. $\square$

The following example illustrates the aforementioned theory.

Example 2.3. To illustrate Theorem 2.2 if we take $p = 3$ and $q = 5$, we have 3 mutually orthogonal $(2K_{1,4} \cup 3K_{1,2} \cup K_{1,1})$-squares $L_s(x, y), s \in \mathbb{Z}_3$ and $x, y \in \mathbb{Z}_{15}$ we get squares, $L_s(x, y), s \in \mathbb{Z}_3$ as follows:

$$L_0(x,y) = \begin{bmatrix}
14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 \\
14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 \\
9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 \\
14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 \\
11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14
\end{bmatrix},$$

$$
L_1(x,y)=\begin{bmatrix}
14 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13\\
10 & 11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3\\
11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0}\\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3\\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3 & 4\\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0}\\
11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3\\
10 & 11 & 12 & 13 & 14 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\
14 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0}\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \mathbf{0}
\end{bmatrix},
$$

$$
L_2(x,y)=\begin{bmatrix}
14 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13\\
9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1\\
8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\\
12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\
14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13\\
14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13\\
12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11\\
8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1\\
9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\
14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13\\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1\\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2\\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1
\end{bmatrix},
$$

which are represented by graphs in Figure 2.1 where $G^0_{f+sx} \cong (2K_{1,4} \cup 3K_{1,2} \cup K_{1,1})$ is the graph corresponding to the zero entry in the square $L_s(x,y), s \in \mathbb{Z}_3$.

In Theorem 2.2 when $p = q$, we obtain a novel construction of p-mutually orthogonal $(K_{1,p} \cup (\frac{p-1}{2})K_{1,2})$-squares of order p^2 as demonstrated in the following theorem.

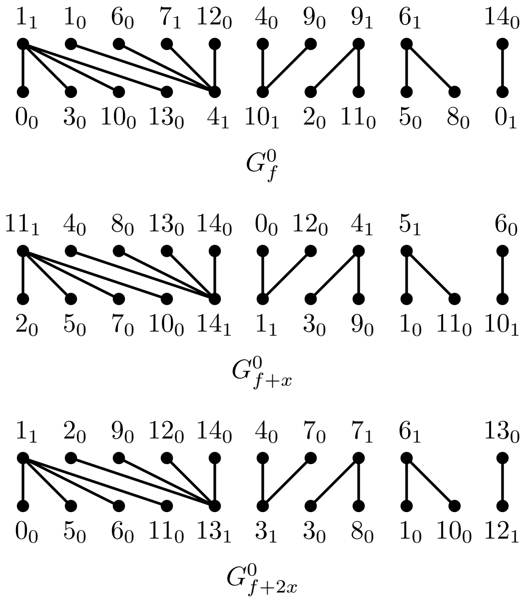

Figure 2.1: MOGS of $K_{15,15}$ by $(2K_{1,4} \cup 3K_{1,2} \cup K_{1,1})$.

Theorem 2.4. *Let p be an odd prime number. Then*

$$N\big(p^2, K_{1,p} \cup p(\tfrac{p-1}{2})K_{1,2}\big) \geq p.$$

Proof. Using the technique presented in Theorem 2.1, the result follows by applying Equations (2), (3) and (4) for all different $0 \leq s, r \leq p-1$ and $x, y \in \mathbb{Z}_{p^2}$. Now we show that the subgraph G^0_f produced from the 0-entries of L_0 is isomorphic to $\big(K_{1,p} \cup p(\frac{p-1}{2})K_{1,2}\big)$. A similar argument may be applied to the other subgraphs G^s_f which produced from the i-entries of L_s, $0 \leq s \leq p-1$, $i \in \mathbb{Z}_{p^2}$. It is obvious that every row includes one zero, there is precisely one column that has p zeros, there are precisely $(\frac{p-1}{2})$ columns that have two zeros and $(\frac{p^2+p-2}{2})$ columns have no zero. $\square$

Theorem 2.4 can be exemplified through the following example.

Example 2.5. Serving as a direct instantiation of Theorem 2.4 for $p = 3$ we observe that $N(9, K_{1,3} \cup 3K_{1,2}) \geq 3$, indicating the existence of three mutually orthogonal $K_{1,3} \cup 3K_{1,2}$-squares L_s of order 9, $s \in \mathbb{Z}_3$, these

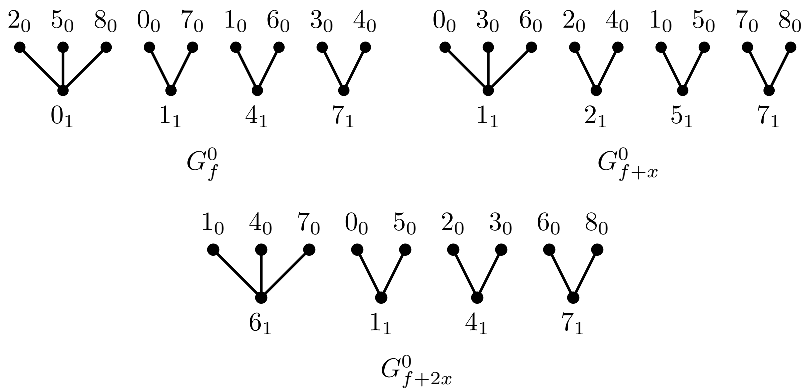

Figure 2.2: 3-mogs of $K_{9,9}$ by $(K_{1,3} \cup 3K_{1,2})$.

squares are defined as follows:

$$L_0 = \begin{bmatrix} 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 \\ 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 \\ 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 \\ 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\ 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\ 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\ 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \end{bmatrix}.$$

Hence, we get 3 mutually orthogonal $K_{1,3} \cup 3K_{1,2}$-squares, which are represented by the graphs in Figure 2.2, where $G_{f+sx}^0 \cong K_{1,3} \cup 3K_{1,2}, i \in \mathbb{Z}_9$ is the graph corresponding to the entry 0 in the square $L_s, s \in \mathbb{Z}_3$.

Note that every figure in Figure 2.2 and its translations represents edge decompositions of $K_{9,9}$ by $(K_{1,3} \cup 3K_{1,2})$. That is equivalent L_i squares

$i = 0, 1, 2$ and in each square, for all $x \in \mathbb{Z}_9$, every row contains one x, there are exactly one column with three x-entries, three columns have two x-entries and five columns have no x-entry.

In the next theorem we consider the more general MOGS with respect to the previous results, Theorems 2.1 and 2.2, by a similar way.

Theorem 2.6. *Let p, q, r be distinct odd prime integers. Then*

$$N(pqr, G) \geq k = \min\{p, q, r\},$$

where

$$\begin{aligned} G \cong & \left(\tfrac{p-1}{2}\right)\left(\tfrac{q-1}{2}\right)\left(\tfrac{r-1}{2}\right)K_{1,8} \\ & \cup \left(\left(\tfrac{p-1}{2}\right)\left(\tfrac{q-1}{2}\right) + \left(\tfrac{p-1}{2}\right)\left(\tfrac{r-1}{2}\right) + \left(\tfrac{q-1}{2}\right)\left(\tfrac{r-1}{2}\right)\right)K_{1,4} \\ & \cup \left(\tfrac{p+q+r-3}{2}\right)K_{1,2} \cup K_{1,1}. \end{aligned}$$

Proof. The result follows by applying Equations (2), (3) and (4) for all different $0 \leq s, r \leq k-1$ and $x, y \in \mathbb{Z}_{pqr}$. Now, we prove that the subgraph G_f^0 produced from the 0-entries of L_0 is isomorphic to G. A similar reasoning may be made to the other page in L_s, $0 \leq s \leq k-1$. It is obvious that every row includes one zero, there are precisely

$$\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)\left(\frac{r-1}{2}\right)$$

columns that have eight zeros, there are exactly

$$\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right) + \left(\frac{p-1}{2}\right)\left(\frac{r-1}{2}\right) + \left(\frac{q-1}{2}\right)\left(\frac{r-1}{2}\right)$$

columns have four zeros, there are exactly

$$\left(\frac{p+q+r-3}{2}\right)$$

columns have two zeros, one column has one zero and

$$\left(\frac{7pqr - pq - pr - qr + 3p + 3q + 3r - 1}{2}\right)$$

columns have no zero. □

The following theorem introduces a novel construction of p-mutually orthogonal $(\frac{p-1}{2})K_{1,2p} \cup K_{1,p} \cup (\frac{p-1}{2})\ p^2K_{1,2}$- squares of order p^3.

Theorem 2.7. *Let p be an odd prime number. Then*

$$N\big(p^3, (\tfrac{p-1}{2})K_{1,2p} \cup K_{1,p} \cup (\tfrac{p-1}{2})\ p^2K_{1,2}\big) \geq p.$$

Proof. Using the technique presented in Theorem 2.1, the result follows by applying Equations (2), (3) and (4) for all different $0 \leq s, r \leq p-1$ and $x, y \in \mathbb{Z}_{p^3}$. Now we wish to show that the subgraph G_f^0 produced from the 0-entries of L_0 is isomorphic to $\big((\tfrac{p-1}{2})K_{1,2p} \cup K_{1,p} \cup (\tfrac{p-1}{2})\ p^2K_{1,2}\big)$. Also, a similar argument can be applied to the other pages in L_s, $0 \leq s \leq p-1$. It is obvious that every row includes one zero, there are precisely $(\frac{p-1}{2})$ columns have $2p$ zeros, there is precisely one column that has p zeros, there are exactly $(\frac{p-1}{2})\ p^2$ columns with two zeros and $(\frac{p^3+p^2-p-1}{2})$ columns have no zero. □

As a straightforward application of Theorem 2.7 for $p = 3$ we get

$$N(27, K_{1,6} \cup K_{1,3} \cup 9K_{1,2}) \geq 3,$$

which means that there are three mutually orthogonal $K_{1,6} \cup K_{1,3} \cup 9K_{1,2}$-squares L_s of order 27, $s \in \mathbb{Z}_3$, which are defined as $L_s(x, y) = y - f(x) - sx$, where $x, y \in \mathbb{Z}_{27}$ and for all $s \in \{0, 1, 2\}$. It is easily verifed that for all different $0 \leq k, r \leq 2$ the pair (L_k, L_r) is orthogonal under the condition of Equation (3). Now we show that the subgraph G_f^0 produced from the 0-entries of L_0 is isomorphic to $(K_{1,6} \cup K_{1,3} \cup 9K_{1,2})$. A similar argument may also be made for the remaining pages in L_1, L_2. It is obvious that every row includes one zero, there is exactly one column having 6 zeros, one column having 3 zeros and nine columns having two zeros and 16 columns having no zero.

The ensuing result expands upon the earlier discoveries within a wider framework.

Theorem 2.8. *Let $n = p_1p_2p_3 \cdots p_r$ where $p_1, p_2, p_3, \ldots, p_r$ are distinct odd prime integers. Then $N(n, G_n) \geq k = \min\{p_1, p_2, p_3, \ldots, p_r\}$ where,*

$$\mathbf{G}_r \cong \bigcup_{m=1}^{r} \left(\left\{ \sum_{\substack{\alpha_1, \alpha 2, \ldots, \alpha_m \in \{1,2,\ldots,r\} \\ \alpha_1 < \alpha_2 < \cdots < \alpha_m}} \left[\prod_{i=1}^{m} \left(\tfrac{p_{\alpha_i}-1}{2} \right) \right] \right\} K_{1,2^m} \right) \cup K_{1,1}.$$

Proof. Using the technique presented in Theorem 2.1, the result follows by applying Equations (2), (3) and (4) for all different $0 \leq s, r \leq k-1$ and $x, y \in \mathbb{Z}_n$. Now we show that the subgraph G_f^0 produced from the 0-entries of L_0 is isomorphic to $\mathbf{G}_r$. A similar argument may also be made for the remaining pages in $L_s, 0 \leq s \leq k-1$. It is obvious that each row includes one zero, there are precisely

$$\left\{ \sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_m \in \{1,2,\dots,r\} \\ \alpha_1<\alpha_2<\cdots<\alpha_m}} \left[\prod_{i=1}^{m} \left(\frac{p_{\alpha_i}-1}{2} \right) \right] \right\}$$

columns having (2^m) 0-entry, where, $1 \leq m \leq r$, precisely one column has one zero and there are precisely

$$n - \left\{ \sum_{m=1}^{r} \left\{ \sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_m \in \{1,2,\dots,r\} \\ \alpha_1<\alpha_2<\cdots<\alpha_m}} \left[\prod_{i=1}^{m} \left(\frac{p_{\alpha_i}-1}{2} \right) \right] \right\} + 1 \right\}$$

columns have no zero. □

Theorem 2.8 may be explained by the following example, let $n = p_1p_2p_3p_4$ where p_1, p_2, p_3, p_4 are distinct odd prime integers. Then

$$N(p_1p_2p_3p_4, \mathbf{G}_4) \geq k = \min\{p_1, p_2, p_3, p_4\}$$

where,

$$\mathbf{G}_4 \cong \bigcup_{m=1}^{4} \left(\left\{ \sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_m \in \{1,2,\dots,r\} \\ \alpha_1<\alpha_2<\cdots<\alpha_m}} \left[\prod_{i=1}^{m} \left(\frac{p_{\alpha_i}-1}{2} \right) \right] \right\} K_{1,2^m} \right) \cup K_{1,1}.$$

This formula can be expanded as follows:

$$\begin{aligned} G \;\cong\;& \left(\left(\tfrac{p_1-1}{2}\right)\left(\tfrac{p_2-1}{2}\right)\left(\tfrac{p_3-1}{2}\right)\left(\tfrac{p_4-1}{2}\right)\right) K_{1,16} \\ &\cup \left(\begin{array}{c} (\tfrac{p_1-1}{2})(\tfrac{p_2-1}{2})(\tfrac{p_3-1}{2}) + (\tfrac{p_1-1}{2})(\tfrac{p_2-1}{2})(\tfrac{p_4-1}{2}) \\ +(\tfrac{p_1-1}{2})(\tfrac{p_3-1}{2})(\tfrac{p_4-1}{2}) + (\tfrac{p_2-1}{2})(\tfrac{p_3-1}{2})(\tfrac{p_4-1}{2}) \end{array} \right) K_{1,8} \\ &\cup \left(\begin{array}{c} (\tfrac{p_1-1}{2})(\tfrac{p_2-1}{2}) + (\tfrac{p_1-1}{2})(\tfrac{p_3-1}{2}) + (\tfrac{p_1-1}{2})(\tfrac{p_4-1}{2}) \\ +(\tfrac{p_2-1}{2})(\tfrac{p_3-1}{2}) + (\tfrac{p_2-1}{2})(\tfrac{p_4-1}{2}) + (\tfrac{p_3-1}{2})(\tfrac{p_4-1}{2}) \end{array} \right) K_{1,4} \\ &\cup \left(\tfrac{p_1-1}{2} + \tfrac{p_2-1}{2} + \tfrac{p_3-1}{2} + \tfrac{p_4-1}{2}\right) K_{1,2} \\ &\cup K_{1,1}. \end{aligned}$$

3 Large constructions of MOGS

In this section, we will generate larger mutually orthogonal arrays MOGS by combining smaller component MOGS. Specifically, we employ extension methods to introduce novel constructions for $N(n, G) = k \geq 3$, where G denotes the disjoint union of specific complete bipartite graphs. All subsequent results are grounded on Proposition 1.4 which leverages the Kronecker product and the existence of MOGS for certain classes of graphs. These serve as ingredients for the Kronecker product, enabling the derivation of new MOGS.

Theorem 3.1. *Let p, q be odd prime numbers. Then*

$$N(pq, G) \geq k = \min\{p, q\},$$

where,

$$G \cong \left(\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)K_{1,4} \cup \left(\frac{p+q-2}{2}\right)K_{1,2} \cup K_{1,1}\right).$$

Proof. Utilizing Theorem 2.1 we can construct p-mutually orthogonal for $K_{p,p}$ by $(\frac{p-1}{2})K_{1,2} \cup K_{1,1}$-squares denoted by $N_s, s \in \mathbb{Z}_p$ and q-mutually orthogonal for $K_{q,q}$ by $(\frac{q-1}{2})K_{1,2} \cup K_{1,1}$-squares $M_s, s \in \mathbb{Z}_q$. If $\min\{p, q\} = k$, then by applying Proposition 1.4 take the Kronecker product for squares N_s, M_s. Now, we can construct k mutually orthogonal G_f-squares and we have $L_s(r,t) = p(y - f(x) - sx) + (b - f(a) - sa)$, $x, y \in \mathbb{Z}_q, a, b \in \mathbb{Z}_p, r, t \in \mathbb{Z}_{pq}$ and $0 \leq s \leq k-1$, such that $r \equiv a \pmod p, t \equiv b \pmod p$, and simply need to show that they are mutually orthogonal. Since $N_s, s \in \mathbb{Z}_p$ and $M_s, s \in \mathbb{Z}_q$ are mutually orthogonal of order p and q respectively then. It is easily verifed that for all different $0 \leq k, r \leq k-1$ the pair (L_s, L_r) is orthogonal under the condition for all $\alpha, \beta \in \mathbb{Z}_{pq}$

$$\begin{aligned}&\Big(L_s(\alpha,\beta), L_r(\alpha,\beta)\Big)\\ &\quad= \Big(p(y-f(x)-sx)+(b-f(a)-sa), p(y-f(x)-rx)+(b-f(a)-ra)\Big).\end{aligned}$$

Hence the squares $L_s, 0 \leq s \leq k-1$ are mutually orthogonal. It remains to prove the isomorphism of $G^i_{f+sx}, 0 \leq s \leq k-1, i \in \mathbb{Z}_{pq}$ to the subgraph $(\frac{p-1}{2})(\frac{q-1}{2})K_{1,4} \cup (\frac{p+q-2}{2})K_{1,2} \cup K_{1,1}$ of $K_{pq,pq}$. Now, we show that G^0_f formed from the 0-entries in L_0 induce a copy of $(\frac{p-1}{2})(\frac{q-1}{2})K_{1,4} \cup (\frac{p+q-2}{2})K_{1,2} \cup K_{1,1}$. It is clear that every row contains one, there are exactly $(\frac{p-1}{2})(\frac{q-1}{2})$ columns have four 0-entry, there are exactly $(\frac{p+q-2}{2})$ columns that have two zeros, one column has one zero and $(\frac{3pq-p-q-1}{2})$ columns have no zero. Additionally, we can simply apply a similar argument to the remaining pages in $L_s, s \in \mathbb{Z}_k$. □

Example 3.2. To demonstrate Theorem 3.1, if we take $p = 3$ and $q = 5$, then $N(15, 2K_{1,4} \cup 3K_{1,2} \cup K_{1,1}) \geq 3 = \min\{3, 5\}$. From Theorem 2.1 we have 3-mutually orthogonal $(K_{1,2} \cup K_{1,1})$-squares of $K_{3,3}$ denoted by $M_s, s \in \mathbb{Z}_3$ and 5-mutually orthogonal $(2K_{1,2} \cup K_{1,1})$-squares of $K_{5,5}$ denoted by $N_s, s \in \mathbb{Z}_5$. So, by using Proposition 1.4, we get 3-mutually orthogonal (G^i_{f+sx})-squares, $i \in \mathbb{Z}_{15}$ denoted by $L_s, s \in \mathbb{Z}_3$, where $G^i_{f+sx} \cong (K_{1,2} \cup K_{1,1})[(2K_{1,2} \cup K_{1,1})] \cong (2K_{1,4} \cup 3K_{1,2} \cup K_{1,1})$, is the graph corresponding to the entry i in the square $L_s, s \in \mathbb{Z}_3$.

$$M_0 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \ M_1 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \ M_2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix},$$

$$N_0 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 \end{bmatrix},$$

$$N_4 = \begin{bmatrix} 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix},$$

$$L_0 = \begin{bmatrix} 14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 \\ 11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 10 \\ 14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 \\ 11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 10 \\ 14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13 \\ 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 10 \\ 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 10 \\ 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{bmatrix},$$

$$L_1=\begin{bmatrix}
14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8\\
10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\
14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8\\
11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 5\\
11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 5\\
9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3\\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4\\
9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3\\
6 & 7 & 8 & 9 & 5 & 11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0\\
6 & 7 & 8 & 9 & 5 & 11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0\\
9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3\\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 & 2 & 3 & 4\\
9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3\\
6 & 7 & 8 & 9 & 5 & 11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0\\
6 & 7 & 8 & 9 & 5 & 11 & 12 & 13 & 14 & 10 & 1 & 2 & 3 & 4 & 0
\end{bmatrix},$$

$$L_2=\begin{bmatrix}
14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8\\
14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8\\
12 & 13 & 14 & 10 & 11 & 2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6\\
13 & 14 & 10 & 11 & 12 & 3 & 4 & 0 & 1 & 2 & 8 & 9 & 5 & 6 & 7\\
12 & 13 & 14 & 10 & 11 & 2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6\\
4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13\\
4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 14 & 10 & 11 & 12 & 13\\
2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6 & 12 & 13 & 14 & 10 & 11\\
3 & 4 & 0 & 1 & 2 & 8 & 9 & 5 & 6 & 7 & 13 & 14 & 10 & 11 & 12\\
2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6 & 12 & 13 & 14 & 10 & 11\\
14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8\\
14 & 10 & 11 & 12 & 13 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8\\
12 & 13 & 14 & 10 & 11 & 2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6\\
13 & 14 & 10 & 11 & 12 & 3 & 4 & 0 & 1 & 2 & 8 & 9 & 5 & 6 & 7\\
12 & 13 & 14 & 10 & 11 & 2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6
\end{bmatrix}.$$

The following corollary is constructed p-mutually orthogonal of K_{p^2,p^2} by using subgraphs distinct from those derived Theorem 2.4.

Corollary 3.3. *Let p be an odd prime number. Then*

$$N\Big(p^2, (\tfrac{p-1}{2})^2 K_{1,4} \cup (p-1)K_{1,2} \cup K_{1,1}\Big) \geq p.$$

Proof. As direct application to Theorem 3.1 if we take $p = q$. Then we have

$$N\Big(p^2, (\tfrac{p-1}{2})^2 K_{1,4} \cup (p-1)K_{1,2} \cup K_{1,1}\Big) \geq p.$$

□

Now, we offer the following result, which generalizes Theorem 2.1.

Proposition 3.4. *Let p^α be an odd number with integer power $\alpha \geq 1$ of a prime number $p > 2$ and G_α be a subgraph of K_{p^α,p^α}. Then $N(p^\alpha, G_\alpha) \geq p$, where*

$$G_\alpha \cong \bigcup_{r=0}^{\alpha} \left(\frac{p-1}{2}\right)^{\alpha-r} \binom{\alpha}{r} K_{1,2^{\alpha-r}}.$$

Proof. The proof utilizes mathematical induction over the integer power α and Proposition 1.4. Ultimately, we construct p mutually orthogonal G_α-squares and we have $L_s(r,t) = p(y-f(x)-sx)+(b-f(a)-sa)$, $x, y \in \mathbb{Z}_{p^{\alpha-1}}$, $a, b \in \mathbb{Z}_p$, $r, t \in \mathbb{Z}_{p^\alpha}$ and $0 \leq s \leq p-1$ such that $r \equiv a \pmod{p}, t \equiv b \pmod{p}$. It is still necessary to prove isomorphism of p mutually orthogonal G_α-squares. Now we show that the page, formed from the 0-entries in L_0 induce a copy of G_α. Also, we can easily apply a similar argument to the other pages in L_s, $s \in \mathbb{Z}_p$. It is obvious that each row includes one zero, there are exactly $\left(\frac{p-1}{2}\right)^{\alpha-m} \binom{\alpha}{m}$ columns have $2^{\alpha-m}$ zeros, $0 \leq m \leq \alpha$, and $\left(p^\alpha - \sum_{m=0}^{\alpha} \left(\frac{p-1}{2}\right)^{\alpha-m} \binom{\alpha}{m}\right)$ columns have no zero. □

In the next theorem we consider the more general MOGS with respect to the previous result Theorem 3.1 by a similar way, we have:

Theorem 3.5. *Let p, q, r be odd prime numbers. Then $N(pqr, \mathbf{G}_3) \geq k = \min\{p, q, r\}$, where*

$$\begin{aligned}\mathbf{G}_3 \cong &\left(\tfrac{p-1}{2}\right)\left(\tfrac{q-1}{2}\right)\left(\tfrac{r-1}{2}\right)K_{1,8} \\ &\cup \left(\left(\tfrac{p-1}{2}\right)\left(\tfrac{q-1}{2}\right) + \left(\tfrac{p-1}{2}\right)\left(\tfrac{r-1}{2}\right) + \left(\tfrac{q-1}{2}\right)\left(\tfrac{r-1}{2}\right)\right)K_{1,4} \\ &\cup \left(\tfrac{p+q+r-3}{2}\right)K_{1,2} \cup K_{1,1}.\end{aligned}$$

Proof. Let p, q, r be odd prime integers and let $f(x) = x^2 + 2x + 1$ then by using Theorem 2.2 we can construct k_1-mutually orthogonal for $K_{pq,pq}$ by $\left(\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)K_{1,4} \cup \left(\frac{p+q-2}{2}\right)K_{1,2} \cup K_{1,1}\right)$-squares, where $k_1 = \min\{p, q\}$ and r-mutually orthogonal $K_{r,r}$ of $\left(\frac{r-1}{2}\right)K_{1,2} \cup K_{1,1}$-squares. If $\min\{k_1, r\} = k$, then by using Proposition 1.4 we can construct k mutually orthogonal $\mathbf{G}_3$-squares and we have

$$\begin{aligned}\mathbf{G}_3 &\cong \left(\left(\tfrac{p-1}{2}\right)\left(\tfrac{q-1}{2}\right)K_{1,4} \cup \left(\tfrac{p+q-2}{2}\right)K_{1,2} \cup K_{1,1}\right)\left[\left(\tfrac{r-1}{2}\right)K_{1,2} \cup K_{1,1}\right)] \\ &\cong \left(\tfrac{p-1}{2}\right)\left(\tfrac{q-1}{2}\right)\left(\tfrac{r-1}{2}\right)K_{1,8} \cup \left(\left(\tfrac{p-1}{2}\right)\left(\tfrac{q-1}{2}\right) + \left(\tfrac{p-1}{2}\right)\left(\tfrac{r-1}{2}\right)\right. \\ &\quad \left. + \left(\tfrac{q-1}{2}\right)\left(\tfrac{r-1}{2}\right)\right)K_{1,4} \cup \left(\tfrac{p+q+r-3}{2}\right)K_{1,2} \cup K_{1,1}.\end{aligned}$$

$L_s(\alpha, \beta) = r(L_s(x, y)) + (b - f(a) - sa)$, $x, y \in \mathbb{Z}_{pq}$, $a, b \in \mathbb{Z}_r$, $\alpha, \beta \in \mathbb{Z}_{pqr}$ and $0 \leq s < k-1$, such that $\alpha \equiv a \pmod{r}$, $\beta \equiv b \pmod{r}$. Now we show that the page, formed from the 0-entries in L_0, is isomorphic to $\mathbf{G}_3$. A similar argument may also be made for the remaining pages in L_s, $0 \leq s < k-1$. It is clear that every row contains one zero, there are exactly

$$\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)\left(\frac{r-1}{2}\right)$$

columns that have eight zeros, there are exactly

$$\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right) + \left(\frac{p-1}{2}\right)\left(\frac{r-1}{2}\right) + \left(\frac{q-1}{2}\right)\left(\frac{r-1}{2}\right)$$

columns that have four zeros, there are exactly

$$\left(\frac{p+q+r-3}{2}\right)$$

columns that have two zeros, one column that has one zero and

$$\left(\frac{7pqr - pq - pr - qr + 3p + 3q + 3r - 1}{2}\right)$$

columns that have no zero. □

As a direct construction of this theorem we consider the following example.

Example 3.6. To illustrate Theorem 3.5. If $p = 3, q = 5, r = 7$ then $N(3.5.7, G) \geq 3$, where, $G \cong 6K_{1,8} \cup 11K_{1,4} \cup 6K_{1,2} \cup K_{1,1}$. Let $f(x) = x^2 + 2x + 1$ be the starter function of the subgraph $6K_{1,8} \cup 11K_{1,4} \cup 6K_{1,2} \cup K_{1,1}$ of $K_{105,105}$. Applying Definition 1.2 and Equation (2) with $n = 105$, we have $L_s(x, y) = y - f(x) - sx$ for all $s \in \{0, 1, 2\}$. So there exist 3-MOGS of $K_{105,105}$ by $G \cong 6K_{1,8} \cup 11K_{1,4} \cup 6K_{1,2} \cup K_{1,1}$ with respect to $\mathbb{Z}_{105}$.

The subsequent corollary establishes p-mutually orthogonal arrays for K_{p^3,p^3} by using subgraphs distinct from those derived in Theorem 2.7.

Corollary 3.7. *Let p be an odd prime number. Then*

$$N\big(p^3, (\tfrac{p-1}{2})^3 K_{1,8} \cup 3(\tfrac{p-1}{2})^2 K_{1,4} \cup 3(\tfrac{p-1}{2}) K_{1,2} \cup K_{1,1}\big) \geq p.$$

Proof. A direct application to Theorem 3.5 occurs when $p = q = r$. □

The upcoming result extends the Theorems 3.1, 3.5 to cases where n has an odd prime factorization.

Theorem 3.8. *Let n be an odd ordered prime factorizations number such that $n = p_1 \cdots p_2 \cdots p_3 \cdots p_r$. Then $N(n, G_n) \geq k = \min\{p_1, p_2, p_3, \ldots, p_r\}$ where,*

$$\mathbf{G}_r \cong \bigcup_{s=1}^{r} \left(\left\{ \sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_m \in \{1,2,\ldots,r\} \\ \alpha_1 < \alpha_2 < \cdots < \alpha_m}} \left[\prod_{i=1}^{m} \left(\frac{p_{\alpha_i} - 1}{2} \right) \right] \right\} K_{1,2^m} \right) \cup K_{1,1}.$$

Proof. The proof can be derived by applying Proposition 1.4 and Theorem 3.1 repeatedly for r-times. □

4 Conclusion

This study focuses on the construction of mutually orthogonal graph squares for complete bipartite graphs by disjoint unions of certain complete bipartite graphs. Our approach involves utilizing the starter functions to derive our results. Through this study, we have obtained novel MOGS results, detailed in the accompanying Table 4.1.

Table 4.1: Summary of the results.

n	G	$N(p, G)$
p	$(\frac{p-1}{2})K_{1,2} \cup K_{1,1}$	$= p$
$pq, \quad p \neq q$	$(\frac{p-1}{2})(\frac{q-1}{2})K_{1,4} \cup (\frac{p+q-2}{2})K_{1,2} \cup K_{1,1}$	$\geq k = \min\{p, q\}$
p^2	$(\frac{p-1}{2})^2 K_{1,4} \cup (p-1)K_{1,2} \cup K_{1,1}$	$\geq p$
p^2	$K_{1,p} \cup p\ (\frac{p-1}{2})K_{1,2}$	$\geq p$
p^3	$(\frac{p-1}{2})K_{1,2p} \cup K_{1,p} \cup\ (\frac{p-1}{2})\ p^2 K_{1,2})$	$\geq p$
p^α	$G_\alpha \cong \bigcup_{r=0}^{\alpha} \left(\frac{p-1}{2}\right)^{\alpha-r} \binom{\alpha}{r} K_{1,2^{\alpha-r}}$	$\geq p$
pqr	$\mathbf{G}_3, \quad p \neq q \neq r$	$\geq k = \min\{p, q, r\}$
$p_1p_2 \cdots p_r$	$\mathbf{G}_r, \quad p_1 \neq p_2 \neq \cdots \neq p_r$	$\geq k = \min\{p_1, \ldots, p_r\}$

Acknowledgment

We are grateful to the BICA editor for their guidance and support and to the referee for their valuable feedback which improved this article.

References

[1] B. Alspach, K. Heinrich and G. Liu, *Orthogonal factorizations of graphs*, in "Contemporary design theory", J.H. Dinitz and D.R. Stinson, eds. Wiley, New York, (1992), 13–40.

[2] G. Appa, D. Magos and I. Mourtos, An LP-based proof for the non-existence of a pair of orthogonal Latin squares of order 6, *Oper. Res. Lett.*, **32** (2004) 336–344.

[3] C.J. Colbourn and J.H. Dinitz, Mutually orthogonal Latin squares: a brief survey of constructions, *J. Statist. Plann. Inference*, **95** (2001), 9–48.

[4] C.J. Colbourn and J.H. Dinitz (eds.), *Handbook of combinatorial designs, second edition*, Chapman & Hall/CRC, London, Boca Raton, FL, 2007.

[5] P.J. Dukes and C.M. van Bommel, Mutually orthogonal Latin squares with large holes, *J. Statist. Plann. Inference*, **159** (2015), 81–89.

[6] A. El-Mesady, Y.S. Hamed and H. Shabana, On the decomposition of circulant graphs using algorithmic approaches. *Alex. Eng. J.*, **61** (2022), 8263–8275.

[7] A. El-Mesady, Y.S. Hamed and K.M. Abualnaja, A novel application on mutually orthogonal graph squares and graphorthogonal arrays, *AIMS Math.*, **7** (2022), 7349–7373.

[8] R. El-Shanawany, *Orthogonal double covers of complete bipartite graphs*, Ph.D. Thesis, Universitat Rostock, 2002.

[9] R. El-Shanawany, On mutually orthogonal graph-path squares, *Open J. Discrete Math.*, **6** (2016), 7–12.

[10] R. El-Shanawany, On mutually orthogonal disjoint copies of graph square, *Note Math.*, **36** (2016), 89–98.

[11] R. El-Shanawany and A. El-Mesady, Mutually orthogonal graph squares for disjoint union of stars, *Ars Combin.*, **149** (2020), 83–91.

[12] R. El-Shanawany, S. Nada, A. Elrokh and E. Sallam, A novel construction of mutually orthogonal three disjoint union of certain trees squares, *Appl. Math. Inf. Sci.* **17** (2033), 7–11.

[13] E. Gilbert, F. MacWilliams and N. Sloane, Codes which detect deception, *Bell System Tech. J.* **53** (1974), 405–424.

[14] H.F. MacNeish, Euler squares, *Ann. of Math.*, **23** (1922), 221–227.

[15] R. Sampathkumar and S. Srinivasan, Mutually orthogonal graph squares, *J. Combin. Des.* **17** (2009), 369–373.

[16] I.M. Wanless and B.S. Webb, The existence of Latin squares without orthogonal mates, *Des. Codes Cryptogr.* **40** (2006), 131–135.

[17] X. Wang, N. Jin, M. Cao and Y. Wen, Construction of authentication codes based on orthogonal array and Latin square *Comput. Appl. Math.*, **41** (2022), Paper no. 161, 22pp.

R. El-Shanawany
Department of Physics and Engineering Mathematics, Faculty of Electronic Engineering, Menofia University
ramadan_elshanawany380@yahoo.com

M.E. Abdel-Aal
Department of Mathematics, Faculty of Science, Benha University
mohamed_el77@yahoo.com,mohamed.abdelghani@fsc.bu.edu.eg