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Some imbalanced hypergraph Zarankiewicz numbers

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Abstract. The Zarankiewicz number $z(m, n; a, b)$ is the maximum number of edges $|E|$ among all bipartite graphs $G = (X \dot\cup Y, E)$ satisfying $|X| = m$, $|Y| = n$, and that no a vertices of X and b vertices of Y induce a copy of the complete bipartite graph $K_{a,b}$ as a subgraph of G. For $m \geq (a-1)\binom{n}{b}$, Čulík proved $z(m, n; a, b) = (a-1)\binom{n}{b} + (b-1)m$. We extend this result to hypergraphs of a similarly imbalanced variety. Our key will be a construction employing Baranyai's theorem on hyperclique matching decompositions.

1 Introduction

Fix $a, b, m, n \in \mathbb{N}$. The *Zarankiewicz number* $z(m, n; a, b)$ is the maximum number of edges $|E|$ among all bipartite graphs $G = (X \dot\cup Y, E)$ satisfying $|X| = m$, $|Y| = n$, and that no a vertices of X and b vertices of Y induce a copy of the complete bipartite graph $K_{a,b}$ as a subgraph of G. Its determination or estimation is the 1951 problem of Zarankiewicz [15], which remains open today. Kővári, Sós and Turán [10] proved the seminal bound

$$z(m, n; a, b) < (a-1)^{1/b}(n-b+1)m^{1-(1/b)} + (b-1)m \tag{1}$$

(see also Füredi [8] and Nikiforov [12]). The best diagonal bounds for fixed but general $a \geq 2$ are

$$\Omega\Big(m^{2-\frac{2}{a+1}}\Big) \overset{[7]}{\leq} z(m, m; a, a) \overset{(1)}{\leq} O\Big(m^{2-\frac{1}{a}}\Big),$$

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although for $a \in \{2,3\}$ the lower bound admits substantial improvement (see below), and for $a \geq 5$ it admits polylogarithmic improvement (see Bohman and Keevash [3]). Cases admitting asymptotics or formulas are extremely rare. Kővári, Sós and Turán [10] proved

$$z(n,n;2,2) = \big(1 \pm o(1)\big)n^{3/2} \qquad \text{and} \qquad z\big(p^2+p,\, p^2;\, 2,\, 2\big) = p^3 + p^2$$

for primes p. Reiman proved that for q any power of a prime,

$$z\big(q^2+q+1,\, q^2+q+1;\, 2,\, 2\big) = \big(q^2+q+1)(q+1).$$

Brown and Füredi [4, 8]; Mörs [11] and Alon, Mellinger, Mubayi and Verstraëte [1] respectively showed

$$\begin{aligned} z(m,m,3,3) &= \big(1 \pm o(1)\big)m^{5/3}, \\ z(m,m,2,b+1) &= \big(b^{1/2} \pm o(1)\big)n^{3/2} \ \big(\text{for } b \text{ fixed}\big) \qquad \text{and} \\ z(m,n;2,b) &= \big(1 \pm o(1)\big)mn^{1/2} \ \big(\text{for } b \text{ fixed and } m = \big(1 \pm o(1)\big)n^{b/2}\big). \end{aligned}$$

An early, elementary, and exact result of Čulík considers a rather severe imbalance between m and n.

Theorem 1.1 (Čulík [6]). *When $m \geq (a-1)\binom{n}{b}$, the formula*

$$z(m,n;a,b) = (a-1)\tbinom{n}{b} + (b-1)m$$

holds.

Exact but fairly technical extensions of Theorem 1.1 for suitable $m = \Theta(n^b)$ were given by Guy [9], Roman [14], and more recently by Chen, Horsley and Mammoliti [5].

We extend Theorem 1.1 to hypergraphs of a similarly imbalanced variety. We first outline our considerations coarsely. The conventional Zarankiewicz number $z(m,n;a,b)$ is the maximum number of edges $|E|$ among all bipartite graphs $G = (X \,\dot\cup\, Y, E)$ satisfying $|X| = m$, $|Y| = n$, and that no a vertices from X and b vertices from Y induce a copy of the complete bipartite graph $K_{a,b}$ as a subgraph of G. The parameter of this paper considers k-partite k-graphs H having a fixed vertex partition $V(H) = V_1 \,\dot\cup \cdots \dot\cup\, V_k$ into classes of prescribed sizes. (Here, the edges of H are k-tuples meeting each class V_i, over $1 \leq i \leq k$, precisely once.) Our parameter seeks the maximum number of edges that H can have when no a_i vertices of V_i, over $1 \leq i \leq k$, induce a copy of the complete k-partite k-graph $K^{(k)}_{a_1,\dots,a_k}$ (having $\prod_{i=1}^k a_i$ many k-tuple edges) as a subhypergraph of H. Čulík's result determines the conventional Zarankiewicz number exactly in the case that V_1 is significantly larger than V_2. Our paper achieves an analogous result when each V_i, over $1 \le i \le k-1$, is significantly larger than V_{i+1}. To make these considerations precise, we prepare some notation. Henceforth, fix $k \in \mathbb{N}$, a set $V = V_1 \dot\cup \cdots \dot\cup V_k$ and partition thereof, an ordering

$$\boldsymbol{V}_k = (V_1, \dots, V_k), \qquad \text{and} \qquad \boldsymbol{a}_k = (a_1, \dots, a_k) \in \mathbb{N} \times \cdots \times \mathbb{N} = \mathbb{N}^k.$$

Let $\binom{\boldsymbol{V}_k}{k}$ be the set of all $\kappa \in \binom{V}{k}$ satisfying $|\kappa \cap V_i| = 1$ for all $i \in [k] = \{1, \dots, k\}$. Any subset $H = H(k) \subseteq \binom{\boldsymbol{V}_k}{k}$ is a k-partite k-graph with partition $\boldsymbol{V}_k$. We say H is *$\boldsymbol{a}_k$-avoidant* when every $(A_1, \dots, A_k) \in \binom{V_1}{a_1} \times \cdots \times \binom{V_k}{a_k}$ admits $(\alpha_1, \dots, \alpha_k) \in A_1 \times \cdots \times A_k$ satisfying $\{\alpha_1, \dots, \alpha_k\} \notin H$. Define

$$\mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k) = \Big\{ H \subseteq \tbinom{\boldsymbol{V}_k}{k} : \ H \text{ is } \boldsymbol{a}_k\text{-avoidant} \Big\}$$

and

$$z(\boldsymbol{V}_k, \boldsymbol{a}_k) = \max \big\{ |H| : \ H \in \mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k) \big\}.$$

Note that $z\big((V_1, V_2), (a_1, a_2)\big)$ is the conventional Zarankiewicz number $z\big(|V_1|, |V_2|; a_1, a_2\big)$. Note also that $z\big((V_1), (a_1)\big) = a_1 - 1$ holds trivially.

We prove the following hypergraph version of Theorem 1.1.

Theorem 1.2. *Every integer $k \ge 2$ satisfies*

$$z(\boldsymbol{V}_k, \boldsymbol{a}_k) \le z(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}) \tbinom{|V_k|}{a_k} + (a_k - 1)|V_1| \cdots |V_{k-1}|. \tag{2}$$

Equality holds when all $1 \le i \le k-1$ satisfy

$$|V_i| \ge a_i \tbinom{|V_{i+1}|}{a_{i+1}} + a_i^2 \tag{3}$$

and also when $k = 2$ and more simply $|V_1| \ge (a_1 - 1)\binom{|V_2|}{a_2}$. In these cases,

$$z(\boldsymbol{V}_k, \boldsymbol{a}_k) = \sum_{i=1}^{k} \Big((a_i - 1) \Big(\prod_{h=1}^{i-1} |V_h| \Big) \prod_{j=i+1}^{k} \tbinom{|V_j|}{a_j} \Big). \tag{4}$$

In particular, we construct $Z(k) \in \mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k)$ where $|Z(k)|$ is the upper bound of (2). Moreover, when additionally $|V_k| \ge a_k + a_k^2$ and $1 \le r \le \lfloor |V_k|/a_k \rfloor - a_k$ is an integer, we construct an r-sequence

$$\boldsymbol{Z}_k = \big(Z_1(k), \dots, Z_r(k) \big) \in \mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k) \times \cdots \times \mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k) \tag{5}$$

of pairwise edge-disjoint entries satisfying that $|Z_1(k)| = \cdots = |Z_r(k)|$ is the upper bound of (2).

We say a few words on our proofs of (2)–(5). Section 2 gives a standard double-counting argument for (2). Iterative equality in (2) immediately gives (4). The challenge in proving Theorem 1.2 lies in the equality under (3) and, crucially, its relationship with the r-sequence of (5). In particular, we recursively construct $Z(k) \in \mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k)$ where $|Z(k)|$ is the upper bound of (2). To construct $Z(k)$, we require access to a very long sequence

$$\boldsymbol{Z}_{k-1} = \big(Z_1(k-1), \ldots, Z_s(k-1)\big) \in \mathcal{Z}(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}) \times \cdots \times \mathcal{Z}(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1})$$

of optimal and pairwise edge-disjoint entries. This sequence is itself built recursively. Thus, to maintain our induction on $k \geq 2$ for Theorem 1.2, we *must* in fact construct the r-sequence of (5). We complete these details in Section 4. We then prove (3) in Section 5 as a relaxation of Section 4.

The main novelty of the paper lies entirely in ensuring edge-disjointness in (5). Here, we use a subtle application of Baranyai's theorem [2] on hyperclique matching decompositions (see Section 3 and Lemmas 4.1 and 4.2.

2 Proof of Theorem 1.2: the upper bound in (2)

Fix $H \in \mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k)$ and let $\boldsymbol{\mathcal{V}}(k-1) = V_1 \times \cdots \times V_{k-1}$.

For $\boldsymbol{v}_{k-1} = (v_1, \ldots, v_{k-1}) \in \boldsymbol{\mathcal{V}}(k-1)$, define

$$N_H(\boldsymbol{v}_{k-1}) = \big\{v_k \in V_k : \{v_1, \ldots, v_{k-1}, v_k\} \in H\big\}$$

and

$$\deg_H(\boldsymbol{v}_{k-1}) = \big|N_H(\boldsymbol{v}_{k-1})\big|.$$

For $v_k \in V_k$, define

$$N_H(v_k) = \big\{\{v_1, \ldots, v_{k-1}\} : \{v_1, \ldots, v_{k-1}, v_k\} \in H\big\}.$$

Clearly, $v_k \in N_H\big((v_1, \ldots, v_{k-1})\big)$ precisely when $\{v_1, \ldots, v_{k-1}\} \in N_H(v_k)$. Double-counting gives

$$|H| = \sum_{\boldsymbol{v}_{k-1} \in \boldsymbol{\mathcal{V}}(k-1)} \deg_H(\boldsymbol{v}_{k-1}) = \sum_{v_k \in V_k f} \deg_H(v_k). \tag{6}$$

Consider the set S of $(\boldsymbol{v}_{k-1}, A_k) \in \boldsymbol{\mathcal{V}}(k-1) \times \binom{V_k}{a_k}$ with $A_k \subseteq N_H(\boldsymbol{v}_{k-1})$. Double-counting gives

$$|S| = \sum_{\boldsymbol{v}_{k-1} \in \boldsymbol{\mathcal{V}}(k-1)} \binom{\deg_H(\boldsymbol{v}_{k-1})}{a_k}$$
$$= \sum_{A_k \in \binom{V_k}{a_k}} \Big| \bigcap_{\alpha_k \in A_k} N_H(\alpha_k) \Big| \leq z(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}) \binom{|V_k|}{a_k} \quad (7)$$

because each such $\bigcap_{\alpha_k \in A_k} N_H(\alpha_k) \subseteq \binom{\boldsymbol{V}_{k-1}}{k-1}$ is $\boldsymbol{a}_{k-1}$-avoidant. Define $\boldsymbol{\mathcal{V}}_-(k-1)$, $\boldsymbol{\mathcal{V}}_0(k-1)$, and $\boldsymbol{\mathcal{V}}_+(k-1)$ to be the sets of all $\boldsymbol{v}_{k-1} \in \boldsymbol{\mathcal{V}}(k-1)$ for which $\deg_H(\boldsymbol{v}_{k-1}) - a_k$ is, respectively, negative, zero and positive. Then

$$z\big(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}\big) \binom{|V_k|}{a_k} \overset{(7)}{\geq} \sum_{\star \in \{-,0,+\}} \sum_{\boldsymbol{v}_{k-1} \in \boldsymbol{\mathcal{V}}_\star(k-1)} \binom{\deg_H(\boldsymbol{v}_{k-1})}{a_k}$$
$$\geq \big|\boldsymbol{\mathcal{V}}_0(k-1)\big| + \sum_{\boldsymbol{v}_{k-1} \in \boldsymbol{\mathcal{V}}_+(k-1)} \deg_H(\boldsymbol{v}_{k-1})$$
$$\overset{(6)}{=} |H| - (a_k - 1)\big|\boldsymbol{\mathcal{V}}_0(k-1)\big| - \sum_{\boldsymbol{v}_{k-1} \in \boldsymbol{\mathcal{V}}_-(k-1)} \deg_H(\boldsymbol{v}_{k-1})$$
$$\geq |H| - (a_k - 1)\Big(\big|\boldsymbol{\mathcal{V}}_0(k-1)\big| + \big|\boldsymbol{\mathcal{V}}_-(k-1)\big|\Big)$$
$$\geq |H| - (a_k - 1)\big|\boldsymbol{\mathcal{V}}(k-1)\big|.$$

Thus,

$$|H| \leq (a_k - 1)\big|\boldsymbol{\mathcal{V}}(k-1)\big| + z\big(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}\big)\binom{|V_k|}{a_k},$$

where

$$\big|\boldsymbol{\mathcal{V}}(k-1)\big| = |V_1| \cdots |V_{k-1}|.$$

3 Permutations, matchings, and Baranyai's theorem

Fix $d \in \mathbb{N}$ and a finite set X. Let $\binom{X}{d}!$ denote the symmetric group on $\binom{X}{d}$. We say $\Pi \subseteq \binom{X}{d}!$ is *respectful* when $\pi(D) \neq \pi'(D')$ for all distinct $\pi, \pi' \in \Pi$ and for all $D, D' \in \binom{X}{d}$ with nonempty intersection. We wish to show that there exist respectful families $\Pi \subseteq \binom{X}{d}!$ of at least a certain size.

Fact 3.1. *There exists a respectful family* $\Pi \subseteq \binom{X}{d}!$ *satisfying* $|\Pi| \geq \lceil |X|/d \rceil - d$.

Fact 3.1 will be a fairly easy corollary of Baranyai's theorem [2] on decompositions of $\binom{X}{d}$ into perfect matchings. For this, recall that a *matching* $\mathcal{D} \subset \binom{X}{d}$ is pairwise disjoint and is *perfect* when $X = \bigcup_{D \in \mathcal{D}} D$. A family $\mathbb{D}$ of matchings $\mathcal{D} \subset \binom{X}{d}$ *decomposes* $\binom{X}{d}$ when $\binom{X}{d} = \bigcup_{\mathcal{D} \in \mathbb{D}} \mathcal{D}$ is a partition.

Theorem 3.2 (Baranyai [2])**.** *$\binom{X}{d}$ admits a decomposition into perfect matchings if and only if d divides $|X|$.*

The following approximate version of Theorem 3.2 is a corollary thereof.

Corollary 3.3. *$\binom{X}{d}$ admits a decomposition into matchings each of size at least $\lceil |X|/d \rceil - d$.*

For completeness, we derive Corollary 3.3 from Theorem 3.2. We then use Corollary 3.3 to prove Fact 3.1.

Proof of Corollary 3.3

The result follows from Theorem 3.2 if d divides $|X|$, so assume otherwise. Let $|X| = dq + r$ for an integer $1 \leq r < d$. Let W be a $(d-r)$-set disjoint from X and let $Y = W \dot{\cup} X$. Fix a decomposition $\mathbb{D}$ of $\binom{Y}{d}$ into perfect matchings. From each $\mathcal{D} \in \mathbb{D}$, remove all $D \in \mathcal{D}$ meeting W to form a (sub)matching $\mathcal{D}^* \subset \mathcal{D}$ and a family $\mathbb{D}^* = \big\{\mathcal{D}^* : \mathcal{D} \in \mathbb{D}\big\}$. Every matching $\mathcal{D}^* \in \mathbb{D}^*$ resides entirely in X and has size

$$\big|\mathcal{D}^*\big| \geq |\mathcal{D}| - |W| = \big(|Y|/d\big) - (d-r) \geq \lceil |X|/d \rceil - d.$$

Now, $\mathbb{D}^*$ decomposes $\binom{X}{d}$. Indeed, fix $D \in \binom{X}{d}$. Since $\binom{X}{d} \subset \binom{Y}{d}$, some $\mathcal{D} \in \mathbb{D}$ holds D. But $D \subseteq X$ so $D \in \mathcal{D}^* \in \mathbb{D}^*$. Moreover, disjoint $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{D}$ yield disjoint submatchings $\mathcal{D}_1^*, \mathcal{D}_2^* \in \mathbb{D}_*$. □

Proof of Fact 3.1

Set $m = \lceil |X|/d \rceil - d$. We define a respectful $\Pi = \big\{\pi_i : 0 \leq i \leq m-1\big\} \subset \binom{X}{d}!$. Let $\mathbb{D}$ be the decomposition of $\binom{X}{d}$ guaranteed by Corollary 3.3. We define each $\pi_i \in \Pi$ piecewise on each $\mathcal{D} = \{D_j : j \in \mathbb{Z}_{|\mathcal{D}|}\} \in \mathbb{D}$ in a cyclic way (treating i as an element of $\mathbb{Z}_{|\mathcal{D}|}$):

$$\pi_i(D_j) = D_{i+j} \in \mathcal{D}. \tag{8}$$

Then $\pi_i(D_j) = \pi_{i'}(D_j)$ holds only when $i = i'$ because $|\mathcal{D}| \geq m$ by Theorem 3.3. Moreover, π_i is defined on each $D \in \binom{X}{d}$ because some $\mathcal{D} \in \mathbb{D}$ holds D. To prove Fact 3.1, fix $\pi_a, \pi_b \in \Pi$, $D \in \mathcal{D} \in \mathbb{D}$, and $D' \in \mathcal{D}' \in \mathbb{D}$, and write $D = D_j$ and $D' = D'_{j'}$ for some $j \in \mathbb{Z}_{|\mathcal{D}|}$ and $j' \in \mathbb{Z}_{|\mathcal{D}'|}$.

Bijectivity. Let $a = b = i$ and $\pi_i(D_j) = \pi_i(D'_{j'})$. Then $\mathcal{D} = \mathcal{D}'$ from (8) because $\pi_i(D_j) \in \mathcal{D}$ and $\pi_i(D'_{j'}) \in \mathcal{D}'$ aligned in the pairwise disjoint $\mathbb{D}$. Also from (8) is

$$D_{i+j} = \pi_i(D_j) = \pi_i(D_{j'}) = D_{i+j'}$$

so $j \equiv j' \pmod{|\mathcal{D}|}$ and $D_j = D'_{j'}$.

Respectfulness. Let $a \neq b$ and $D_j \cap D'_{j'} \neq \emptyset$. First, let $D_j = D'_{j'}$, whence $\mathcal{D} = \mathcal{D}'$. Then $\pi_a(D_j) = D_{a+j}$ and $\pi_b(D_j) = D_{b+j}$ are distinct from $a \neq b$ and $a + j \not\equiv b + j \pmod{|\mathcal{D}|}$. Next, let $D_j \neq D'_{j'}$. From their meeting follow $\mathcal{D} \neq \mathcal{D}'$ (as matchings), $\mathcal{D} \cap \mathcal{D}' = \emptyset$ (in $\mathbb{D}$), and $\pi_a(D_j) \neq \pi_b(D'_{j'})$ $\big($in $\mathcal{D}$ and $\mathcal{D}'\big)$. $\square$

4 Proof of Theorem 1.2: the sequence in (5)

Throughout this proof, we assume that $k \geq 2$ and that the following strengthening of (3) holds:

$$|V_i| \geq a_i \binom{|V_{i+1}|}{a_{i+1}} + a_i^2 \qquad \text{and} \qquad |V_k| \geq a_k + a_k^2 \tag{9}$$

for all $1 \leq i \leq k-1$. For the purposes of (5), fix an integer

$$1 \overset{(9)}{\leq} r_k \leq \big(|V_k|/a_k\big) - a_k. \tag{10}$$

We inductively construct a sequence

$$\boldsymbol{Z}_k = \big(Z_\iota(k) : \iota \in I\big) \in \mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k) \times \cdots \times \mathcal{Z}(\boldsymbol{V}_k, \boldsymbol{a}_k) \tag{11}$$

of $|I| = r_k$ specially indexed (explained later in context) and pairwise edge-disjoint entries each satisfying

$$\big|Z_\iota(k)\big| = z(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}) \binom{|V_k|}{a_k} + (a_k - 1)|V_1| \cdots |V_{k-1}|. \tag{12}$$

The choice of r_k in (10) is suitable for an application of Fact 3.1, so we are guaranteed a henceforth fixed and respectful family $\Pi_k \subseteq \binom{V_k}{a_k}!$ of size

$$|\Pi_k| = r_k \overset{(10)}{\leq} \big(|V_k|/a_k\big) - a_k \leq \big\lceil |V_k|/a_k \big\rceil - a_k. \tag{13}$$

For $k \geq 3$, the choice $r_{k-1} = \binom{|V_k|}{a_k}$ inductively satisfies (10) because

$$1 \leq r_{k-1} = \binom{|V_k|}{a_k} \overset{(9)}{\leq} \big(|V_{k-1}|/a_{k-1}\big) - a_{k-1}. \tag{14}$$

We proceed to our inductive construction of (11).

The base $k = 2$

We construct the following r_2-sequence $\boldsymbol{Z}_2 = \big(Z_\pi(2) : \pi \in \Pi_2\big)$ (cf. (13)):

(i) fix a partition

$$V_1 = R \ \dot\cup \ \dot{\bigcup}\Big\{Z_{A_2} : A_2 \in \tbinom{V_2}{a_2}\Big\},$$

where each $A_2 \in \binom{V_2}{a_2}$ satisfies $\big|Z_{A_2}\big| = a_1 - 1$, which is possible by

$$|V_1| \overset{(9)}{\geq} a_1\tbinom{|V_2|}{a_2} + a_1^2 \geq (a_1 - 1)\tbinom{|V_2|}{a_2};$$

(ii) fix a partition

$$V_2 = Q \ \dot\cup \ \dot{\bigcup}\big\{Y_\pi : \pi \in \Pi_2\big\},$$

where each $\pi \in \Pi_2$ satisfies $\big|Y_\pi\big| = a_2 - 1$, which is possible[1] for $a_2 \geq 2$ by

$$|V_2|/(a_2 - 1) \geq |V_2|/a_2 \geq \big(|V_2|/a_2\big) - a_2 \overset{(10)}{\geq} r_2 \overset{(13)}{=} |\Pi_2|;$$

(iii) for each $\pi \in \Pi_2$, define the edge-disjoint union (of complete bipartite graphs[2])

$$Z_\pi(2) = K\big[R,\, Y_\pi\big] \ \dot\cup \ \dot{\bigcup}\Big\{K\big[Z_{\pi(A_2)},\, A_2\big] : A_2 \in \tbinom{V_2}{a_2}\Big\}.$$

We will repeatedly use the observation that, for every $(v_1, \pi) \in V_1 \times \Pi_2$, the neighborhood in $Z_\pi(2)$ of v_1 is

$$N_{Z_\pi(2)}(v_1) = \begin{cases} Y_\pi & \text{when } v_1 \in R, \\ A_2 & \text{when } v_1 \in Z_{\pi(A_2)} \text{ for } A_2 \in \binom{V_2}{a_2}. \end{cases} \tag{15}$$

[1] Trivially, $Q = V_2$ when $a_2 = 1$.
[2] Here, and for sets X and Y unrelated to any above, $K[X, Y] = \big\{\{x, y\} : x \in X,\, y \in Y\big\}$.

We now show that $\boldsymbol{Z}_2 = \big(Z_\pi(2) : \pi \in \Pi_2\big)$ satisfies the properties of (11). For that, fix $\pi \neq \pi' \in \Pi_2$.

Claim. $\big|Z_\pi(2)\big| = z(\boldsymbol{V}_1, \boldsymbol{a}_1)\binom{|V_2|}{a_2} + (a_2 - 1)|V_1|$, *so* $Z_\pi(2)$ *satisfies* (12) *with* $k = 2$.

Proof. By (i)–(iii),

$$\begin{aligned}\big|Z_\pi(2)\big| &= \big|Y_\pi\big||R| + \sum\nolimits_{A_2 \in \binom{V_2}{a_2}} |A_2|\big|Z_{\pi(A_2)}\big| \\ &= (a_2 - 1)\Big(|V_1| - (a_1 - 1)\tbinom{|V_2|}{a_2}\Big) + a_2(a_1 - 1)\tbinom{|V_2|}{a_2},\end{aligned}$$

which is $(a_1 - 1)\binom{|V_2|}{a_2} + (a_2 - 1)|V_1|$, and where $z(\boldsymbol{V}_1, \boldsymbol{a}_1) = a_1 - 1$ holds trivially. □

Claim. $Z_\pi(2) \in \mathcal{Z}(\boldsymbol{V}_2, \boldsymbol{a}_2)$.

Proof. Fix $(A_1, A_2) \in \binom{V_1}{a_1} \times \binom{V_2}{a_2}$ and $\alpha_1 \in A_1 \setminus Z_{\pi(A_2)} \neq \emptyset$. We seek $\alpha_2 \in A_2 \setminus N_{Z_\pi(2)}(\alpha_1)$ $\big($cf. (15)$\big)$. For $\alpha_1 \in R$, pick $\alpha_2 \in A_2 \setminus Y_\pi \neq \emptyset$. For $\alpha_1 \in Z_{\pi(A_2')}$ with $A_2' \in \binom{V_2}{a_2} \setminus \{A_2\}$, pick $\alpha_2 \in A_2 \setminus A_2' \neq \emptyset$. □

Lemma 4.1. $Z_\pi(2)$ *and* $Z_{\pi'}(2)$ *are edge-disjoint.*

Proof. Fix $v_1 \in V_1$. We show $N_{Z_\pi(2)}(v_1) \cap N_{Z_{\pi'}(2)}(v_1) = \emptyset$ $\big($cf. (15)$\big)$. For $v_1 \in R$, these sets are Y_π and $Y_{\pi'}$ and are disjoint by $\pi \neq \pi'$. For $v_1 \in Z_{\pi(A_2)} = Z_{\pi'(A_2')}$ with $A_2, A_2' \in \binom{V_2}{a_2}$, the equality $\pi(A_2) = \pi'(A_2')$ holds in a respectful family $\Pi_2 \subseteq \binom{V_2}{a_2}!$ with $\pi \neq \pi'$, so the a_2-sets $A_2 = N_{Z_\pi(2)}(v_1)$ and $A_2' = N_{Z_{\pi'}(2)}(v_1)$ must be disjoint. □

The inductive step $k \geq 3$

This step is a formal generalization of the base step. First, we invoke induction on $\mathcal{Z}(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1})$ with $r_{k-1} = \binom{|V_k|}{a_k}$ from (14) to construct an r_{k-1}-sequence

$$\boldsymbol{Z}_{k-1} = \Big(Z_{A_k}(k-1) : A_k \in \tbinom{V_k}{a_k}\Big) \in \mathcal{Z}(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}) \times \cdots \times \mathcal{Z}(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}) \tag{16}$$

of pairwise edge-disjoint entries each satisfying

$$\big|Z_{A_k}(k-1)\big| = z(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}). \tag{17}$$

This appeal to induction uses the tacit feature from (9) that $|V_{k-1}| \geq a_{k-1}\binom{|V_k|}{a_k} + a_{k-1}^2 \geq a_{k-1} + a_{k-1}^2$. Next, we complete (11) by constructing the following r_k-sequence $\boldsymbol{Z}_k = \big(Z_\pi(k) : \pi \in \Pi_k\big)$ (cf. (13)):

(i) fix the edge-partition (of the complete $(k-1)$-partite $(k-1)$-graph)

$$K^{(k-1)}[V_1, \dots, V_{k-1}] = R(k-1) \ \dot\cup \ \dot{\bigcup} \Big\{ Z_{A_k}(k-1) : \ A_k \in \tbinom{V_k}{a_k} \Big\},$$

which is possible by (16);

(ii) fix a partition

$$V_k = Q_k \ \dot\cup \ \dot{\bigcup} \big\{ Y_\pi : \ \pi \in \Pi_k \big\},$$

where each $\pi \in \Pi_k$ satisfies $|Y_\pi| = a_k - 1$, which is possible[3] for $a_k \geq 2$ by

$$|V_k|/(a_k - 1) \geq |V_k|/a_k \geq \big(|V_k|/a_k\big) - a_k \overset{(10)}{\geq} r_k \overset{(13)}{=} \big|\Pi_k\big|;$$

(iii) for each $\pi \in \Pi_k$, we will define the following edge-disjoint unions

$$Z_\pi(k) = K^{(k)}\big[R(k-1), \, Y_\pi\big] \ \dot\cup \ \dot{\bigcup} \Big\{ K^{(k)}\big[Z_{\pi(A_k)}(k-1), \, A_k\big] : \ A_k \in \tbinom{V_k}{a_k} \Big\}$$

to consist of all $\{v_1, \dots, v_{k-1}, v_k\}$ satisfying either

$$\big(\{v_1, \dots, v_{k-1}\}, v_k\big) \in R(k-1) \times Y_\pi$$

or

$$\big(\{v_1, \dots, v_{k-1}\}, v_k\big) \in Z_{\pi(A_k)}(k-1) \times A_k$$

for some $A_k \in \binom{V_k}{a_k}$.

We will repeatedly use that, for every $(v_1, \dots, v_{k-1}, \pi) \in V_1 \times \dots \times V_{k-1} \times \Pi_k$, the neighborhood in $Z_\pi(k)$ of $(v_1, \dots, v_{k-1})$ is

$$N_{Z_\pi(k)}\big((v_1, \dots, v_{k-1})\big) = \begin{cases} Y_\pi & \text{when } \{v_1, \dots, v_{k-1}\} \in R(k-1), \\ A_k & \text{when } \{v_1, \dots, v_{k-1}\} \in Z_{\pi(A_k)}(k-1) \text{ for } A_k \in \binom{V_k}{a_k}. \end{cases} \tag{18}$$

We now show that $\boldsymbol{Z}_k = \big(Z_\pi(k) : \ \pi \in \Pi_k\big)$ satisfies the properties of (11). For that, fix $\pi \neq \pi' \in \Pi_k$.

Claim. $\big|Z_\pi(k)\big| = z\big(\boldsymbol{V}_{k-1}, \boldsymbol{a}_{k-1}\big)\binom{|V_k|}{a_k} + (a_k - 1)|V_1| \cdots |V_k|$, *so* $Z_\pi(k)$ *satisfies* (12).

[3] Trivially, $Q_k = V_k$ when $a_k = 1$.

Proof. By (i)–(iii),

$$\big|Z_\pi(k)\big| = |Y_\pi|\big|R(k-1)\big| + \sum\nolimits_{A_k\in\binom{V_k}{a_k}} |A_k|\big|Z_{\pi(A_k)}(k-1)\big|$$
$$\stackrel{(17)}{=} (a_k-1)\Big(|V_1|\cdots|V_{k-1}| - \tbinom{|V_k|}{a_k} z\big(\boldsymbol{V}_{k-1},\boldsymbol{a}_{k-1}\big)\Big) + a_k z\big(\boldsymbol{V}_{k-1},\boldsymbol{a}_{k-1}\big)\tbinom{|V_k|}{a_k}$$

which is (12). □

Claim. $Z_\pi(k)\in\mathcal{Z}(\boldsymbol{V}_k,\boldsymbol{a}_k)$.

Proof. Fix $(A_1,\dots,A_k)\in\binom{V_1}{a_1}\times\cdots\times\binom{V_k}{a_k}$. Some $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_{k-1})\in A_1\times\cdots\times A_{k-1}$ satisfies

$$\{\alpha_1,\dots,\alpha_{k-1}\}\notin Z_{\pi(A_k)}(k-1)$$

as the latter avoids $\boldsymbol{a}_{k-1}$ (cf. (16)). We seek $\alpha_k\in A_k\setminus N_{Z_\pi(k)}(\boldsymbol{\alpha})$ $\big($cf. (18)$\big)$. For $\{\alpha_1,\dots,\alpha_{k-1}\}\in R(k-1)$, pick $\alpha_k\in A_k\setminus Y_\pi$. For $\{\alpha_1,\dots,\alpha_{k-1}\}\in Z_{\pi(A'_k)}(k-1)$ with $A'_k\in\binom{V_k}{a_k}\setminus\{A_k\}$, pick $\alpha_k\in A_k\setminus A'_k$. □

Lemma 4.2. *$Z_\pi(k)$ and $Z_{\pi'}(k)$ are edge-disjoint.*

Proof. Fix $\boldsymbol{v}=(v_1,\dots,v_{k-1})\in V_1\times\cdots\times V_{k-1}$. We show $N_{Z_\pi(k)}(\boldsymbol{v})\cap N_{Z_{\pi'}(k)}(\boldsymbol{v})=\emptyset$ $\big($cf. (18)$\big)$. For $\{v_1,\dots,v_{k-1}\}\in R(k-1)$, these sets are Y_π and $Y_{\pi'}$ and are disjoint by $\pi\neq\pi'$. For $\{v_1,\dots,v_{k-1}\}\in Z_{\pi(A_k)}(k-1)=Z_{\pi'(A'_k)}(k-1)$ with $A_k,A'_k\in\binom{V_k}{a_k}$, the equality $\pi(A_k)=\pi'(A'_k)$ holds in a respectful family $\Pi_k\subseteq\binom{V_k}{a_k}!$ with $\pi\neq\pi'$, so the a_k-sets $A_k=N_{Z_\pi(k)}(\boldsymbol{v})$ and $A'_k=N_{Z_{\pi'}(k)}(\boldsymbol{v})$ must be disjoint. □

5 Proof of Theorem 1.2: equality under (3)

Recall the hypothesis (3): $|V_i|\geq a_i\binom{|V_{i+1}|}{a_{i+1}}+a_i^2$ for all $1\leq i\leq k-1$. Under (3), we show that there exists $Z(k)\in\mathcal{Z}(\boldsymbol{V}_k,\boldsymbol{a}_k)$ where $|Z(k)|$ is the upper bound of (2). The proof here is the case $r_k=1$ in Section 4. However, for that we may simply take $\Pi_k=\{\iota_k\}$, where $\iota_k\in\binom{V_k}{a_k}!$ is the identity mapping. Here, Π_k is respectful by default so no appeal to Baranyai's theorem is needed. As such, (10) and (13) are unnecessary so we may remove the condition $|V_k|\geq a_k+a_k^2$ from Section 4. Note, moreover, that when $k=2$ in Section 4, the statement (i) needs only $|V_1|\geq(a_1-1)\binom{|V_2|}{a_2}$ rather than the stronger $|V_1|\geq a_1\binom{|V_2|}{a_2}+a_1^2$ of (3). In other words, this relaxation recovers Čulík's result (Theorem 1.1).

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