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Classroom notes

Real-world examples of applications of vertex coloring and edge betweenness centrality

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Abstract. Examples of real-world applications of mathematics are rare in undergraduate mathematics and statistics curricula. In this paper, we present applications of vertex coloring and edge betweenness centrality, subjects that appear in courses in graph theory and complex networks. To show students the relevance of these topics in a real-world environment, we provide applications to existing networks.

1 An application of vertex coloring: the channel assignment problem

In this section we present an application of vertex coloring to the channel assignment problem.

1.1 Background and motivation for the channel assignment problem

The FM radio station KKLS broadcasts in Sioux Falls, South Dakota, on frequency 104.7. No other radio station in the area can use this frequency, or its signal would interfere with that of KKLS. However, this frequency

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can be re-used by radio stations that are sufficiently far from Sioux Falls. In fact, this frequency is also used by radio station K284BA in Rapid City, South Dakota, which is approximately 325 miles from Sioux Falls.

The Channel Assignment Problem seeks the fewest number of frequencies (or channels) that can be assigned to a collection of radio stations in a specified geographic area, without introducing signal interference. We begin by showing how this problem can be modeled using vertex coloring in graphs.

1.2 An application to real-world data

Assume that there are six radio stations to which frequencies must be assigned, and that the distances between pairs of cities are given in Table 1.1. According to [8] two Class A radio stations with towers no taller than 100 meters can use the same frequency if they are located at least 65 miles apart. The Channel Assignment Problem seeks the fewest number of frequencies that can be used by the six given stations.

Table 1.1: Distances between cities (in miles).

	WBZO Bay Shore	WJGK Newburgh	WGY-FM Albany	New Rochelle	North Salem	Hudson
WBZO Bay Shore		66.913	135.644	30.903	45.42	109.264
WJGK Newburgh	66.913		80.372	42.573	25.563	52.995
WGY-FM Albany	135.644	80.372		120.199	91.474	27.618
New Rochelle	30.903	42.573	120.199		31.245	92.673
North Salem	45.42	25.563	91.474	31.245		64.43
Hudson	109.264	52.995	27.618	92.673	64.43	

We model this problem using a *conflict graph*, developed from the table of distances between cities obtained using [9], and shown in Figure 1.1. Each radio station is represented by a vertex in the graph, and an edge exists between two vertices if the distance between the corresponding radio stations is less than or equal to 65 miles.

The solution to the Channel Assignment Problem for these six radio stations is precisely the solution to the vertex coloring problem, applied to this conflict graph. Vertex coloring is a well-known NP-complete problem in graph theory. As such, there exists no efficient algorithm for solving this problem (on general graphs) in polynomial time. Instead, it is typical to seek bounds on the graph's chromatic number, which is the fewest number

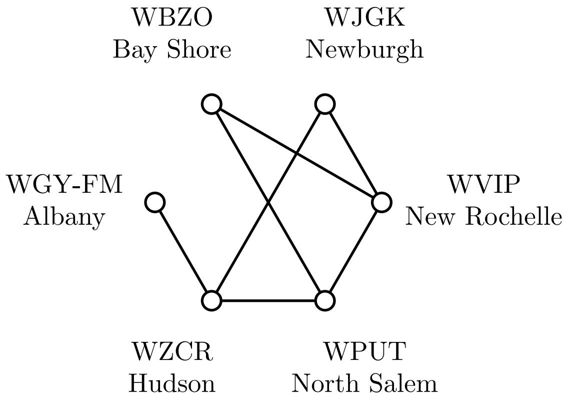

Figure 1.1: A conflict graph.

of colors in a vertex coloring of the graph. However, for this particular graph, we can precisely solve the vertex coloring problem by first observing the presence of a triangle, which means the chromatic number is at least three, and then finding a vertex coloring that uses three colors. The coloring in Figure 1.2 is one such coloring.

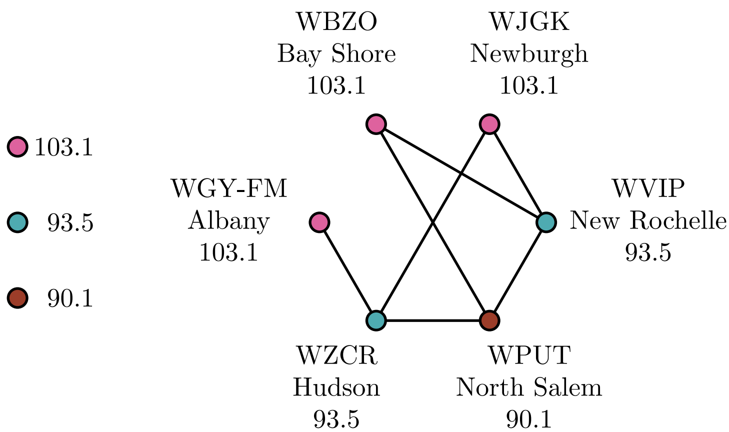

Figure 1.2: Coloring of conflict graph with frequencies.

Thus, one solution to the Channel Assignment Problem for these six radio stations is to assign one frequency to WGY-FM Albany, WBZO Bay Shore, and WJGK Newburgh; a second frequency to WZCR Hudson and WVIP New Rochelle; and a third frequency to WPUT North Salem. In fact, this solution is consistent with the actual channel assignments for these radio stations in the real-world. Stations WGY-FM Albany, WBZO Bay Shore, and WJGK Newburgh all broadcast on frequency 103.1; WZCR Hudson and WVIP New Rochelle broadcast on frequency 93.5; and WPUT North Salem broadcasts on frequency 90.1.

This assignment minimizes the total number of frequencies without introducing any signal interference between the radio stations.

The U.S. Federal Communications Commission (FCC) is responsible for regulating communications through radio, television, cable, satellite, and wire transmissions. They maintain a list of all radio stations in the U.S.; this list is publicly available at [8] and includes for each station, a class designation and frequency. Additionally, the Legal Information Institute of Cornell Law School [10] provides minimum permissible distances between pairs of FM radio stations, for the various station classes. More information on minimum distances can be found on the FCC website [7].

Exercise 1.1. Solve the Channel Assignment Problem for the "Class B" FM radio stations in New Hampshire.

Solution: We seek the minimum number of frequencies that can be assigned to the set of "Class B" FM radio stations in New Hampshire. According to the FCC website, there are seven such radio stations located in six cities (with two stations located in Manchester); Table 1.2 gives the distances between each city pair, where distances are calculated "as the crow flies" [9].

Table 1.2: New Hampshire stations (distance in miles).

	Claremont	Concord	Dover	Keene	Manchester	Portsmouth
Claremont		42	74	30	51	82
Concord	42		33	42	15	41
Dover	74	33		73	32	10
Keene	30	42	73		41	77
Manchester	51	15	32	41		35
Portsmouth	82	41	10	77	35	

According to the website from Cornell Law School, there must be at least 150 miles between "Class B" stations that share the same frequency, else their signals will interfere. Since the distance between each pair of cities is less than 150 miles, the conflict graph is a complete graph, and thus, a vertex coloring requires a different color for each vertex. That is, each radio station requires a different frequency. We note that on the FCC website, each of the seven New Hampshire “Class B” FM stations indeed has a different frequency.

Exercise 1.2. Solve the Channel Assignment Problem for the “Class A” FM radio stations in South Dakota.

Solution: We seek the minimum number of frequencies that can be assigned to the set of “Class A” FM radio stations in South Dakota. According to the FCC website, there are 23 such radio stations located in 18 cities (with two stations located in each of Aberdeen, Rapid City, and Spearfish, and three stations in Sioux Falls). According to website [10], there must be at least 71 miles between “Class A” stations that share the same frequency, else their signals will interfere. However, that value does not reflect shorter towers. Shorter towers have a shorter range and thus can be placed closer together than taller towers. See [7] for additional details, including that “Class A” stations located at least 65 miles apart can share the same frequency. We therefore use 65 as the minimum distance threshold in establishing conflicts. Table 1.3 gives the distances between each city pair and highlights those city pairs with conflicts in the corresponding conflict graph.

There are two different approaches we can take to solve this problem. In the first approach, we start by constructing the conflict graph for the set of 18 cities (rather than 23 stations). We observe that this graph contains several triangles as subgraphs. Thus, the chromatic number of the graph (of 18 cities) is at least three. In fact, there exists a three-coloring, as shown in the below table.

Color	City
1	Aberdeen, Brandon, Loomis, Onida
2	Brookings, Fort Pierre, Hot Springs, Ipswich
2	Lake Andes, Sisseton, Spearfish, Vermillion
3	Miller, Pierre, Rapid City, Sioux Falls, Wagner, Watertown

We can then use the above solution for 18 cities to produce an assignment for the 23 radio stations. We observe that we can use existing labels for the second station in Aberdeen, the second station in Rapid City, and the second station in Spearfish. However, the two additional stations in Sioux Falls each require a new color, since all three Sioux Falls stations must have different colors, and since none of those colors can duplicate the two

Table 1.3: South Dakota stations (distances in miles).

	Aberdeen	Brandon	Brookings	Fort Pierre	Hot Springs	Ipswich	Lake Andes	Loomis	Miller	Onida	Pierre	Rapid City	Sioux Falls	Sisseton	Spearfish	Vermillion	Wagner	Watertown
Aberdeen		159	111	120	283	26	159	116	69	93	118	251	158	70	271	201	165	78
Brandon	159		55	195	345	176	103	77	135	189	195	333	8	145	366	59	93	94
Brookings	111	55		177	339	132	121	77	109	163	176	319	57	90	359	111	117	39
Fort Pierre	120	195	177		167	100	123	119	69	29	2	143	190	186	172	204	136	165
Hot Springs	283	345	339	167		260	249	270	235	191	168	46	338	352	76	333	262	331
Ipswich	26	176	132	100	260		160	123	64	72	99	226	173	97	245	212	167	101
Lake Andes	159	103	121	123	249	160		49	97	131	123	243	95	188	281	86	14	140
Loomis	116	77	77	119	270	123	49		67	116	118	256	72	139	289	92	50	91
Miller	69	135	109	69	235	64	97	67		55	68	212	131	123	240	158	105	96
Onida	93	189	163	29	191	72	131	116	55		27	162	184	161	187	205	143	145
Pierre	118	195	176	2	168	99	123	118	68	27		144	189	184	173	203	136	163
Rapid City	251	333	319	143	46	226	243	256	212	162	144		326	322	42	328	256	306
Sioux Falls	158	8	57	190	338	173	95	72	131	184	189	326		147	360	54	85	95
Sisseton	70	145	90	186	352	97	188	139	123	161	184	322	147		342	199	189	53
Spearfish	271	366	359	172	76	245	281	289	240	187	173	42	360	342		366	294	332
Vermillion	201	59	111	204	333	212	86	92	158	205	203	328	54	199	366		72	147
Wagner	165	93	117	136	262	167	14	50	105	143	136	256	85	189	294	72		139
Watertown	78	94	39	165	331	101	140	91	96	145	163	306	95	53	332	147	139	

(distinct) colors assigned to Brandon and Brookings, as we observe that the triplet of Brandon, Brookings, and Sioux Falls forms a complete subgraph in the original graph. The table below shows a coloring/labeling of the 23 stations, using five colors.

Color	City
1	Aberdeen-1, Brandon, Loomis, Onida, Sisseton-1, Rapid City-1
2	Brookings, Fort Pierre, Hot Springs, Ipswich, Lake Andes
2	Sisseton-2, Spearfish, Vermillion
3	Aberdeen-2, Miller, Pierre, Rapid City-2, Sioux Falls-1
3	Wagner, Watertown
4	Sioux Falls-2
5	Sioux Falls-3

An alternative solution approach starts with a conflict graph on 23 vertices, one for each radio station. We observe a complete graph composed of Brandon, Brookings, Sioux Falls-1, Sioux Falls-2, and Sioux Falls-3. This means that the chromatic number is at least five, and the five-coloring in the above table validates that five is in fact the minimum.

Thus, the minimum number of frequencies we can assign to these 23 stations in South Dakota is five. We note that the FCC website shows 16 distinct frequencies utilized for this collection of stations. We should acknowledge that while minimizing the number of frequencies is one possible objective around the assignment of channels, other considerations could influence the actual assignments, including station preferences for channels at one end or near the middle of the channel spectrum.

Exercise 1.3. Solve the Channel Assignment Problem for the "Class A" FM radio stations in South Dakota, under the assumption that stations located less than 71 miles apart cannot share the same frequency.

Solution: We seek the minimum number of frequencies that can be assigned to the set of "Class A" FM radio stations in South Dakota, assuming now a minimum distance of 71 miles between stations that share the same frequency. We can again utilize Table 1.3 while noting that the following city pairs now pose conflicts with respect to the revised distance threshold: Aberdeen - Miller, Fort Pierre - Miller, Loomis - Miller, and Miller - Pierre.

Once again, there are two different approaches we can take to solve this problem. In the first approach, we start by constructing the conflict graph for the set of 18 cities (rather than 23 stations). We observe that this graph contains as a subgraph, a complete graph with four vertices, by noting that Fort Pierre, Miller, Onida, and Pierre have pairwise conflicts. Thus, the chromatic number of the graph (of 18 cities) is at least four. In fact, there exists a four-coloring, as shown in the below table.

Color	City
1	Brookings, Fort Pierre, Lake Andes, Rapid City
1	Vermillion
2	Aberdeen, Hot Springs, Loomis, Pierre, Sioux Falls
3	Brandon, Ipswich, Onida, Sisseton, Spearfish, Wagner
4	Miller, Watertown

We can then use the above solution for 18 cities to produce an assignment for the 23 radio stations. We observe that we can use existing labels for the second station in Aberdeen, the second station in Rapid City, the second station in Spearfish, and one of the two additional stations in Sioux Falls. However, the final station in Sioux Falls requires a fifth color, since all three Sioux Falls stations must have different colors, and since none of those colors can duplicate the two (distinct) colors assigned to Brandon and Brookings, as we observe that the triplet of Brandon, Brookings, and Sioux Falls forms a complete subgraph in the original graph. The table below shows a coloring/labeling of the 23 stations, using five colors.

Color	City
1	Aberdeen-1, Brookings, Fort Pierre, Lake Andes
1	Rapid City-1, Vermillion
2	Aberdeen-2, Hot Springs, Loomis, Pierre, Sioux Falls-1
3	Brandon, Ipswich, Onida, Sisseton, Spearfish-1, Wagner
4	Miller, Watertown, Rapid City-2, Sioux Falls-2
5	Sioux Falls-3, Spearfish-2

An alternative solution approach starts with a conflict graph on 23 vertices, one for each radio station. We observe a complete graph composed of Brandon, Brookings, Sioux Falls-1, Sioux Falls-2, and Sioux Falls-3. This means that the chromatic number is at least five, and the five-coloring from the above table validates that five is in fact the minimum. Thus, the minimum number of frequencies we can assign to these 23 stations in South Dakota is five.

2 Edge betweenness centrality

In this section we use the metric of edge betweenness centrality to analyze the network of National LambdaRail.

2.1 Background and motivation for edge betweenness centrality

Betweenness centrality was introduced by Anthonisse [1] and Freeman [3] to identify vertices that are most important within a graph. Later Girvan and Newman introduced an analagous metric to identify the importance of an edge in a graph, known as edge betweenness centrality [5].

In a telecommunication network composed of physical spans that connect pairs of cities, certain spans play a particularly critical role in facilitating information flow in the network. This is quantified by edge betweenness centrality, which, in graph theoretic terms, is the frequency with which an edge appears on a shortest path between two distinct vertices in the graph. We can apply this concept to a telecommunication network by modeling the network as a graph, where each edge of the graph represents a physical span, and each vertex represents a city in the network.

We begin with a definition of edge betweenness centrality. Next, we compute the edge betweenness centrality for some simple graphs, and we derive general results for several classes of well-known graphs. Then, we explore the application of edge betweenness centrality to the fiber-optic network of National LambdaRail and draw connections to a related concept of betweenness centrality defined for vertices.

Definition 2.1. The *betweenness centrality* of an edge e in a graph G is denoted $c_B(e)$ and measures the frequency with which e appears on a shortest path between pairs of vertices in G. Let V be the set of vertices of G, and let e be an edge in G. Let $\sigma(x,y)$ be the number of shortest paths between distinct vertices x and y in G, and let $\sigma(x,y|e)$ be the number of shortest paths between x and y that contain e. Then

$$c_B(e) = \sum_{(x,y)\in V} \frac{\sigma(x,y|e)}{\sigma(x,y)}$$

Betweenness centrality of an edge can be defined more simply as the number of shortest paths in a graph that go through a particular edge. Betweenness centrality can be applied to both weighted and unweighted graphs.

We next present a series of exercises for students that cover elementary properties of edge betweenness. For these exercises, we use the definition of betweenness centrality as presented in Definition 2.1, and we assume that all graphs are unweighted, so that a shortest path between two vertices is a path with the fewest number of edges.

Exercise 2.2. For every edge e in a graph G, show that $c_B(e) \geq 1$.

Solution: If x and y are the vertices representing the endpoints of e, then there is a shortest path between x and y that contains e, namely e itself, hence $c_B(e)$ is at least one.

Exercise 2.3. Compute the betweenness centrality for the edge e with endpoints w and x in Figure 2.1.

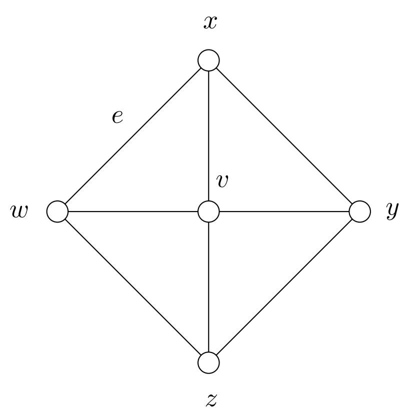

Figure 2.1: A graph for Exercise 2.3.

Solution: The pairs of vertices that include edge e on a shortest path between them are as follows: $(w, x), (w, y), (x, z)$. Thus

$$c_B(e) = \frac{\sigma(w, x|e)}{\sigma(w, x)} + \frac{\sigma(w, y|e)}{\sigma(w, y)} + \frac{\sigma(x, z|e)}{\sigma(x, z)} = 1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$$

We next consider the edge betweenness centrality for some well-known classes of graphs.

Exercise 2.4. For an edge e in a complete graph G, what is $c_B(e)$?

Solution: Since G is complete, there is an edge between every pair of vertices. Thus, for every pair of vertices, there is exactly one shortest path between them, and that shortest path is the edge connecting them. Thus, $c_B(e) = 1$ for every edge e in a complete graph.

Exercise 2.5. For an edge e in a chordless cycle G with three vertices, what is $c_B(e)$?

Solution: A chordless cycle with three vertices is a complete graph, so as shown in Exercise 2.3, $c_B(e) = 1$ for every edge e in G.

Exercise 2.6. If G is a chordless cycle with at least four vertices, show that $c_B(e) \geq 2$ for every edge e in G.

Solution: We consider first the case where G is a chordless cycle with exactly four vertices. Let e be a edge in G with endpoints x and y. Let w be the vertex of G adjacent to x but different from y, and let z be the vertex of G adjacent to y but different from x. Then

$$\begin{aligned} c_B(e) &= \frac{\sigma(x,y|e)}{\sigma(x,y)} + \frac{\sigma(x,z|e)}{\sigma(x,z)} + \frac{\sigma(x,w|e)}{\sigma(x,w)} + \frac{\sigma(y,z|e)}{\sigma(y,z)} + \frac{\sigma(y,w|e)}{\sigma(y,w)} + \frac{\sigma(w,z|e)}{\sigma(w,z)} \\ &= 1 + 0.5 + 0 + 0 + 0.5 + 0 = 2. \end{aligned}$$

Now, we consider the case where G is a chordless cycle with at least five vertices. Let e be a edge in G with endpoints x and y. Let w be a vertex of G adjacent to x but different from y. Then there are at least two pairs of vertices, namely (x, y) and (w, y), each of which has exactly one shortest path in G, and that shortest path includes e. Hence, $c_B(e) \geq 2$.

Recall that an edge is a *bridge* or *cut-edge* in a graph G if its removal disconnects the graph.

Exercise 2.7. Let e be a bridge in a graph G. Let H be the disconnected graph resulting from the removal of e from G. Let V_1 and V_2 be the vertex sets of each of the connected components of H. Show that $c_B(e)$, the edge betweenness centrality of e in G, is $|V_1||V_2|$; that is, the product of the cardinalities of vertex sets V_1 and V_2.

Solution: For every vertex pair $(x, y) \in G$, where $x \in V_1$ and $y \in V_2$, e is on every shortest path between x and y in G. Moreover, e is on no shortest path between vertices that are either both in V_1 or both in V_2. The total number of vertex pairs in G with one endpoint in V_1 and one endpoint in V_2 is $|V_1||V_2|$. Hence, $c_B(e) = |V_1||V_2|$.

2.2 A real-world application of edge betweenness centrality

A network of National LambdaRail (NLR) is shown in Figure 2.2.

Figure 2.2: National LambdaRail FrameNet map.

We can model this network as a weighted graph, where the city office locations are vertices, and the physical spans connecting them are edges. We assign to each edge a weight equal to the length of the corresponding span in the NLR network. We can then use MatlabBGL [4] to calculate the betweenness centrality for each edge in the graph, and in so doing, determine which physical spans of the network are likely to have the most or least traffic passing through them.

Table 2.1 gives the edge betweenness centralities calculated by MatlabBGL, using the simpler definition that counts the number of shortest paths going through each network edge.

Table 2.1: Edge betweenness centralities for the NLR network.

	SEAT	DENV	SUNN	LOSA	ELPA	ALBU	KANS	HOUS	TULS	BATO	CHIC	JACK	ATLA	RALE	WASH	PHIL	NEWY	BOST	PITT	CLEV
SEAT	0	16	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
DENV	16	0	0	0	0	29	52	0	0	0	0	0	0	0	0	0	0	0	0	0
SUNN	3	0	0	20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LOSA	0	0	20	0	35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ELPA	0	0	0	35	0	26	0	32	0	0	0	0	0	0	0	0	0	0	0	0
ALBU	0	29	0	0	26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
KANS	0	52	0	0	0	0	0	0	25	0	64	0	0	0	0	0	0	0	0	0
HOUS	0	0	0	0	32	0	0	0	18	39	0	0	0	0	0	0	0	0	0	0
TULS	0	0	0	0	0	0	25	18	0	0	0	0	0	0	0	0	0	0	0	0
BATO	0	0	0	0	0	0	0	39	0	0	0	36	0	0	0	0	0	0	0	0
CHIC	0	0	0	0	0	0	64	0	0	0	0	0	14	0	0	0	0	10	0	39
JACK	0	0	0	0	0	0	0	0	0	36	0	0	41	0	0	0	0	0	0	0
ATLA	0	0	0	0	0	0	0	0	0	0	14	41	0	42	0	0	0	0	0	0
RALE	0	0	0	0	0	0	0	0	0	0	0	0	42	0	35	0	0	0	0	0
WASH	0	0	0	0	0	0	0	0	0	0	0	0	0	35	0	41	0	0	33	0
PHIL	0	0	0	0	0	0	0	0	0	0	0	0	0	0	41	0	26	0	0	0
NEWY	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	26	0	9	0	0
BOST	0	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	9	0	0	0
PITT	0	0	0	0	0	0	0	0	0	0	0	0	0	0	33	0	0	0	0	36
CLEV	0	0	0	0	0	0	0	0	0	0	39	0	0	0	0	0	0	0	36	0

We observe that the span between Kansas City (KANS) and Chicago (CHIC) has the greatest betweenness centrality; other spans with relatively high betweenness centralities include Kansas City (KANS) to Denver (DENV), Atlanta (ATLA) to Raleigh (RALE), Jacksonville (JACK) to Atlanta (ATLA), and Washington DC (WASH) to Philadelphia (PHIL). Spans with the lowest betweenness centralities are Seattle (SEAT) to Sunnyvale (SUNN) and New York (NEWY) to Boston (BOST). A disruption to spans with high betweenness centrality could have a significant impact on the overall connectivity and efficiency of the network.

Exercise 2.8. Refer to the map in Figure 2.2. Suppose the span between Kansas City and Chicago becomes severed and cannot be used. Use the result from Exercise 2.7 to compute the edge betweenness centrality of the span between Houston and Baton Rouge.

Solution: If the span between Kansas City and Chicago becomes severed, then the span between Houston and Baton Rouge becomes a bridge, and its edge betweenness centrality is the product of the number of cities in the two components of the network that result from the removal of the span between Houston and Baton Rouge. Hence, the edge betweenness centrality of the span between Houston and Baton Rouge is $11 \times 9 = 99$.

2.3 A connection to vertex betweenness centrality

A related concept to edge betweenness centrality is *vertex betweenness centrality*, which measures the frequency with which (or number of times that) a given vertex in a graph appears on a shortest path between pairs of vertices in the graph. In [2] we applied this concept to the National LambdaRail network to identify cities that are most critical for communication between other cities. In that paper, it was determined that the two cities with the highest vertex betweenness centralities are Kansas City and Chicago, and those with the lowest are Seattle and Boston. We immediately observe the correlation between edges with high (or low) edge betweenness centralities, and their endpoints having similarly high (or low) vertex betweenness centralities. In fact, for the NLR network, there is a strong correlation between the vertex and edge betweenness centralities, in general. This is attributed to the network's structure, namely a union of cycles. If a graph is a cycle, then there is a one-to-one correspondence between its vertices and edges. A graph composed of a union of cycles would have similar vertex and edge betweenness centralities. We can see this more clearly when we count the number of occurrences of a vertex among the edges that appear along shortest paths in the calculation of the edge betweenness centralities. This can be achieved by summing the rows from the Edge Betweenness Centrality matrix (see Table 2.2); the resulting values are in the "SUM" column. The column "vbc" gives the betweenness centrality for each vertex (city); we observe the strong correlation between the SUM and vbc values.

3 Future directions

Edge betweenness centrality can be applied to other types of networks. In a social network, where each edge represents a person, edges with high betweenness centrality may represent individuals who could effectively connect different social groups. That is, it may be valuable to tap into these individuals for interventions or other strategies to expand communications between different groups comprising the network. In biological networks, such as protein-protein interaction networks (see [6]), edges with high betweenness centrality can represent particularly important interactions.

Variations of the Channel Assignment Problem could include considerations of additional variables that would influence the assignment of frequencies to stations. For example, it is likely the case that frequencies in relatively

Table 2.2: Edge betweenness centrality and vertex betweenness centrality values.

City	SUM	vbc
KANS	141	122
CHIC	127	108
WASH	109	90
DENV	97	78
ATLA	97	78
ELPA	93	74
HOUS	89	70
JACK	77	58
RALE	77	58
BATO	75	56
CLEV	75	56
PITT	69	50
PHIL	67	48
LOSA	55	36
ALBU	55	36
TULS	43	24
NEWY	35	16
SUNN	23	4
SEAT	19	0
BOST	19	0
Correlation		1

close proximity must differ by a certain amount (for example two stations in close proximity may not be able to broadcast on 97.9 and 98.1). Another point of note is that in our paper, we have used distances between cities, but it may be more precise to use the distances between the radio towers, if this information were publicly available.

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