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Some matrix constructions of non-symmetric regular group divisible designs

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Abstract. Saurabh and Sinha (*Bull. Inst. Combin. Appl.* **95** (2022) and *Bull. Inst Combin. Appl.* **97** (2023)) obtained solutions of L_2-type designs, semi-regular group divisible and symmetric regular group divisible designs in the range of $r, k \leq 10$ using certain combinatorial matrices. Here by using matrix approaches, solutions of non-symmetric regular group divisible (RGD) designs listed in (Clatworthy, Tables of two–associate–class partially balanced designs, U.S. Department of Commerce, National Bureau of Standards, Washington, DC Report No. NBS-AMS-63, 1973) are obtained except for a few. As special case we obtain a series of μ-resolvable balanced incomplete block designs and quasidouble solutions of some RGD designs.

1 Introduction

Clatworthy [3] tabulated 110 semi-regular and 209 regular group divisible designs along with their solutions and resolvability status under the range of $r, k \leq 10$. Later Sinha [23] and Saurabh and Sinha [20] updated the table of group divisible designs. Saurabh and Sinha [17, 19] obtained solutions of L_2-type designs, semi-regular group divisible and symmetric regular group divisible designs in the range of $r, k \leq 10$ using certain combinatorial matrices.

Here solutions of the non-symmetric regular group divisible (RGD) designs listed in Clatworthy [3] are obtained using matrix approaches except few. As special case we obtain a series of μ-resolvable balanced incomplete block

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design and quasidouble solutions of some RGD designs. Future work will be done on the matrix solutions of the remaining non-symmetric RGD designs listed in Clatworthy [3], Sinha [23] and Saurabh and Sinha [20]. Some relevant definitions in the context of the paper are as follows:

Let $v = mn$ elements be arranged in an $m \times n$ array. An RGD *design* is an arrangement of the $v = mn$ elements in b blocks each of size k such that:

1. every element occurs at most once in a block;
2. every element occurs in r blocks;
3. every pair of elements which are in the same row of the $m \times n$ array occur together in λ_1 blocks whereas every remaining pair of elements occur together in λ_2 blocks and
4. $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$.

The integers $v = mn, b, r, k, \lambda_1$ and λ_2 are known as parameters of the GD design and they satisfy the relations:

$$bk = vr \quad \text{and} \quad (n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1).$$

Let N be the incidence matrix of a GD design then the structure of NN' is given as (see Saurabh and Sinha [19] for GD association schemes):

(i) $NN' = (r - \lambda_1)(I_m \otimes I_n) + (\lambda_1 - \lambda_2)(I_m \otimes J_n) + \lambda_2(J_m \otimes J_n)$ or

(ii) $NN' = (r - \lambda_2)(I_n \otimes I_m) + \lambda_2 (J_n \otimes J_m) + (\lambda_1 - \lambda_2)\{(J_n - I_n) \otimes I_m\}$.

If the incidence matrix N of a block design $D(v, b, r, k)$ may be partitioned in to submatrices as: $N = (N_1|N_2|\cdots|N_t)$ where each $N_i (1 \leq i \leq t)$ is a $v \times \frac{v\mu}{k}$ matrix such that each row sum of N_i is μ then the design is μ-resolvable.

A generalized Bhaskar Rao design GBRD$(v, b, r, k, \lambda; G)$ over a group G is a $v \times b$ array with entries from $G \cup \{0\}$ such that:

1. each row has exactly r group element entries;
2. each column has exactly k group element entries;
3. for each pair of distinct rows $(x_1, x_2, \ldots, x_b)$ and $(y_1, y_2, \ldots, y_b)$, the multi-set $\{x_i y_i^{-1} : i = 1, 2, \ldots, b;\ x_i, y_i \neq 0\}$ contains each group element exactly $\frac{\lambda}{|G|}$ times.

When $|G| = 2$, such a design is known as Bhaskar Rao design. A *difference matrix* $D(k, \lambda g; G)$ over a group G of order g is a $\mathsf{GBRD}(k, \lambda g, \lambda g, k, \lambda g; G)$ i. e. difference matrices are precisely GBRDs with non-zero entries. Further for $k = \lambda g$, the difference matrix is said to be *generalized Hadamard matrix*, $\mathsf{GH}(\lambda g; G)$ over G of order λg with index λ, see de Launey [5].

Further replacing the group entries by 1 and leaving the others 0 in a $\mathsf{GBRD}(v, b, r, k, \lambda; G)$, we obtain the incidence matrix of a $\mathsf{BIBD}(v, b, r, k, \lambda)$. Since for a $\mathsf{BIBD}(v, b, r, k, \lambda)$ it is well known that $bk = vr$ and $r(k-1) = \lambda(v-1)$, a $\mathsf{GBRD}(v, b, r, k, \lambda; G)$ is denoted by $\mathsf{GBRD}(v, k, \lambda; G)$.

Notations: I_n is the identity matrix of order n, $J_{v\times b}$ is the $v \times b$ matrix all of whose entries are 1 and $J_{v\times v} = J_v$, A' is the transpose of matrix A, $A \otimes B$ is the Kronecker product of two matrices A and B, $0_{m\times n}$ is a zero matrix of order $m \times n$ and e_n is an $n \times 1$ column matrix with all entries 1. A $(0, 1)$-matrix: $\alpha = \text{CIRC}\ (0\ 1\ 0 \ldots 0)$ is a permutation circulant matrix of order n such that $\alpha^n = I_n$. For details on circulant matrices, see Davis [4]. SRX and RX numbers are from Clatworthy [3]. The design number R$X(a/b/c\ldots)$ occurs between RX and $\mathsf{R}(X+1)$, see Freeman [10] and Dey [8]. Also $m\#X$ denotes m-multiple of the design number X.

$EA\,(p^n) \approx C_p \times C_p \times \cdots \times C_p$ (n times) denotes the elementary abelian group of order p^n and $C_p = EA(p)$ is a cyclic group of order p where p is a prime. $S_n, A_n$and D_n are permutation, alternating and dihedral groups respectively. For the definition and construction methods of balanced incomplete block design (BIBD) or a 2-(v, k, λ) design, see Dey [9].

2 Earlier constructions

Replacing the elements of a group G of order g by the corresponding $g \times g$ permutation matrices and 0 entry by the $g \times g$ null matrix in a $\mathsf{GBRD}(v, b, r, k, \lambda; G)$ we obtain:

Theorem 2.1 (Gibbons and Mathon [11]).
The existence of a $\mathsf{GBRD}(v, b, r, k, \lambda; G)$ *over a group* G *implies the existence of a* GD *design with parameters:*

$$v^* = vg,\quad b^* = bg,\ r^* = r,\quad k^* = k,\ \lambda_1 = 0,\ \lambda_2 = \frac{\lambda}{g},\ m = v,\ n = g.$$

Further replacing the elements of a group $G(= D_n/S_n/A_n)$ by the corresponding $n \times n$ permutation matrices and 0 entry by $n \times n$ null matrix in $\mathsf{GBRD}(v, b, r, k, \lambda; G)$ we obtain:

Theorem 2.2 (Saurabh and Sinha [18]).
The existence of a $\mathsf{GBRD}(v, b, r, k, \lambda; D_n/S_n/A_n)$ *implies the existence of a* GD *design with parameters:*

$$v^* = nv,\ b^* = nb,\ r^* = r,\ k^* = k,\ \lambda_1 = 0,\ \lambda_2 = \tfrac{\lambda}{n},\ m = v,\ n \geq 3.$$

Theorem 2.3 (Saurabh, Sinha and Singh [16]).
There exists a GD *design with parameters:*

$$\begin{aligned} &v^* = vs,\ b^* = stv,\ r^* = t(k+s-1),\ k^* = k+s-1,\\ &\lambda_1 = (s-2)t,\ \lambda_2 = \lambda,\ m = v,\ n = s, m \geq 2,\ s \geq 2,\ t = \tfrac{r}{\alpha}, \end{aligned}$$

where v, k, λ *are the parameters of an* α*-resolvable* BIBD *with*

$$\lambda = \frac{t[(k+s-1)(k+s-2)-(s-1)(s-2)]}{s(v-1)}.$$

Theorem 2.4 (Saurabh and Prasad [15]).
The existence of a BIBD *with parameters:* v', r', k', b', λ' *implies the existence of a* RGD *with parameters:* $v = 3v'$, $r = b' + r'$, $k = v' + k'$, $b = 3b'$, $\lambda_1 = \lambda' + b'$, $\lambda_2 = r'$, $m = 3$, $n = v$.

3 The constructions

Theorem 3.1. *There exists a resolvable* RGD *design with parameters:*

$$v = q^2,\ b = q(qt+s),\ r = qt+s,\ k = q,\ \lambda_1 = s,\ \lambda_2 = t,\ m = n = q,\ (1)$$

where q *is a prime or prime power.*

Proof. Let $\mathcal{G}$ be an elementary abelian group of order q. It is well known that a $\mathsf{GH}(q; \mathcal{G})$ always exists (see de Launey [7]). Further let M be a matrix obtained by deleting the first column of a normalized $\mathsf{GH}(q; \mathcal{G})$ and let M_1 be a (0, 1)-block matrix obtained by replacing the group entries of M by the corresponding $q \times q$ permutation matrices.

Let N_1 be a block matrix obtained by taking t copies of M_1. Consider the following block matrices:

(i) $N_2 = t$ copies of the block matrix $e_q \otimes I_q$ of order $q^2 \times q$ arranged column-wise and

(ii) $N_3 = s$ copies of the block matrix $e_q \otimes \Gamma_q$ of order $q^2 \times q$ arranged column-wise where $\Gamma_i (1 \leq i \leq q)$ is a $q \times q$ matrix whose i^{th} column contains only 1's and 0 elsewhere.

Further let $N = [N_1|N_2|N_3]$. Then

$$NN' = qt(I_q \otimes I_q) + (s-t)(I_q \otimes J_q) + t(J_q \otimes J_q).$$

Hence N represents a RGD design with parameters (1). Since each row sum of the block matrices $e_q \otimes I_q, e_q \otimes \Gamma_q$ and $q \times q$ permutation matrices are one, the design is resolvable. □

Example 3.2. Consider a generalized Hadamard matrix $\mathsf{GH}(4; EA(4))$ over the elementary abelian group $EA(4) = C_2 \times C_2 = \{1, a, b, c\}$, where $a^2 = b^2 = c^2 = 1$, $ab = ba = c$, $ac = ca = b$, and $bc = cb = a$. Then for $s = 2$, $t = 1$ we obtain a resolvable RGD R118 design with parameters: $v = 16$, $r = 6$, $k = 4$, $b = 24$, $\lambda_1 = 2$, $\lambda_2 = 1$, $m = n = 4$ and incidence matrix N given by:

$$N = \begin{pmatrix} \Gamma_1 & \Gamma_1 & I_4 & I_4 & I_4 & I_4 \\ \Gamma_2 & \Gamma_2 & I_4 & I_2 \otimes (J-I)_2 & (J-I)_2 \otimes I_2 & (J-I)_2 \otimes (J-I)_2 \\ \Gamma_3 & \Gamma_3 & I_4 & (J-I)_2 \otimes (J-I)_2 & I_2 \otimes (J-I)_2 & (J-I)_2 \otimes I_2 \\ \Gamma_4 & \Gamma_4 & I_4 & (J-I)_2 \otimes I_2 & (J-I)_2 \otimes (J-I)_2 & I_2 \otimes (J-I)_2 \end{pmatrix}.$$

Theorem 3.3. *The existence of a* BIBD *with parameters:*

$$v', b', r', k' = 2, \lambda$$

implies the existence of RGD *designs with parameters for* $s, t \geq 1$*:*

$$\text{(a)} \quad \begin{array}{l} v = 2v',\ b = 2sb' + t(v')^2,\ r = sr' + v't, \\ k = 2,\ \lambda_1 = s\lambda,\ \lambda_2 = t,\ m = 2,\ n = v' \end{array} \tag{2}$$

$$\text{(b)} \quad \begin{array}{l} v = 2v',\ b = 2sb' + v'(t + s\,(v'-1)),\ r = s(r' + \lambda\,(v'-1)) + t, \\ k = 2, \lambda_1 = t,\ \lambda_2 = s\lambda,\ m = v',\ n = 2. \end{array} \tag{3}$$

Proof. Let M be the incidence matrix of the BIBD with parameters:

$$v', b', r', k' = 2, \lambda.$$

Consider the following block matrices:

1. $N_1 = s$ copies of block matrix $I_2 \otimes M$ arranged columnwise;

2. $N_2 = t$ copies of block matrix $e_2 \otimes I_{v'}$ arranged columnwise and
3. $N_3 = t$ copies of block matrix: $L = \begin{pmatrix} I_{v'} & I_{v'} & I_{v'} & \cdots & I_{v'} \\ \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{v'-1} \end{pmatrix}$, where $\alpha = \textsc{circ}(0\ 1\ 0 \ldots 0)$ is a permutation circulant matrix of order v'.

Let $N = [N_1|N_2|N_3]$. Then it may be verified that

$$NN' = (s(r' - \lambda) + v't)(I_2 \otimes I_{v'}) + t((J_2 - I_2) \otimes J_{v'}) + s\lambda(I_2 \otimes J_{v'}).$$

Hence N represents a RGD design with parameters (2).

Further let N_1, N_2 and L be same as above and $N_3 = s\lambda$ copies of L. Let $N = [N_1|N_2|N_3]$. Then it is easy to verify that

$$NN' = (s(r' + \lambda v' - 2\lambda) + t)(I_2 \otimes I_{v'}) + s\lambda(J_2 \otimes J_{v'}) + (t - s\lambda)\{(J_2 - I_2) \otimes I_{v'}\}.$$

Hence N represents a RGD design with parameters (3.7). □

Example 3.4. Consider a BIBD with parameters:

$$v' = 4, b' = 6, r' = 3, k' = 2, \lambda = 1.$$

Then for $s = 2, t = 1$, we obtain a RGD design R32 with parameters:

$$v = 8, r = 10, k = 2, b = 40, \lambda_1 = 2, \lambda_2 = 1, m = 2, n = 4$$

whose incidence matrix N is given as:

$$N = \left(\begin{array}{cc|cc|c|ccc} M & 0_{4\times6} & M & 0_{4\times6} & I_4 & I_4 & I_4 & I_4 \\ 0_{4\times6} & M & 0_{4\times6} & M & I_4 & \alpha & \alpha^2 & \alpha^3 \end{array}\right),$$

where M represents a BIBD with parameters:

$$v' = 4, b' = 6, r' = 3, k' = 2, \lambda = 1$$

and $\alpha = \textsc{circ}(0\ 1\ 0\ 0)$ is a permutation circulant matrix of order four.

Theorem 3.5. *There exists a $(p-1)$-resolvable* RGD *design with parameters:*

$$\begin{aligned} &v = p^2,\ r = (p-1)(s+tp),\ k = p-1, \\ &b = p^2(s+tp),\ \lambda_1 = s(p-2),\ \lambda_2 = t(p-2),\ m = n = p, \end{aligned} \tag{4}$$

where p is a prime.

Proof. Consider the following block matrices:

(i) $N_1 = s$ copies of block matrix $I_p \otimes (J-I)_p$ arranged columnwise;

(ii) $N_2 = t$ copies of block matrix $(J-I)_p \otimes I_p$ arranged columnwise and

(iii) $N_3 = t$ copies of block circulant matrix

$$(M_1|M_2|\cdots|M_{p-1}) = \\ \Big(\text{CIRC}(0_p\ \alpha\ \alpha^2 \cdots \alpha^{p-1})\Big|\text{CIRC}(0_p\ \alpha^2 (\alpha^2)^2\ \cdots (\alpha^2)^{p-1})\Big|\cdots \\ \cdots\Big|\text{CIRC}(0_p\ \alpha^{p-1}\ (\alpha^{p-1})^2 \cdots (\alpha^{p-1})^{p-1})\Big)$$

arranged columnwise where $\alpha = \text{CIRC}(0\ 1\ 0\ 0 \ldots 0)$ is a permutation circulant matrix of order p.

Let $N = [N_1|N_2|N_3]$. Then

$$NN' = (s+tp(p-1))\,(I_p \otimes I_p) + (p-2)(s-t)\,(I_p \otimes J_p) + t(p-2)\,(J_p \otimes J_p).$$

Hence N represents a RGD design with parameters (4). Further since each row sum of the block matrices $I_p \otimes (J-I)_p$, $(J-I)_p \otimes I_p$ and $M_i (1 \le i \le p-1)$ is $(p-1)$, the design is $(p-1)$-resolvable. □

For $s = t = 1$ in Theorem 3.5, we obtain:

Corollary 3.6. *There exists a* $(p-1)$*-resolvable* BIBD *with parameters:*

$$v = p^2,\ r = p^2 - 1,\ k = p-1,\ b = p^2(p+1),\ \lambda = p-2.$$

Example 3.7. For $p = 3$, we obtain a 2-resolvable BIBD with parameters:

$$v = 9,\ r = 8,\ k = 2,\ b = 36,\ \lambda = 1$$

whose incidence matrix N is given below:

$$N = [N_1|N_2|N_3] \\ = \left[\begin{array}{ccc|ccc|cccccc} (J-I)_3 & 0_3 & 0_3 & 0_3 & I_3 & I_3 & 0_3 & \alpha & \alpha^2 & 0_3 & \alpha^2 & \alpha \\ 0_3 & (J-I)_3 & 0_3 & I_3 & 0_3 & I_3 & \alpha^2 & 0_3 & \alpha & \alpha & 0_3 & \alpha^2 \\ 0_3 & 0_3 & (J-I)_3 & I_3 & I_3 & 0_3 & \alpha & \alpha^2 & 0_3 & \alpha^2 & \alpha & 0_3 \end{array}\right].$$

Theorem 3.8. *There exists a* RGD *design with parameters:*

$$v = 3n,\ b = 3n^2,\ r = 2n,\ k = 2,\ \lambda_1 = 0,\ \lambda_2 = 1,\ m = 3,\ n. \qquad (5)$$

Proof. Let $\alpha = \textsc{circ}(0\ 1\ 0 \dots 0 \dots 0)$ be a permutation circulant matrix of order n. Consider the following block matrices:

$$\text{(i)}\ \ N_1 = \begin{pmatrix} 0_n & 0_n & 0_n & \cdots & 0_n & 0_n \\ \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{n-1} & I_n \\ I_n & I_n & I_n & \cdots & I_n & I_n \end{pmatrix};$$

$$\text{(ii)}\ \ N_2 = \begin{pmatrix} I_n & I_n & I_n & \cdots & I_n & I_n \\ 0_n & 0_n & 0_n & \cdots & 0_n & 0_n \\ \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{n-1} & I_n \end{pmatrix} \text{ and}$$

$$\text{(iii)}\ \ N_3 = \begin{pmatrix} \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{n-1} & I_n \\ I_n & I_n & I_n & \cdots & I_n & I_n \\ 0_n & 0_n & 0_n & \cdots & 0_n & 0_n \end{pmatrix}.$$

Further let $N = (N_1|N_2|N_3)$. Then we have

(i) Since each column sum of N is 2, the block size is $k = 2$;

$$\text{(ii)}\ \ NN' = (N_1|N_2|N_3)\begin{pmatrix} N_1' \\ N_2' \\ N_3' \end{pmatrix} = \begin{pmatrix} 2nI_n & J_n & J_n \\ J_n & 2nI_n & J_n \\ J_n & J_n & 2nI_n \end{pmatrix}$$

$$= 2n\,(I_3 \otimes I_n) + \{(J - I)_3 \otimes J_n\}.$$

Hence N represents a GD design with parameters (5). □

Example 3.9. Using Theorem 3.8 we obtain a solution of a RGD design R34: $v = 9,\ r = 6,\ \ k = 2,\ b = 27,\ \lambda_1 = 0,\ \lambda_2 = 1,\ m = n = 3$ whose incidence matrix is:

$$N = (N_1|N_2|N_3) = \left(\begin{array}{ccc|ccc|ccc} 0_3 & 0_3 & 0_3 & I_3 & I_3 & I_3 & \alpha & \alpha^2 & I_3 \\ \alpha & \alpha^2 & I_3 & 0_3 & 0_3 & 0_3 & I_3 & I_3 & I_3 \\ I_3 & I_3 & I_3 & \alpha & \alpha^2 & I_3 & 0_3 & 0_3 & 0_3 \end{array}\right),$$ where $\alpha = \textsc{circ}(0\ 1\ 0)$.

Theorem 3.10. *There exists a p-resolvable* RGD *design with parameters:*

$$\begin{aligned} &v = p^2 + p,\ \ r = p^2,\ \ k = p,\ \ b = p^2(p+1), \\ &\lambda_1 = 0,\ \ \lambda_2 = p - 1,\ \ m = p + 1,\ \ n = p, \end{aligned} \tag{6}$$

where $p \geq 3$ is a prime.

Proof. Let $\alpha = \text{CIRC}(0\ 1\ 0\ 0\ldots 0)$ be a permutation circulant matrix of order p. Consider the following block matrices:

(i) $N_1 = (M_1|M_2|\cdots|M_{p-1})$

$$= \Big(\text{CIRC}(0_p\ I_p\ \alpha\ \alpha^2 \cdots \alpha^{p-1})\Big|\text{CIRC}(0_p\ I_p\ \alpha^2 (\alpha^2)^2\ \cdots (\alpha^2)^{p-1})\Big|\cdots$$
$$\cdots\Big|\text{CIRC}(0_p\ I_p\ \alpha^{p-1}\ (\alpha^{p-1})^2 \cdots (\alpha^{p-1})^{p-1})\Big)$$

be a block circulant matrix arranged columnwise;

(ii) N_2 be a block matrix $(J-I)_{p+1} \otimes I_p$ arranged columnwise.

Let $N = (N_1|N_2)$. Then $NN' = p^2\,(I_{p+1} \otimes I_p) + (p-1)\,\{(J-I)_{p+1} \otimes J_p\}$. Hence N represents a RGD design with parameters (6). Since each row sum of the block matrices $M_i (1 \le i \le p-1)$ and N_2 is p, the design is p-resolvable. □

Example 3.11. Let $\alpha = \text{CIRC}(0\ 1\ 0)$ be a permutation circulant matrix. Consider the following block matrix:

$$N = (N_1|N_2|N_3)$$
$$= \left(\begin{array}{cccc|cccc|cccc} 0_3 & I_3 & \alpha & \alpha^2 & 0_3 & I_3 & \alpha^2 & \alpha & 0_3 & I_3 & I_3 & I_3 \\ \alpha^2 & 0_3 & I_3 & \alpha & \alpha & 0_3 & I_3 & \alpha^2 & I_3 & 0_3 & I_3 & I_3 \\ \alpha & \alpha^2 & 0_3 & I_3 & \alpha^2 & \alpha & 0_3 & I_3 & I_3 & I_3 & 0_3 & I_3 \\ I_3 & \alpha & \alpha^2 & 0_3 & I_3 & \alpha^2 & \alpha & 0_3 & I_3 & I_3 & I_3 & 0_3 \end{array}\right),$$

which represents a 3-resolvable RGD design R75 with parameters:

$$v = 12,\ r = 9,\ \ k = 3,\ b = 36,\ \lambda_1 = 0,\ \lambda_2 = 2,\ m = 4,\ n = 3.$$

Theorem 3.12. *There exists a* RGD *design with parameters:*

$$\begin{array}{l} v = q^2 - q,\ b = q(2q-1),\ r = 2q-1,\ k = q-1, \\ \lambda_1 = q-2,\ \lambda_2 = 1,\ m = q-1,\ n = q, \end{array} \tag{7}$$

where q is a prime or prime power.

Proof. Let $N_1 = I_{q-1} \otimes (J-I)_q$ be a square matrix of order $q^2 - q$ and $\mathcal{G}$ be an elementary abelian group. Further let N_2 be a $(q^2-q) \times q^2$ matrix that is produced by substituting group elements from the respective permutation matrices and a null matrix for the zero entry in a $\mathsf{GH}\,(q;\mathcal{G})$ with one row removed.

Let $N = (N_1|N_2)$. Then

$$NN' = 2q\,(I_{q-1} \otimes I_q) + (q-3)\,(I_{q-1} \otimes J_q) + J_{q-1} \otimes J_q.$$

Hence N represents a RGD design with parameters (7). □

Example 3.13. For $q = 4$, N is the incidence matrix of a RGD design R72 with parameters:

$$v = 12,\ r = 7,\ k = 3,\ b = 28,\ \lambda_1 = 2,\ \lambda_2 = 1,\ m = 3,\ n = 4.$$

$$N = \left(\begin{array}{ccc|cccc} (J-I)_4 & 0_4 & 0_4 & I_4 & I_4 & I_4 & I_4 \\ 0_4 & (J-I)_4 & 0_4 & I_4 & I_2 \otimes (J-I)_2 & (J-I)_2 \otimes I_2 & (J-I)_2 \otimes (J-I)_2 \\ 0_4 & 0_4 & (J-I)_4 & I_4 & (J-I)_2 \otimes (J-I)_2 & I_2 \otimes (J-I)_2 & (J-I)_2 \otimes I_2 \end{array}\right).$$

4 Quasidouble solutions

Trivially, by taking m copies of a given $D(v, r, k, b)$ design, we obtain a $D(v, mr, k, bm)$ design which is called the m-multiple of the design. A $D(v, mr, k, bm)$ design is called a quasi-multiple if it is not a m-multiple of any $D(v, r, k, b)$ design. We use the abbreviation quasidouble instead of 2-multiple or quasi 2-multiple, respectively. The interested reader can find many constructions for small quasimultiple affine and projective planes in Buratti [2] and Jungnickel [12, 13]. An interesting quasidouble of the affine plane of order four has been recently discovered by Pavone [14].

Quasidouble solutions of some RGD designs are given below. For these designs an m-multiple solution is reported in Clatworthy [3]. The notation GBRD$(v, k, \lambda; G)$ used below may be found in deLauney [6].

(1) R23 : $v = 6,\ r = 8,\ k = 2,\ b = 24,\ \lambda_1 = 0,\ \lambda_2 = 2,\ m = 3,\ n = 2.$
Solution: Replace $0 \to 0_2, 1 \to I_2, -1 \to (J-I)_2$ in a GBRD$(3, 2, 4; Z_2)$.

(2) R55 : $v = 8,\ r = 6,\ k = 3,\ b = 16,\ \lambda_1 = 0,\ \lambda_2 = 2,\ m = 4,\ n = 2.$
Solution: Replace $0 \to 0_2, 1 \to I_2, -1 \to (J-I)_2$ in a GBRD$(4, 3, 4; Z_2)$.

(3) R113 : $v = 14,\ r = 8,\ k = 4,\ b = 28,\ \lambda_1 = 0,\ \lambda_2 = 2,\ m = 7,\ n = 2.$
Solution: Replace $0 \to 0_2, 1 \to I_2, -1 \to (J-I)_2$ in a GBRD$(7, 4, 4; Z_2)$.

(4) R116 : $v = 15,\ r = 8,\ k = 4,\ b = 30,\ \lambda_1 = 0,\ \lambda_2 = 2,\ m = 5,\ n = 3.$
Solution: Replace $0 \to 0_3, 1 \to I_3, \alpha \to \textsc{circ}(0\ 1\ 0\ ), \alpha^2 \to$ $(\textsc{circ}(0\ 0\ 1\ )$ in a GBRD$(5, 4, 6; C_3)$ where $C_3 = \{1, \alpha, \alpha^2\}$ is a cyclic group of order 3.

(5) R136 : $v = 8,\ r = 10,\ k = 5,\ b = 16,\ \lambda_1 = 4,\ \lambda_2 = 6,\ m = 4,\ n = 2.$
Solution: $J_{8\times 16} - N$ where N is the incidence matrix of R55.

(6) R147 : $v = 12,\ r = 10,\ k = 5,\ b = 24,\ \lambda_1 = 0,\ \lambda_2 = 4,\ m = 6,\ n = 2.$
Solution: Replace $0 \to 0_2, 1 \to I_2, -1 \to (J - I)_2$ in a GBRD$(6, 5, 8; Z_2)$.

(7) R154 : $v = 24,\ r = 10,\ k = 5,\ b = 48,\ \lambda_1 = 0,\ \lambda_2 = 2,\ m = 6,\ n = 4.$
Soultion: Consider the following GBRD$(6, 5, 8; C_4)$ as given in Gibbons and Mathon [11]:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^3 & 1 & \alpha & \alpha^2 & 0 & 0 & 1 & 1 \\ 1 & \alpha^2 & \alpha & \alpha^3 & \alpha^2 & \alpha & 0 & 0 & 1 & \alpha^3 & \alpha^2 & \alpha^2 \\ 1 & \alpha^3 & \alpha^3 & \alpha^2 & 0 & 0 & 1 & \alpha^2 & \alpha & \alpha & \alpha & \alpha^2 \\ 1 & 1 & 0 & 0 & \alpha^3 & \alpha^2 & \alpha^2 & \alpha & \alpha^3 & \alpha & \alpha^2 & \alpha \\ 0 & 0 & 1 & \alpha & \alpha^3 & \alpha & 1 & \alpha^2 & \alpha^2 & \alpha^3 & \alpha^3 & \alpha \end{pmatrix}$$

where $C_4 = \{1, \alpha, \alpha^2, \alpha^3\}$ is a cyclic group of order 4. Replace $0 \to 0_4, 1 \to I_4, \alpha \to \text{CIRC}(0\,1\,0\,0), \alpha^2 \to \text{CIRC}(0\,0\,1\,0), \alpha^3 \to \text{CIRC}(0\,0\,0\,1)$ in A to obtain a quasidouble solution of R154.

5 Tables of designs

This section contains Tables 5.1 and 5.2 of non-symmetric RGD designs listed in Clatworthy [3] constructed using the present theorems. The complement of designs and the designs obtained by duplication are not included in the Tables.

Table 5.1: Non-symmetric RGD designs

No.	RX:$(v, r, k, b, \lambda_1, \lambda_2, m, n)$	Source
1	R1: $(4, 4, 2, 8, 2, 1, 2, 2)$	Th. 3.1; GH$(2; C_2)$; $s = 2, t = 1$
2	R19: $(6, 6, 2, 18, 2, 1, 3, 2)$	Th. 3.3(b); $s = 1, t = 2$; 2-(3,2,1) design
3	R20: $(6, 7, 2, 21, 2, 1, 2, 3)$	Th. 3.3(a); $s = 2, t = 1$
4	R21: $(6, 7, 2, 21, 3, 1, 3, 2)$	Th. 3.3(b); $s = 1, t = 3$
5	R22: $(6, 8, 2, 24, 4, 1, 3, 2)$	Th. 3.3(b); $s = 1, t = 4$
6	R24: $(6, 8, 2, 24, 1, 2, 2, 3)$	Th. 3.3(a); $s = 1, t = 2$
7	R25: $(6, 9, 2, 27, 3, 1, 2, 3)$	Th. 3.3(a); $s = 3, t = 1$

Table 5.1: Non-symmetric RGD designs

No.	RX:$(v,r,k,b,\lambda_1,\lambda_2,m,n)$	Source
8	R26: $(6,9,2,27,5,1,3,2)$	Th. 3.3(b); $s=1,t=5$
9	R27: $(6,9,2,27,1,2,3,2)$	Th. 3.3(b); $s=2,t=1$
10	R28: $(6,10,2,30,6,1,3,2)$	Th. 3.3(b); $s=1,t=6$
11	R29: $(8,6,2,24,0,1,4,2)$	Th.6(b); $s=1,t=0$; 2-(4,2,1) design
12	R30: $(8,8,2,32,2,1,4,2)$	Th. 3.3(b); $s=1,t=2$
13	R31: $(8,9,2,36,3,1,4,2)$	Th. 3.3(b); $s=1,t=3$
14	R32: $(8,10,2,40,2,1,2,4)$	Th.6(a); $s=2,t=1$
15	R33: $(8,10,2,40,4,1,4,2)$	Th. 3.3(b); $s=1,t=4$
16	R34: $(9,6,2,27,0,1,3,3)$	Th. 3.8
17	R35: $(9,10,2,45,2,1,3,3)$	Th. 3.5
18	R36: $(10,8,2,40,0,1,5,2)$	Th. 3.3(b); $s=1,t=0$
19	R38: $(12,8,2,48,0,1,3,4)$	Th. 3.8
20	R39: $(12,9,2,54,0,1,4,3)$	Th. 2.1; GBRD$(4,2,3;C_3)$
21	R40: $(12,10,2,60,0,1,6,2)$	Th. 3.3(b); $s=1,t=0$; 2-(6,2,1) design
22	R41: $(15,10,2,75,0,1,3,5)$	Th. 3.8
23	R46: $(6,7,3,14,2,3,3,2)$	R42+SR19
24	R48: $(6,8,3,16,4,3,3,2)$	2#R42+ SR18
25	R51: $(6,9,3,18,2,4,3,2)$	R42+SR20
26	R59: $(9,5,3,15,2,1,3,3)$	Th. 3.1; GH$(3;C_3)$; $s=2,t=1$
27	R70: $(12,5,3,20,0,1,6,2)$	Th. 2.1; GBRD$(6,3,2;C_2)$
28	R72: $(12,7,3,28,2,1,3,4)$	Th. 3.12
29	R75: $(12,9,3,36,0,2,4,3)$	Th. 3.10
30	R76: $(12,10,3,40,4,1,3,4)$	Th. 3.12; $N=(2\#N_1\|N_2)$
31	R79: $(14,6,3,28,0,1,7,2)$	Th. 2.1; GBRD$(7,3,2;C_2)$
32	R86: $(16,6,3,32,0,1,4,4)$	Th. 2.1; GBRD$(4,3,4;EA(4))$
33	R88: $(18,8,3,48,0,1,9,2)$	Th. 2.1; GBRD$(9,3,2;C_2)$
34	R90: $(20,9,3,60,0,1,10,2)$	Th. 2.1; GBRD$(10,3,2;C_2)$
35	R91: $(21,9,3,63,0,1,7,3)$	Th. 2.1; GBRD$(7,3,3;C_3)$

Table 5.1: Non-symmetric RGD designs

No.	RX:$(v, r, k, b, \lambda_1, \lambda_2, m, n)$	Source
36	R92: $(24, 9, 3, 72, 0, 1, 4, 6)$	Th. 2.1; GBRD$(4, 3, 6; D_3)$/GBRD$(4, 3, 6; C_6)$
37	R93: $(24, 10, 3, 80, 0, 1, 6, 4)$	Th. 2.1; GBRD$(6, 3, 4; EA(4))$
38	R106: $(10, 8, 4, 20, 0, 3, 5, 2)$	Th. 2.3
39	R118: $(16, 6, 4, 24, 2, 1, 4, 4)$	Th. 3.1; GH$(4; EA(4))$; $s = 2, t = 1$
40	R124: $(20, 9, 4, 45, 3, 1, 4, 5)$	Th. 3.12
41	R125: $(24, 7, 4, 42, 0, 1, 8, 3)$	Th. 2.1; GBRD$(8, 4, 3; C_3)$
42	R128: $(26, 8, 4, 52, 0, 1, 13, 2)$	Th. 2.3
43	R129: $(27, 8, 4, 54, 0, 1, 9, 3)$	Th. 2.1; GBRD$(9, 4, 3; C_3)$
44	R131a: $(30, 8, 4, 60, 0, 1, 5, 6)$	Saurabh and Sinha [21]
45	R132a: $(36, 10, 4, 90, 0, 1, 6, 6)$	Saurabh and Sinha [21]
46	R150: $(15, 10, 5, 30, 2, 3, 5, 3)$	Th. 2.3
47	R152a: $(22, 10, 5, 44, 0, 2, 11, 2)$	Th. 2.1; GBRD$(11, 5, 2; C_2)$ Freeman [10]
48	R155: $(25, 7, 5, 35, 2, 1, 5, 5)$	Th. 3.1; GH$(5; C_5)$; $s = 2, t = 1$
49	R160: $(39, 10, 5, 78, 2, 1, 13, 3)$	Th. 2.3
50	R167: $(12, 9, 6, 18, 7, 3, 3, 4)$	Th. 2.4
51	R184: $(49, 9, 7, 63, 2, 1, 7, 7)$	Th. 3.1; GH$(7; C_7)$; $s = 2, t = 1$
52	R185: $(49, 10, 7, 70, 3, 1, 7, 7)$	Th. 3.1; GH$(7; C_7)$; $s = 3, t = 1$
53	R192: $(64, 10, 8, 80, 2, 1, 8, 8)$	Th. 3.1; GH$(8; EA(8))$; $s = 2, t = 1$

The generalized Bhaskar Rao designs and generalized Hadamard matrices used in Table 5.1 may be found in de Launey [6] and de Launey [7]. The RGD designs R2, R3, ..., R17 may be obtained using GH$(2; C_2)$; R60, R61, ..., R68 may be obtained using GH$(3; C_3)$; R119,..., R123 may be obtained using GH$(4; EA(4))$ and R156, R157, R158 may be obtained GH$(5; C_5)$ for different values of s and t in Theorem 3.1. RGD designs may also be obtained from BIBDs as follows:

Let 'm' groups of a GD design be treated as blocks. Then annexing 'c' copies of these 'm' groups to 'b' blocks of a BIBD with parameters:

$$v = mk,\ b,\ r,\ k,\ \lambda;$$

we obtain:

Proposition 5.1. *The existence of a* BIBD *with parameters:*

$$v = mk, b, r, k, \lambda$$

implies the existence of a RGD *design with parameters:*

$$v' = mk,\ b' = b + cm,\ r' = r + c,\ k' = k,\ \lambda_1 = \lambda + c,\ \lambda_2 = \lambda,\ m,\ n = k.$$

Further let resolution class of a resolvable BIBD with parameters:

$$v = mk,\ b,\ r,\ k,\ \lambda$$

be repeated 'c' times. Then removing these repeated classes, we obtain:

Proposition 5.2. *The existence of a* BIBD *with parameters:*

$$v = mk,\ b,\ r,\ k,\ \lambda$$

having a resolution class repeated 'c' times implies the existence of a RGD *design with parameters:* $v' = mk$, $b' = b - cm$, $r' = r - c$, $k' = k$, $\lambda_1 = \lambda - c$, $\lambda_2 = \lambda$, $m = k$, $n = k$.

Table 5.2 lists RGD designs obtained from BIBDs using above propositions:

Table 5.2: BIBDs and Corresponding RGD designs

Using Proposition 5.1

No.	BIBD(v, r, k, b, λ)	c	Derived RGD design: $(v, r, k, b, \lambda_1, \lambda_2, m, n)$
1	BIBD$(10, 9, 2, 45, 1)$	1	R37: $(10, 10, 2, 50, 2, 1, 5, 2)$
2	BIBD$(6, 5, 3, 10, 2)$	1	R43: $(6, 6, 3, 12, 3, 2, 2, 3)$
		2	R45: $(6, 7, 3, 14, 4, 2, 2, 3)$
		3	R47: $(6, 8, 3, 16, 5, 2, 2, 3)$
		4	R49: $(6, 9, 3, 18, 6, 2, 2, 3)$
		5	R53: $(6, 10, 3, 20, 7, 2, 2, 3)$
3	BIBD$(15, 7, 3, 35, 1)$	1	R82: $(15, 8, 3, 40, 2, 1, 5, 3)$
		2	R84: $(15, 9, 3, 45, 3, 1, 5, 3)$

Table 5.2: BIBDs and Corresponding RGD designs

		3	R85: $(15, 10, 3, 50, 4, 1, 5, 3)$
4	BIBD$(8, 7, 4, 14, 3)$	1	R98: $(8, 8, 4, 16, 4, 3, 2, 4)$
		2	R100: $(8, 9, 4, 18, 5, 3, 2, 4)$
		3	R102: $(8, 10, 4, 20, 6, 3, 2, 4)$
5	BIBD$(28, 9, 4, 63, 1)$	1	R131: $(28, 10, 4, 70, 2, 1, 7, 4)$
6	BIBD$(10, 9, 5, 18, 4)$	1	R141: $(10, 10, 5, 20, 5, 4, 2, 5)$
7	BIBD$(28, 9, 7, 36, 2)$	1	R181: $(28, 10, 7, 40, 3, 2, 4, 7)$

Using Proposition 5.2 ($c = 1$)

8	BIBD$(6, 5, 2, 15, 1)$	R18: $(6, 4, 2, 12, 0, 1, 3, 2)$
9	BIBD$(8, 7, 2, 28, 1)$	R29: $(8, 6, 2, 24, 0, 1, 4, 2)$
10	BIBD$(12, 11, 2, 66, 1)$	R40: $(12, 10, 2, 60, 0, 1, 6, 2)$
11	BIBD$(6, 10, 3, 20, 4)$	R52: $(6, 9, 3, 18, 3, 4, 2, 3)$
12	BIBD$(12, 11, 3, 44, 2)$	R78: $(12, 10, 3, 40, 1, 2, 4, 3)$
13	BIBD$(15, 7, 3, 35, 1)$	R81: $(15, 6, 3, 30, 0, 1, 5, 3)$
14	BIBD$(12, 11, 4, 33, 3)$	R111: $(12, 10, 4, 30, 2, 3, 3, 4)$
15	BIBD$(28, 9, 4, 63, 1)$	R130: $(28, 8, 4, 56, 0, 1, 7, 4)$

6 Concluding remarks

In this paper solutions of the non-symmetric regular group divisible (RGD) designs listed in Clatworthy [3] are obtained using matrix approaches except few. As special case we obtain a series of μ-resolvable balanced incomplete block design and quasidouble solutions of some RGD designs. Most of the solutions are obtained by inserting permutation circulant matrices in certain combinatorial matrices. The matrix solutions of the remaining non-symmetric RGD designs listed in Clatworthy [3], Sinha [23] and Saurabh and Sinha [20] will be taken up as a future work.

Quasidouble solutions are important from coding theoretic point of view. Clatworthy [3] reported a unique solution for R23 which is the duplicate of R18. Here a quasidouble solution is obtained for R23 and hence the uniqueness claim is violated.

Let the incidence matrix N of a block design $D(v, r, k, b)$ has a decomposition $[N_{ij}]_{1 \leq i \leq v, 1 \leq j \leq b}$ where N_{ij} are submatrices of suitable sizes. The decomposition is row-wise tactical if each row sum of N_{ij} is r_{ij}, column-wise tactical if each column sum of N_{ij} is k_{ij} and tactical if it is both row-wise as well as column-wise tactical. Further the decomposition is uniform if $r_{ij} = \alpha, k_{ij} = \beta$, for all i, j. If each N_{ij} is an $n \times n$ matrix, $D(v, r, k, b)$ is called square tactical decomposable design, STD (n). See Singh and Saurabh [22].

Tactical decomposable designs are of interest because of their connections with automorphisms of designs, see Bekar, Mitchel and Piper [1] and Singh and Saurabh [22]. In this paper we have obtained some series of tactical and square tactical decomposable RGD designs.

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