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2-tone coloring of planar graphs

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Abstract. A 2-tone coloring of a graph assigns two distinct colors to each vertex with the restriction that adjacent vertices have no common colors, and vertices at distance two have at most one common color. The 2-tone chromatic number of a graph is the minimum number of colors in any 2-tone coloring. We determine a lower bound for the 2-tone chromatic number of planar graphs that is exact for almost all orders. We also determine an exact formula for the 2-tone chromatic number of a particular class of planar graphs whose 2-tone chromatic number had previously only been approximated.

1 Introduction

The problem of vertex coloring of planar graphs motivated the development of graph theory. (See [3] for basic terminology and notation.) There are many generalizations of vertex coloring, including some that assign more than one color to a vertex and others that restrict what colors may appear on vertices at different distances in a graph. We study a variation of vertex coloring that combines these generalizations.

Definition 1.1 (Fonger et al. [15])**.** Let G be a graph, $k, t \in \mathbb{N}$, $[k] = \{1, 2, \ldots, k\}$, and let $\binom{[k]}{t}$ denote the set of t-element subsets of $[k]$. A function $f\colon V(G) \to \binom{[k]}{t}$ is called a *proper t-tone k-coloring* (or sometimes just a *t-tone coloring*) of G if $|f(u) \cap f(v)| < d(u, v)$ for all distinct vertices u and v of G. A graph is *t-tone k-colorable* if it has a proper t-tone k-coloring. The *t-tone chromatic number of G*, denoted by $\tau_t(G)$, is the smallest positive integer k for which G has a proper t-tone k-coloring.

Note that for $t = 1$, $\tau_1(G) = \chi(G)$, the usual chromatic number of a graph G. This paper is solely concerned with the 2-tone chromatic number.

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The 2-tone chromatic number has been determined for complete multipartite graphs, trees [15], cycles, theta graphs [7], Mobius ladders, wheels, fans, products of complete graphs, some products of cycles [2], Sierpinski triangle graphs, Hanoi graphs [4], most cactus graphs [6], powers of paths, squares of cycles and trees, and maximal outerplanar graphs [5]. General upper bounds were found in [2, 11–14], and lower bounds were studied in [17].

Bal et al. considered [1] the 2-tone chromatic number of the random graph. Several authors [2, 10, 16] have studied 2-tone coloring for graph products. For general t, t-tone coloring has been studied for cycles [12, 18], grids [12], and some hypercubes [19].

We shall often call $f(v)$ the *label* associated with the vertex v of the coloring f, and the elements of $f(v)$ will be called *colors*. Thus, in a 2-tone coloring, each vertex has a label of 2 distinct colors. Adjacent vertices have no common colors, and vertices distance two apart have at most one common color. When the context is clear, the label $\{a, b\}$ will be denoted ab. Vertices distance two apart are called *second-neighbors*. A *color class* is the set of all vertices with the same color in some coloring of the graph.

Some basic results are immediate. If H is a subgraph of G then $\tau_2(H) \leq \tau_2(G)$. We have $2n \leq \alpha(G) \cdot \tau_2(G)$ since each color class is an independent set, so $\tau_2(G) \geq \frac{2 \cdot n(G)}{\alpha(G)}$. We have $\tau_2(K_n) = 2n$, and for the cycle C_n, from [7],

$$\tau_2(C_n) = \begin{cases} 6, & \text{if } n = 3, 4, 7, \\ 5, & \text{otherwise.} \end{cases}$$

If $\tau_2(G) = k$, we call a 2-tone k-coloring of G a *minimum coloring*. Two colorings of a graph are *distinct* if they cannot be made the same by a permutation of the colors and an automorphism of the graph. A 2-tone k-coloring is *unique* if there are no two distinct k-colorings. The minimum colorings of C_n are unique for $n \in \{3, 4, 5, 6, 8, 9\}$, again from [7].

2 Classes of maximal planar graphs

In this section, we determine formulas for the 2-tone number of three basic classes of maximal planar graphs.

Definition 2.1. A *pair k-coloring* of a graph G is a 2-tone k-coloring in which every label is distinct. A graph is *pair k-colorable* if it has a pair

k-coloring. The *pair chromatic number* of G, $pc(G)$, is the smallest k for which G has a pair k-coloring.

Some results on the pair chromatic number are immediate. We have

$$pc(G) \geq \tau_2(G),$$

and if $\operatorname{diam}(G) \leq 2$, then this is an equality. This implies that for a join $G + H$,

$$\tau_2(G + H) = pc(G + H) = pc(G) + pc(H).$$

If H is a subgraph of G, then $pc(H) \leq pc(G)$. If $n > \binom{k}{2}$, then $pc(G) > k$. Equivalently, $pc(G) \geq \frac{1+\sqrt{1+8n}}{2}$.

Theorem 2.2 (Bickle [2])**.** *We have*

$$pc(C_n) = \begin{cases} 5, & \text{if } n = 5, 6, 8, 9, \\ 6, & \text{if } n = 3, 4, 7, 10, 11, \ldots, 15, \\ \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil, & \text{if } n \geq 11, \end{cases}$$

$$pc(P_n) = \begin{cases} 5, & \text{if } 3 \leq n \leq 10, \\ \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil, & \text{if } n \geq 11. \end{cases}$$

As a corollary, we find formulas for the 2-tone number of two classes of maximal planar graphs.

Corollary 2.3. *We have*

$$\tau_2(C_n + \overline{K}_2) = \begin{cases} 8, & \text{if } n = 5, 6, 8, 9, \\ 9, & \text{if } n = 3, 4, 7, 10, 11, \ldots, 15, \\ \left\lceil \frac{7+\sqrt{1+8n}}{2} \right\rceil, & \text{if } n \geq 11, \end{cases}$$

$$\tau_2(P_n + K_2) = \begin{cases} 9, & \text{if } 3 \leq n \leq 10, \\ \left\lceil \frac{9+\sqrt{1+8n}}{2} \right\rceil, & \text{if } n \geq 11. \end{cases}$$

Definition 2.4. The k^{th} *power* G^k of a graph G adds all edges between pairs of vertices with distance at most k. The graphs G^2 and G^3 are the *square* and *cube* of G.

In [5], it was shown that for $n \geq k + 2$, we have $\tau_2\left(P_n^k\right) = 2k + 3$. Thus for P_n^3, which is maximal planar, $\tau_2\left(P_n^3\right) = 9$ when $n \geq 5$.

3 Planar graphs with small 2-tone chromatic number

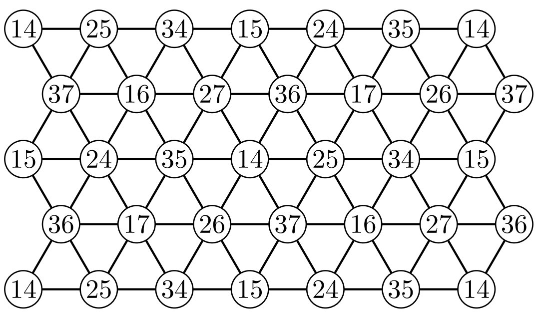
In this section, we determine a lower bound on the 2-tone number of maximal planar graphs that is attained for almost all orders.

For the wheel $C_d + K_1$, it follows [2] from Theorem 2.2 that

$$\tau_2(C_d + K_1) = \begin{cases} 7, & \text{if } d = 5, 6, 8, 9, \\ 8, & \text{if } d = 3, 4, 7, 10 - 15, \\ \left\lceil \frac{5+\sqrt{1+8d}}{2} \right\rceil, & \text{if } d \geq 11. \end{cases}$$

In a maximal planar graph G, the closed neighborhood of any vertex induces a graph that contains a wheel. This provides a lower bound. Let $W_{\max}$ be a wheel in G that has maximum 2-tone number. Then $\tau_2(G) \geq \tau_2(W_{\max}) \geq 7$. When the maximum degree $\Delta(G)$ is large, its center (the K_1) will be a vertex of maximum degree, but when $\Delta(G) \leq 9$, it could be a vertex of degree 3, 4, or 7.

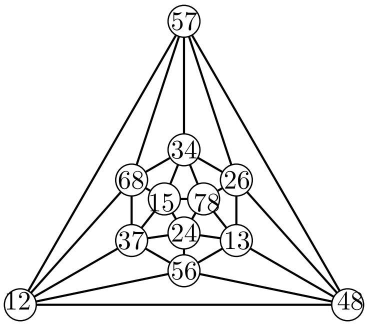
The infinite triangular grid has 2-tone number 7, see [4]. This follows since it contains $K_4 - e$ and by the coloring below, which can be repeated infinitely since the boundaries use the same colors. Colors 1, 2, and 3 form a 3-coloring of this graph, and $\{4, 5, 6, 7\}$ form a 4-coloring. This 2-tone coloring is unique, as can be seen by starting with a wheel.



There is also a maximal toroidal graph with order 12 (bounded by the parallelogram formed by the four vertices colored 15 above) that has 2-tone number 7.

By the formula for $\tau_2(C_d + K_1)$, each vertex of any maximal planar graph G with $\tau_2(G) = 7$ has degree in $\{5, 6, 8, 9\}$. The smallest maximal planar graph with only these degrees is the icosahedron (IC).

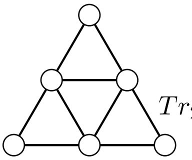
Proposition 3.1. *We have $\tau_2(IC) = 8$ and the 2-tone 8-coloring is unique.*



Proof. The independence number $\alpha(IC) = 3$, so $\tau_2(IC) \geq \frac{2 \cdot n(IC)}{\alpha(IC)} = \frac{2 \cdot 12}{3} = 8$. The upper bound follows from the coloring shown above.

To show that the coloring is unique, note that every color must appear exactly three times. Now deleting a vertex and its neighbors in IC results in $C_5 + K_1$. Thus each color class must have all three vertices at distance 2, at the corners of an induced copy of Tr_2 (see below).

Thus pairs of vertices at distance 3 have no common colors. Thus every 5-cycle around a vertex must use 6 colors, 4 twice and 2 once. Start with a unique 8-coloring (up to symmetry) of $C_5 + K_1$ in IC. The restrictions we have established are enough to eliminate possible colors and labels on the remaining vertices, leaving only one possible coloring up to symmetry. □



Each Tr_2 defined by a color class can be specified by its middle region. In the dual dodecahedron, the 8-coloring gives a set of vertices that turns out to be independent. Thus each 2-tone coloring corresponds to a (maximum) independent set of the dodecahedron.

Deleting three edges that form a triangle in IC results in a 2-tone 7-colorable graph. Thus deleting two adjacent vertices does also, but

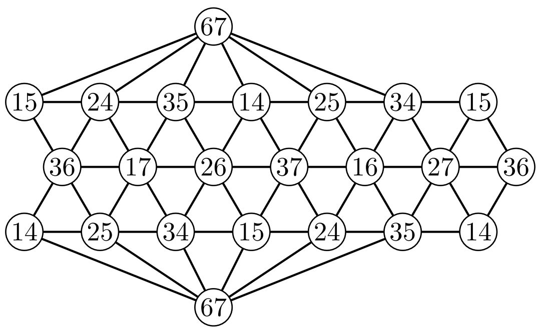
$$\tau_2(IC - v) = 8,$$

because

$$\tau_2(IC - v) \geq \frac{2 \cdot n(IC - v)}{\alpha(IC - v)} = \frac{2 \cdot 11}{3} > 7.$$

To describe a class of maximal planar graphs with 2-tone number 7, we start with a subgraph of the infinite triangular grid, but connected to itself in one dimension. For $1 \leq j \leq k$, $k \geq 2$, and $1 \leq i \leq 6$, let vertex (i, j) be adjacent to $(i, j+1)$, $(i+1, j)$, and $(i+1, j+1)$, where i is taken modulo 6 (but j is not taken modulo k). This graph has two regions of length 6 in a plane drawing. Add two degree 6 vertices u and v to make this graph maximal planar, and denote it G_{6k+2}.

The graph G_{20} is shown below (the three vertices on the left and right with the same labels must be identified), along with a 2-tone 7-coloring.



Proposition 3.2. *For $k \geq 2$, we have $\tau_2(G_{6k+2}) = 7$. The smallest maximal planar graph with 2-tone number 7 is G_{14}.*

Proof. Any maximal planar graph with 2-tone number 7 must have minimum degree 5. The smallest such graph is the icosahedron, and $\tau_2(IC) = 8$. There is no such graph with order 13, and one with order 14, G_{14} [8, 9].

Clearly $\tau_2(G_{6k+2}) \geq 7$ since these graphs contain $K_4 - e$. Now $G_{6k+2} - u - v$ is contained in the infinite triangular grid, so it has a 2-tone 7-coloring. Finally, we label u and v with 67 or 45, depending on parity (see the example above). $\square$

Suppose we form a graph by adding two vertices inside the non-triangular regions of C_{2k}^2 to produce a maximal planar graph. By the formula for $\tau_2(C_{2k}^2)$ in [5] and the values of $\tau_2(C_d + K_1)$, the only k for which such a graph could be 2-tone 7-colorable are 6, 8, and 9. It is easy to show that

8 and 9 do not work, so G_{14} is the only graph of this type that is 2-tone 7-colorable.

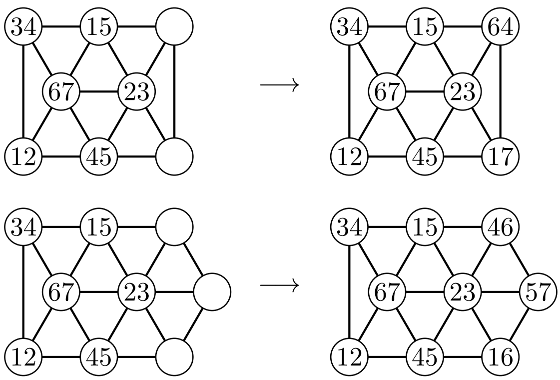
The 2-tone 7-coloring of G_{6k+2} is unique. This is a corollary of a more general result that is presented next.

Lemma 3.3. *If four consecutive vertices of a* 5*-cycle are* 2*-tone* 5*-colored, there is a unique way to extend this* 5*-coloring to the remaining vertex. The same is true for four vertices of a* 6*-cycle, five vertices of an* 8*-cycle, and six vertices of a* 9*-cycle.*

Proof. For each of these cycles, a 2-tone 5-coloring of them corresponds to a 2-tone 5-coloring of the Petersen graph [7]. It is easily verified that four consecutive vertices of a 5-cycle of the Petersen graph are only contained in that 5-cycle, and similarly for the cycles of other lengths. □

Theorem 3.4. *If G is a maximal planar graph with $\tau_2(G) = 7$ and all vertices have degree* 5 *or* 6*, then the* 2*-tone* 7*-coloring is unique.*

Proof. We begin with two adjacent vertices that either both have degree 5 or have degrees 5 and 6, and color their neighborhoods. Note that C_5 and C_6 have unique pair 5-colorings. Thus there is a unique 2-tone 7-coloring for $C_5 + K_1$, which extends uniquely to the wheel centered at the other vertex, in either case.



Let H be the subgraph induced by the colored vertices. We continue to color new vertices on the boundary of H. Each of these vertices has at least 3 colored neighbors since they are contained in colored wheels, and some vertex v has at least four colored neighbors since otherwise H is a wheel.

Since v receives two colors, there are five colors available for its neighbors. By Lemma 3.3, there is a unique extension of the coloring to the uncolored neighbor(s) of v. Thus each vertex of G is colored uniquely, so the 2-tone 7-coloring of G is unique. □

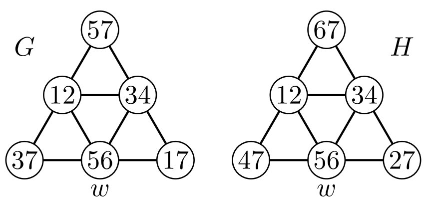
Conjecture 3.5. If G is a maximal planar graph with $\tau_2(G) = 7$, then the 2-tone 7-coloring is unique.

Next we consider three ways of combining two maximal planar graphs with 2-tone number 7 to produce another maximal planar graph with 2-tone number 7. A 3-*sum* identifies triangles of two graphs.

Theorem 3.6. *If two maximal planar graphs G and H with 2-tone number 7 are identified on a triangle, and this does not create any vertex degrees not in $\{5, 6, 8, 9\}$, then the resulting graph $G \cup H$ has 2-tone number 7. If the 7-colorings of G and H are unique, so is the 7-coloring of $G \cup H$.*

Proof. If we identify vertices a and a' in a common triangle of two maximal planar graphs, the degree of the new vertex is $d(a) + d(a') - 2$. For this to be in $\{5, 6, 8, 9\}$, we must have $d(a) = d(a') = 5$ or $\{d(a), d(a')\} = \{5, 6\}$.

Say the common triangle is uvw, with labels 12, 34, and 56, respectively. Note that there are also triangles containing each two of these three vertices in both G and H. Each of the other vertices in these triangles must have color 7. For G, suppose the other labels are 17, 37, and 57. Then in H, the corresponding labels must be 27, 47, and 67.



Assume $d_G(w) = 5$. Then w has one more neighbor in G, which must have label 24. If $d_H(w) = 5$, the remaining neighbor in H must have label 13. If $d_H(w) = 6$, the remaining neighbors in H must have labels 14 and 23. Thus there is no conflict among the neighbors of w.

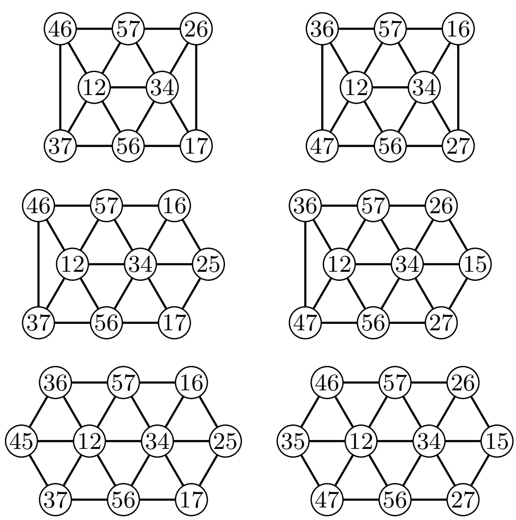
Similarly, there can be no conflict among the neighbors of u or v. Since we had no choice of what colors to assign to vertices of H, the partial coloring we produced must be part of any 7-coloring of H (up to permutation of colors). Thus any 7-colorings of G and H can be combined without conflict. Similarly, if the 7-colorings of G and H are unique, so is the 7-coloring of $G \cup H$. $\square$

Instead of identifying triangles, we could identify copies of $K_4 - e$ in two maximal planar graphs.

Theorem 3.7. *Let G and G' be two maximal planar graphs with $\delta(G) = \delta(G') = 5$ and $\tau_2(G) = \tau_2(G') = 7$ each containing 4-cycle $v_1v_2v_3v_4v_1$ and edge v_1v_3, $5 \leq d_G(v_i) = d_{G'}(v_i) \leq 6$ for $i \in \{1,3\}$ and $10 \leq d_G(v_i) + d_{G'}(v_i) \leq 11$ for $i \in \{2,4\}$. Form H by identifying all pairs of v_i for all i and deleting v_1v_3. Then $\tau_2(H) = 7$.*

Proof. If $d_G(v_1) = d_{G'}(v_1) = 5$, then $d_H(v_1) = 6$, and if $d_G(v_1) = d_{G'}(v_1) = 6$, then $d_H(v_1) = 8$. If $d_G(v_2) = d_{G'}(v_2) = 5$, then $d_H(v_2) = 8$, and if $d_G(v_2) = 5 < d_{G'}(v_2) = 6$, then $d_H(v_2) = 9$. Thus no single wheel of H forbids a 7-coloring.

We assume a 7-coloring of G and try to extend it to G'. Note that there is a vertex u in G' other than v_3 that neighbors v_1 and v_2. There are 4 colors used on v_1 and v_2, and two labels used on their neighbors in G. Thus there is only one possible label for u. Similarly, there are neighbors of v_2 and v_3, v_3 and v_4, and v_4 and v_1 in G' for which there is only one possible label. Whether $d_G(v_1) = 5$ or $d_G(v_1) = 6$, the neighbors of v_1 in G' can be labeled without conflict. The same is true for v_3.



Now the coloring may be extended uniquely to the remaining neighbors of v_2 and v_4 as in the previous proofs (see the above figures above, where in each of three cases G is on the left, G' is on the right, and the four central vertices are identified). Since we had no choice of what colors to assign to vertices of G', the partial coloring we produced must be part of any 7-coloring of H (up to permutation of colors). Thus any 7-colorings of G and G' can be combined without conflict. □

Our third way of combining two maximal planar graphs is to identify the neighborhoods of two vertices with the same degree. First, we need a lemma.

Lemma 3.8. *Let G be a maximal planar graph with $\tau_2(G) = 7$. If v is a vertex whose neighbors all have degree 5 or 6, then v has an even number of second-neighbors.*

Proof. For any 2-tone 7-coloring of G, there is a set S of 2 colors used on v, and a set T of exactly 5 colors used on its neighbors. We start with the unique coloring of v and its neighbors, and color the second-neighbors of v.

Let u be a second-neighbor of v that shares exactly two neighbors x and y with v. Now there are 4 colors on x and y that cannot appear on u, so 3 colors can. This allows 3 possible labels, one of which is used on v. Thus u must use one color from S and one color from T, and there is only once choice for the latter.

Second-neighbors that share exactly two neighbors with v either neighbor each other, or are one apart on the cycle C formed by all second-neighbors of v. Now any second-neighbor z with only one common neighbor with v has four distinct colors used on its neighbors. Thus z has only two possible labels, so it must use one color from S and one color from T. Thus the colors from S alternate around C, so its length is even. □

This lemma can be used to quickly show that some maximal planar graphs (e.g., IC) with degrees in $\{5, 6, 8, 9\}$ are not 7-colorable.

Theorem 3.9. *Let G and G' be two maximal planar graphs with $\tau_2(G) = \tau_2(G') = 7$ containing vertices u and v with $d = d(u) = d(v)$ so that u has neighbors $u_1, \ldots, u_d$ and v has neighbors $v_1, \ldots, v_d$ (in cyclic order) and $5 \leq d(u_i) = d(v_i) \leq 6$ for all i. Form H by deleting u and v, and identifying u_i and v_i for all i. Then $\tau_2(H) = 7$.*

Proof. If $d_G(u_i) = d_{G'}(v_i) = 5$, then $d_H(v_i) = 6$, and if $d_G(u_i) = d_{G'}(v_i) = 6$, then $d_H(v_i) = 8$. Thus no single wheel of H forbids a 7-coloring.

Suppose the second-neighbors of u in G are denoted x_i and these vertices form cycle C in order. We assume a 7-coloring of G and show that it extends to a 7-coloring of H. Say u had label 12. By Lemma 3.8, colors 1 and 2 must alternate on C. Let the second-neighbors of v in G' be denoted y_i and these vertices form cycle C' in order. Label each y_i by swapping 1 for 2 in the label of x_i. This causes no distance 1 conflicts. Now x_i and y_i have distinct labels. Also, x_i and y_{i+1} have distinct labels since x_{i+1} and y_{i+1} have a common color. The same argument works for x_i and y_{i+2}. Thus there are no distance 2 conflicts.

The same coloring can be obtained by extending cycles of wheels centered at v_i. When $d_G(u_i) = 5$, we have 4 colored vertices in H, so by Lemma 3.3, we can extend the coloring uniquely to a 6-cycle. When $d_G(u_i) = 6$, we have 5 colored vertices in H, so we can extend the coloring uniquely to an 8-cycle. Since we had no choice of what colors to assign to vertices of G', the partial coloring we produced must be part of any 7-coloring of G' (up to permutation of colors). Thus any 7-colorings of G and G' can be combined without conflict. □

Let $T(n)$ be the minimum value of $\tau_2(G)$ over all maximal planar graphs of order n. The previous results allow us to determine $T(n)$ for most values of n.

Lemma 3.10. *If there is a 2-tone 7-colorable maximal planar graph of order n, then there are 2-tone 8-colorable maximal planar graphs of orders $n+1$, $n+2$, and $n+3$.*

Proof. Let G be a 2-tone 7-colorable maximal planar graph of order n with vertices a, b, c, d, e colored 12, 34, 56, 57, and 37, respectively, and regions abc, abd, and ace. Add vertices u, v, and w adjacent to the vertices of each respective triangle. Then color u with 78, v with 68, and w with 48. This produces a 2-tone 8-colorable maximal planar graph of order $n+3$. Examples of order $n+1$ and $n+2$ are produced by deleting one or two of $\{u, v, w\}$. □

Theorem 3.11. *We have*

$$T(n) = \begin{cases} 7, & \text{if } n = 14, 18, 20, 21, 24, 25, 26, \geq 28, \\ 8, & \text{if } n = 4, 7, \ldots, 13, 15, 16, 17, \\ 9, & \text{if } n = 5, 6. \end{cases}$$

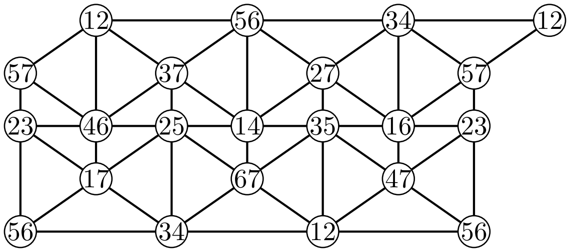
Proof. By Proposition 3.2, a maximal planar graph G has $\tau_2(G) \geq 8$ for $n \leq 13$. Some maximal planar graphs with 2-tone number 8 are K_4, $C_k + \overline{K}_2$ for $k \in \{5, 6, 8, 9\}$, and IC, which proves the theorem for $n \in \{4, 7, 8, 10, 11, 12\}$. By Corollary 2.3, $\tau_2\big(K_3+\overline{K}_2\big) = 9 = \tau_2\big(C_4+\overline{K}_2\big) = \tau_2(P_4 + K_2)$, which covers all possibilities for $n \in \{5, 6\}$. It is easy to add a single degree 3 vertex inside a region of $C_6 + \overline{K}_2$ and IC so that the resulting graphs are 2-tone 8-colorable, which covers $n \in \{9, 13\}$.

Small maximal planar graphs with minimum degree 5 are cataloged at the Combinatorial Object Server [9]. The unique maximal planar graph with minimum degree 5 and order 15 has $\alpha = 4$, so $\tau_2 \geq \frac{2 \cdot n}{\alpha} = \frac{2 \cdot 15}{4} > 7$.

There are three maximal planar graphs with minimum degree 5 and order 16, one of which has a vertex of degree 7. For the others, one has four independent degree 6 vertices, and the other has two pairs of adjacent degree 6 vertices. Both have $\alpha = 4$, so $\tau_2 \geq \frac{2 \cdot n}{\alpha} = \frac{2 \cdot 16}{4} = 8$.

There are four maximal planar graphs with minimum degree 5 and order 17, one of which has a vertex of degree 7. Each of the other three has $\alpha = 5$, and at least one vertex that is not contained in an independent set of size 5. Thus a 7-coloring uses at most $2 \cdot 4 + 5 \cdot 5 = 33$ colors (with repetition), a contradiction. By Lemma 3.10, there are 2-tone 8-colorable graphs of orders 15, 16, and 17.

By Proposition 3.2, there are 2-tone 7-colorable graphs of orders $6k + 2$, $k \geq 2$. The second smallest 2-tone 7-colorable maximal planar graph G_{18} has order 18 (identify the left and right sides of the figure below). It has six degree 6 vertices that induce a cycle.



For larger values of n, we use Theorems 3.6, 3.7, and 3.9 to produce larger 2-tone 7-colorable maximal planar graphs.

Using Theorem 3.6 on G_{6k+2} and G_{14} produces a 2-tone 7-colorable graph with order $6k + 13$ for $k \geq 2$.

Using Theorem 3.6 on G_{6k+2} and G_{18} produces a 2-tone 7-colorable graph with order $6k+17$ for $k \geq 2$.

Using Theorem 3.7 on G_{6k+2} and G_{14} (using copies of $K_4 - e$ with three degree 5 vertices and one degree 6 vertex) produces a 2-tone 7-colorable graph with order $6k+12$ for $k \geq 2$.

Using Theorem 3.7 on G_{6k+2} and G_{18} (using copies of $K_4 - e$ with three degree 5 vertices and one degree 6 vertex) produces a 2-tone 7-colorable graph with order $6k+16$ for $k \geq 2$.

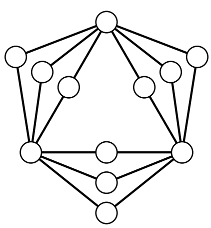
Using Theorem 3.9 on G_{6k+2} and G_{20} with two degree 5 vertices produces a 2-tone 7-colorable graph with order $6k+15$ for $k \geq 3$. (Note that this theorem does not work on G_{14} and G_{20}.)

Using Theorem 3.9 on G_{14} and G_{14} with two degree 5 vertices produces a 2-tone 7-colorable graph with order 21. These cases cover all the remaining values of n. □

The value of $T(n)$ is either 7 or 8 for $n \in \{19, 22, 23, 27\}$.

4 Planar graphs with large 2-tone chromatic number

All the examples we have seen so far have 2-tone number close to $\tau_2(W_{\max})$. Cranston and LaFayette [12] found an example where this is not the case. For each $t \geq 1$, we form H_t from K_3 by replacing each edge $vw \in E(K_3)$ with a copy of $K_{2,t}$, identifying the degree t vertices with v and w.



Cranston and LaFayette [12] showed that

$$\left\lceil \sqrt{3\Delta(H_t) + 0.25} + 0.5 \right\rceil \leq \tau_2(H_t) \leq \left\lceil \sqrt{3\Delta(H_t) + 30.25} + 0.5 \right\rceil.$$

These bounds are close, but not exactly the same. Since they did not find an exact formula for $\tau_2(H_t)$ of these graphs, we strengthen their approach to do so.

Theorem 4.1. *For each $t \geq 1$, we form H_t from K_3 by replacing each edge $vw \in E(K_3)$ with a copy of $K_{2,t}$, identifying the high degree vertices with v and w. For all t we have*

$$\tau_2(H_t) = \begin{cases} \left\lceil \sqrt{3\Delta(H_t) + 6.25} + 1.5 \right\rceil, & \text{if } \Delta \leq 34, \\ \left\lceil \sqrt{3\Delta(H_t) + 24.25} + .5 \right\rceil, & \text{if } \Delta \geq 13. \end{cases}$$

Proof. Let x, y, and z denote the vertices of the original K_3 (they have degrees at least 4 except when $t = 1$). There must be either 3, 4, 5, or 6 colors used on these three vertices. We consider all possible cases (up to permutation of colors) for the labels of x, y, and z.

Suppose the labels on x, y, and z are 12, 13, and 23. Now 1, 2, and 3 cannot be used on any other vertices. Note that all degree 2 vertices are distance 2 from each other, and so need distinct labels. There are $\frac{3}{2}\Delta(H_t) = 3t$ such vertices, so we need $k + 3$ colors, where $\binom{k}{2} \geq 3t$. Thus we use $\lceil 3.5 + \sqrt{3\Delta + 0.25}\ \rceil$ colors.

Suppose the labels on x, y, and z are 12, 13, and 14. Now 1 cannot be used on any other vertex, but 2, 3, and 4 can each be used on a third of the vertices. Labels 23, 24, and 34 cannot be used. Thus we need $k + 1$ colors, where $\binom{k}{2} - 3 \geq 3t$. Thus we use $\lceil 1.5 + \sqrt{3\Delta + 6.25}\ \rceil$ colors.

Suppose the labels on x, y, and z are 12, 13, and 24. Now 1 and 2 cannot be used on any other vertex, but 3 and 4 can each be used on a third of the vertices. Label 34 cannot be used. Thus we need $k + 2$ colors, where $\binom{k}{2} - 1 \geq 3t$. Thus we use $\lceil 2.5 + \sqrt{3\Delta + 2.25}\ \rceil$ colors.

Suppose the labels on x, y, and z are 12, 13, and 45. Now 1 cannot be used on any other vertex, but 2, 3, 4, and 5 can each be used on a third of the vertices. Labels 23, 24, 25, 34, and 35 cannot be used, while 45 can be used again. Thus we need $k + 1$ colors, where $\binom{k}{2} - 5 \geq 3t$. Thus we use at least $\lceil 1.5 + \sqrt{3\Delta + 10.25}\ \rceil$ colors.

Suppose the labels on x, y, and z are 12, 34, and 56. Now these 6 colors can each be used on a third of the vertices. Labels 12, 34, and 56 can be used on degree 2 vertices, but the other 12 labels from [6] cannot. Thus we need k colors, where $\binom{k}{2} - 12 \geq 3t$. Thus we use $\left\lceil 0.5 + \sqrt{3\Delta + 24.25}\, \right\rceil$ colors.

We compare the number of colors from these five cases. We find that $\left\lceil 1.5 + \sqrt{3\Delta + 6.25}\, \right\rceil$ is smallest when $\Delta \leq 34$ and $\left\lceil 0.5 + \sqrt{3\Delta + 24.25}\, \right\rceil$ is smallest when $\Delta \geq 13$. Note that in both of these cases, it is easy to ensure that a coloring with the minimum number of colors actually exists. $\square$

Thus these graphs have $\tau_2(G) \approx \sqrt{3\Delta}$. This can be completed to a maximal planar graph with the same Δ when $t \geq 3$. The following upper bound based on maximum degree is known.

Theorem 4.2 (Cranston and LaFayette [12])**.** *Let G be a planar graph with maximum degree $\Delta = \Delta(G) \geq 3$. Then $\tau_2(G) \leq \left\lfloor \sqrt{4\Delta + 50.25} + 31.1 \right\rfloor$ and $\tau_2(G) \leq \max\left\{41, \left\lfloor \sqrt{4\Delta + 50.25} + 11.5 \right\rfloor \right\}$.*

Cranston and LaFayette [12] also conjecture that there is some C so that $\tau_2(G) \leq \sqrt{3\Delta} + C$ for any planar graph.

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