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A method of constructing pairwise balanced designs containing parallel classes

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Abstract. The obvious way to construct a GDD (group-divisible design) recursively is to use Wilson's Fundamental Construction for GDDs. Then a PBD (pairwise balanced design) is often obtained by adding a new point to each group of the GDD. However, after constructing such a PBD, it might be the case that we then want to identify a parallel class of blocks. In this short note, we explore some possible ways of doing this.

1 Introduction and definitions

We use standard design-theoretic terminology for GDDs (group-divisible designs), PBDs (pairwise balanced designs), and transversal designs (TDs). To begin, we recall a few definitions of these kinds of designs from [1].

Let K and L be sets of positive integers (we can assume that every element of K is at least two). A K*-group-divisible design* (or K-GDD) with group sizes in L is a triple $(X, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

1. X is a set of *points*.
2. $\mathcal{G}$ is a partition of X into *groups* such that $|G| \in L$ for all $G \in \mathcal{G}$.
3. $\mathcal{A}$ consists of a set of *blocks* such that
   (a) $|G \cap A| \leq 1$ for all $G \in \mathcal{G}$ and for all $A \in \mathcal{A}$,
   (b) every pair of points from different groups is contained in exactly one block, and
   (c) $|A| \in K$ for all $A \in \mathcal{A}$.

If $K = \{k\}$, we write k-GDD for simplicity. The *type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We usually employ an exponential notation to describe types: type $t_1^{u_1} t_2^{u_2} \ldots$ denotes u_i occurrences of t_i for $i = 1, 2, \ldots$.

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A $\mathrm{TD}(k,m)$ (or *transversal design*) is a k-GDD of type m^k. Thus every block in a transversal design is a transversal of the k groups. It is well known that a $\mathrm{TD}(k,m)$ is equivalent to $k-2$ mutually orthogonal Latin squares of order m.

Let K be a set of positive integers. A (v,K)*-pairwise balanced design* (or (v,K)-PBD) is a pair $(X,\mathcal{A})$ that satisfies the following properties:

1. X is a set of *points*,
2. $\mathcal{A}$ consists of a set of *blocks* such that every pair of points is contained in exactly one block, and
3. $|A| \in K$ for all $A \in \mathcal{A}$.

If $K = \{k\}$, we write (v,k)-PBD for simplicity. A (v,k)-PBD is also known as a $(v,k,1)$*-balanced incomplete block design* (or $(v,k,1)$-BIBD).

If q is a prime or prime power, then there exist a $(q^2,q,1)$-BIBD (an *affine plane* of order q) and a $(q^2+q+1,q+1,1)$-BIBD (a *projective plane* of order q).

Before describing our constructions for PBDs containing parallel classes, we recall Wilson's Fundamental Construction for GDDs (which we abbreviate to WFC). We follow the presentation from [1, §IV.2.1].

Construction 1.1 (WILSON'S FUNDAMENTAL CONSTRUCTION FOR GDDS)**.** Suppose that $(X,\mathcal{G},\mathcal{A})$ is a GDD and let $w\colon X \to \mathbb{Z}^+$ (w is called a *weighting*). For every block $A \in \mathcal{A}$, suppose there is a K-GDD of type $\{w(x) : x \in A\}$. For all $G \in \mathcal{G}$, define $w_G = \sum_{x \in G} w(x)$. Then there is a K-GDD of type $\{w_G : G \in \mathcal{G}\}$.

2 Constructions

We often construct GDDs recursively using Construction 1.1. Then a PBD can be obtained from the resulting GDD by adding a new point to each group. Suppose, after constructing such a PBD, that we identify a parallel class of blocks. Given a parallel class of blocks, we can then use these blocks as groups to construct a new GDD.

Our next theorem provides one way to accomplish this goal.

Theorem 2.1. *Suppose there is a* $\mathrm{TD}(\ell+1,m)$ *and a* $\mathrm{TD}(\ell,u)$*, where* $u \le m$*. Suppose that there is a* K-GDD *of type* $u^\ell v^1$*, where* $\ell \in K$*. Finally, suppose* *there exists an* $(mu+1, K)$-PBD*. Then there exists a* K-GDD *of type* $\ell^{mu}(tv+1)^1$ *for all* t *such that* $0 \le t \le m-u$*.*

Proof. Start with a TD$(\ell+1, m)$ and delete $m-t$ points from the last group. This yields an $(\{\ell, \ell+1\}, 1)$-GDD of type $m^\ell t^1$. The blocks have sizes ℓ and $\ell+1$, and every block of size $\ell+1$ intersects the last group. Give every point in the first ℓ groups weight u, give every point in the last group weight v, and apply WFC (Construction 1.1). For a block of size ℓ, we fill in a TD(ℓ, u). For a block of size $\ell+1$, we fill in a K-GDD of type $u^\ell v^1$.

The blocks in a TD(ℓ, u) have size ℓ. We have assumed that $\ell \in K$, so we now have a K-GDD of type $(mu)^\ell (tv)^1$. Let ∞ be a new point. Replace every group G of size mu by an $(mu+1, K)$-PBD on $G \cup \{\infty\}$. Also, add ∞ to the last group. This produces an $(mu\ell + tv + 1, K \cup \{tv+1\})$-PBD.

It remains to identify a parallel class of blocks in this PBD. The parallel class will consist of mu blocks of size ℓ and the block of size $tv+1$. Choose u of the points that were deleted from the last group of the TD$(\ell+1, m)$ (note that $u \le m-t$). These u points induce u classes of m blocks of size ℓ, each of which partitions the points in the first ℓ groups of the TD$(\ell+1, m)$. Denote these classes by $\mathcal{P}_i$, $1 \le i \le u$.

When we apply WFC, we replace every point x in the first ℓ groups by a set of u points, say $\{x\} \times \{1, \dots, u\}$. Every block B of size ℓ is replaced by the u^2 blocks in a TD(ℓ, u), in which the groups are $\{x\} \times \{1, \dots, u\}$, $x \in B$. For all $B \in \mathcal{P}_i$ (where $1 \le i \le u$), we can stipulate that $B' = \{(x, i) : x \in B\}$ is one of the blocks in the TD(ℓ, u) constructed from B.

Now define

$$\mathcal{P} = \{B' : B \in \mathcal{P}_i, 1 \le i \le u\}.$$

It is easily seen that $\mathcal{P}$ is a set of mu blocks of size ℓ that form a partition of the first ℓ groups of the K-GDD of type $(mu)^\ell (tv)^1$. In the constructed PBD, there is a unique block B_0 of size $tv+1$ arising from the last group of the TD$(\ell+1, m)$ together with ∞. The blocks in $\mathcal{P}$, along with B_0, comprise the desired parallel class. This parallel class is taken to be the groups in a K-GDD of type $\ell^{mu}(tv+1)^1$. $\square$

Here is a specific application of Theorem 2.1.

Corollary 2.2. *Suppose* $m \equiv 0$ *or* $1 \pmod 5$*,* $m > 10$*, and let* $0 \le t \le m-4$*. Then there exists a* 5-GDD *of type* $5^{4m}(4t+1)^1$*.*

Proof. We apply Theorem 2.1 with $\ell = 5$, $u = v = 4$, and $K = \{5\}$. A K-GDD of type $u^\ell v^1$ is just a 5-GDD of type 4^6, which is obtained from an affine plane of order 5 with a point deleted. A $\mathrm{TD}(\ell, u)$ is obtained from a projective plane of order 4 by deleting a point. A $\mathrm{TD}(6, m)$ exists from [1, §III.3.6]. An $(mu+1, K)$-PBD is just a $(4m+1, 5, 1)$-BIBD, which exists because $m \equiv 0$ or 1 (mod 5) (see [1, §II.3.1]). We obtain a 5-GDD of type $5^{4m}(4t+1)^1$. □

Remark 2.3. The constructed PBD has blocks of size 5 and a block of size $4t+1$. Many results on such PBDs are known, e.g., see [1, §IV.1.2]. But there is apparently less information known on when such a PBD contains a parallel class that includes the block of size $4t+1$.

Here is one small numerical example to illustrate.

Example 2.4. We construct 5-GDDs of type $5^{44}s^1$ for $s = 1, 5, \dots, 29$. It suffices to take $m = 11$, let $t = 0, 1, \dots, 7$, and apply Corollary 2.2.

We next observe that we can improve Theorem 2.1 if we have some information about the existence of disjoint blocks in the $\mathrm{TD}(\ell, u)$.

Theorem 2.5. *Suppose there is a* $\mathrm{TD}(\ell+1, m)$. *Suppose also that there is a* $\mathrm{TD}(\ell, u)$ *containing* α *disjoint blocks, where* $u \leq m$. *Suppose* $\ell \in K$ *and suppose that there is a* K-GDD *of type* $u^\ell v^1$. *Finally, suppose there exists an* $(mu+1, K)$-PBD. *Then there exists a* K-GDD *of type* $\ell^{mu}(tv+1)^1$ *for all* t *such that* $0 \leq t \leq m - \lceil u/\alpha \rceil$.

Proof. The construction of the $(mu\ell+tv+1, K \cup \{tv+1\})$-PBD is the same as in the proof of Theorem 2.1. However, we construct the parallel class of blocks of size ℓ slightly differently. Choose $\lceil u/\alpha \rceil$ of the points that were deleted from the last group of the $\mathrm{TD}(\ell+1, m)$ (note that $\lceil u/\alpha \rceil \leq m - t$). These $\lceil u/\alpha \rceil$ points induce $\lceil u/\alpha \rceil$ classes of m blocks of size ℓ, each of which partitions the points in the first ℓ groups of the $\mathrm{TD}(\ell+1, m)$. Denote these classes by $\mathcal{P}_i$, $1 \leq i \leq \lceil u/\alpha \rceil$.

When we apply WFC, we replace every point x in the first ℓ groups by a set of u points, say $\{x\} \times \{1, \dots, u\}$. Every block $B \in \mathcal{P}_i$ is replaced by the u^2 blocks in a $\mathrm{TD}(\ell, u)$ in which the groups are $\{x\} \times \{1, \dots, u\}$, $x \in B$.

Partition the set $\{1, \dots, u\}$ into $\lceil u/\alpha \rceil$ disjoint sets, say $T_1, \dots, T_{\lceil u/\alpha \rceil}$, each of size at most α. Each $\mathrm{TD}(\ell, u)$ contains α disjoint blocks. For $1 \leq i \leq$

$\lceil u/\alpha \rceil$, for each $B \in \mathcal{P}_i$ and every $j \in T_i$, define $B'_j = \{(x,j) : x \in B\}$. We can stipulate that the α blocks B'_j (for all $j \in T_i$) are blocks in the $\mathrm{TD}(\ell, u)$ constructed from B.

Now define

$$\mathcal{P} = \{B'_j : B \in \mathcal{P}_i, 1 \leq i \leq \lceil u/\alpha \rceil, j \in T_i\}.$$

It is easily seen that $\mathcal{P}$ is a set of mu blocks of size ℓ that form a partition of the first ℓ groups of the K-GDD of type $(mu)^\ell (tv)^1$. The rest of the construction proceeds as before. □

Remark 2.6. The advantage of Theorem 2.5 (with $\alpha > 1$) as compared to Theorem 2.1 is that we can take larger values of t in Theorem 2.5.

In order to apply Theorem 2.5, we need to know something about disjoint blocks in a $\mathrm{TD}(\ell, u)$. This problem has been studied extensively in the case $\ell = 3$, where a set of disjoint blocks is a partial transversal of the associated Latin square. See Wanless [3] for a survey of results on this problem. We briefly mention a few general results for arbitrary ℓ that are well-known and/or follow from elementary counting arguments. We expect that more results along these lines can be proven, but we do not pursue this problem in this note.

Lemma 2.7.

1. *If there is a* $\mathrm{TD}(\ell, u)$, *then* $\ell \leq u + 1$.
2. *A* $\mathrm{TD}(u+1, u)$ *does not contain two disjoint blocks.*
3. *A* $\mathrm{TD}(u, u)$ *contains* u *disjoint blocks.*
4. *If there is a* $\mathrm{TD}(\ell+1, u)$, *then there is a* $\mathrm{TD}(\ell, u)$ *that contains* u *disjoint blocks.*
5. *A* $\mathrm{TD}(\ell, u)$ *with* $u \geq \ell \geq 2$ *contains at least three disjoint blocks, unless* $u = \ell = 2$.
6. *A* $\mathrm{TD}(\ell, u)$ *contains at least* $\left\lceil \frac{u^2}{\ell(u-1)+1} \right\rceil$ *disjoint blocks.*

Proof. Parts 1–4 are well-known, so we only provide a proof of parts 5 and 6. First we prove part 5. A block B in a $\mathrm{TD}(\ell, u)$ intersects $\ell(u-1)$ other blocks. There are u^2 blocks. Hence, there exists a block disjoint from B if and only if $u^2 > 1 + \ell(u-1)$ or $\ell < u + 1$. Since $\ell \leq u$, there are at least two disjoint blocks. Now, assume that B_1 and B_2 are disjoint blocks. There are $\ell(\ell-1)$ blocks that intersect both B_1 and B_2. Since every point is in u blocks, there are $2\ell(u-\ell)$ blocks that contain exactly one point from $B_1 \cup B_2$. It follows that there is a block disjoint from both B_1 and B_2 if and only if $u^2 - 2\ell(u-\ell) - \ell(\ell-1) - 2 > 0$. Fix ℓ and define

$$f(u) = u^2 - 2\ell(u-\ell) - \ell(\ell-1) - 2.$$

We have $f'(u) = 2(u-\ell) \geq 0$ if $u \geq \ell$. Also, $f(\ell) = \ell - 2 > 0$ since $\ell > 2$. It follows that $f(u) > 0$ for all $u \geq \ell$ when $\ell \geq 3$. When $\ell = 2$, we have $f(u) = (u-2)^2$, so $f(u) > 0$ if and only if $u > 2$. This establishes the existence of three disjoint blocks in the TD, unless $u = \ell = 2$.

To prove part 6, let $B_1, \dots, B_r$ be a maximal set of r disjoint blocks in a TD(ℓ, u). Denote $Y = \bigcup_{i=1}^{r} B_i$. Since we started with a maximal set of disjoint blocks, there is no block disjoint from Y. Denote the set of $u^2 - r$ blocks other than $B_1, \dots, B_r$ by $\mathcal{B}'$. For $B \in \mathcal{B}'$, define $a_B = |B \cap Y|$. By assumption, $a_B \geq 1$ for all $B \in \mathcal{B}'$. We have

$$\sum_{B \in \mathcal{B}'} a_B = r\ell(u-1).$$

Therefore the mean of the a_B's is

$$\overline{a} = \frac{r\ell(u-1)}{u^2 - r}.$$

Since $\overline{a} \geq 1$, we have

$$\frac{r\ell(u-1)}{u^2 - r} \geq 1,$$

or

$$r\ell(u-1) \geq u^2 - r.$$

Consequently,

$$r \geq \frac{u^2}{\ell(u-1)+1}.$$

Hence, the TD(ℓ, u) contains at least $\left\lceil \frac{u^2}{\ell(u-1)+1} \right\rceil$ disjoint blocks. $\square$

Remarks 2.8.

1. Wilson's 1974 construction for mutually orthogonal Latin squares [4] explicitly makes use of transversal designs containing disjoint blocks. Also, it is observed in [4] that a TD(ℓ, u) contains at least two disjoint blocks if $\ell \leq u$.
2. We observe that Corollary 2.2 is a special case of Theorem 2.5 in which $\alpha = 1$. In view of Lemma 2.7, we cannot take $\alpha > 1$ in this case because $\ell = u + 1$.
3. The bound proven in part 6 of Lemma 2.7 also follows from a more general result due to Rosenfeld [2, Theorem I b)].

Here is an example of an application of Theorem 2.5 with $\alpha = u$.

Corollary 2.9. *Suppose there exist a* TD$(8, m)$ *and a* $(7m + 1, 7, 1)$-BIBD *and let* $0 \leq t \leq m-1$*. Then there exists a* $\{7, 8\}$-GDD *of type* $7^{7m}(7t+1)^1$.

Proof. We apply Theorem 2.5 with $\ell = u = v = \alpha = 7$ and $K = \{7, 8\}$. A projective plane of order 7 with a point deleted yields an 8-GDD of type 7^8, which is a K-GDD of type $u^\ell v^1$. A TD(ℓ, u) with α disjoint blocks is obtained from an affine plane of order 7 (one parallel class yields the groups, and a second parallel class yields α disjoint blocks). A $(7m+1, 7, 1)$-BIBD, which exists by hypothesis, is an $(mu + 1, K)$-PBD. We obtain a $\{7, 8\}$-GDD of type $7^{7m}(7t + 1)^1$. □

Remark 2.10. It is known (see [1, §II.3.1]) that $(7m + 1, 7, 1)$-BIBDs exist for $m \equiv 0$ or $1 \pmod 6$, $m > 372$. Also, TD$(8, m)$ are known to exist for all $m > 74$. So Corollary 2.9 can be applied for all $m \equiv 0$ or $1 \pmod 6$, $m > 372$.

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