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The extreme case of 3-PGDDs with block size 4 and 2 groups

Apiwat Peereeyaphat, Dinesh G. Sarvate, and
Chariya Uiyyasathian

Abstract. We introduce a generalization of group divisible 3-designs with 2 groups, 3-GDDs with 2 associate classes, into partial group divisible 3-designs, 3-PGDDs with 3 associate classes. A *partial group divisible* 3-*design*, 3-PGDD$(n, 2, k; \lambda, \mu_{21}, \mu_{12})$, is a pair $(G_1 \cup G_2, \mathcal{B})$ where G_1 and G_2 are called *groups* of size n and $\mathcal{B}$ is a collection of k-subsets, called *blocks*, of $G_1 \cup G_2$ such that every 3-subset of G_i occurs in λ blocks in $\mathcal{B}$ and every i elements of G_1 and j elements of G_2 occur together in μ_{ij} blocks in $\mathcal{B}$ for $i \neq j \in \{1, 2\}$. Our study focuses on the case $k = 4$. We study obvious necessary conditions for the existence of a 3-PGDD$(n, 2, 4; \lambda, \mu_{21}, 0)$, and prove that they are sufficient whenever $(n-2)\lambda = n\mu_{21}$. Our construction technique relies on the existence of 3-$(n, 4, \lambda)$ designs and some large sets of triple systems.

1 Introduction

A t-(n, k, λ) *design*, is a pair $(X, \mathcal{B})$ where X is an n-set of points and $\mathcal{B}$ is a collection of k-subsets, called *blocks*, of X with the property that every t-subset of X is contained in exactly λ blocks. The parameter λ is called the *index* of the design. A 2-$(n, 3, \lambda)$ design is called a *triple system*, TS(n, λ). A *group divisible design*, GDD$(n, m, k; \lambda, \mu)$, is a collection of k-subsets, called *blocks*, of an nm-set X where the set X of elements is partitioned into m subsets (called *groups*) of size n, every pair of distinct elements of the same group occurs together in λ blocks, and every pair of elements from different groups occurs together in μ blocks. A GDD has two indices λ and μ. Pairs of elements that occur in the same group are called *first associates*, and pairs of elements that occur in different groups are called

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Corresponding author: Chariya Uiyyasathian <chariya.u@chula.ac.th>

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second associates. Note that a GDD$(n, m, 3; \lambda, \lambda)$ is a 2-design and, in fact, a TS(nm, λ). In 2000, the existence problem for a group divisible design with block size three was settled by Fu, Rodger, and Sarvate [1,2]. In 2004, Hurd and Sarvate began to investigate the existence of a group divisible design with block size four and two groups, see [4,5] for more details.

Several years later, in 2018, Sarvate and Bezire [8] extended the definition of group divisible designs with two groups into a 3-design. First, note that any 3-subset of a set is called a *triple*.

Definition 1.1. A *group divisible 3-design*, 3-GDD$(n, 2, k; \lambda, \mu)$, is a pair $(G_1 \cup G_2, \mathcal{B})$ where G_1 and G_2 are called *groups* of size n and $\mathcal{B}$ is a collection of k-subsets, called *blocks*, of $G_1 \cup G_2$ such that

(i) every triple of each group occurs in λ blocks in $\mathcal{B}$ and

(ii) every triple where two elements are from one group and one element from the other group occurs in μ blocks in $\mathcal{B}$.

Triples of elements of $G_1 \cup G_2$ satisfying (i) and (ii) are called *first associates* and *second associates*, respectively.

Moreover, Sarvate and Bezire [8] showed that the obvious necessary conditions are sufficient for the existence of a 3-GDD$(n, 2, 4; \lambda, \mu)$ except possibly when $n \equiv 1, 3 \pmod 6$, $n \neq 3, 7, 13$, and $\lambda > \mu$. Later on, Sarvate and Cowden [9] gave a construction of a 3-GDD$(n, 2, 4; \lambda, \mu)$ for $n \equiv 1, 7, 9 \pmod{12}$. Nonetheless, the existence problem when $n \equiv 3 \pmod{12}$, $n \neq 3$, and $\lambda \equiv 9 \pmod{12}$ was still left unsolved. In 2023, Tefera et al. [10] obtained some necessary conditions for the existence of a 3-GDD$(n, 3, 5; \lambda, \mu)$, and they further presented some constructions especially when $\mu = 0$.

We next introduce some useful notation. In a group divisible 3-design $(G_1 \cup G_2, \mathcal{B})$, a triple of $G_1 \cup G_2$ is an (s, t)-*triple* if it contains s elements from G_1 and t elements from G_2. Thus, triples of $G_1 \cup G_2$ are classified into four types: $(3, 0)$-triples and $(0, 3)$-triples are first associates while $(2, 1)$-triples and $(1, 2)$-triples are second associates. In addition, types of blocks are defined in the same manner. A block in $\mathcal{B}$ is said to be a $[p, q]$-*block* if it contains p elements from G_1 and q elements from G_2.

In our study, we divide the set of all second associate triples into two distinct associate classes depending on the number of elements belonging to each group. We further introduce two new indices corresponding to each associate class. Hence, we present a generalization of Definition 1.1 with three indices as follows:

Definition 1.2. A *partial group divisible 3-design*, 3-PGDD$(n, 2, k; \lambda, \mu_{21}, \mu_{12})$, is a pair $(G_1 \cup G_2, \mathcal{B})$ where G_1 and G_2 are called *groups* of size n and $\mathcal{B}$ is a collection of k-subsets, called *blocks*, of $G_1 \cup G_2$ such that

(i) every $(3, 0)$-triple and $(0, 3)$-triple occurs in λ blocks in $\mathcal{B}$,

(ii) every $(2, 1)$-triple occurs in μ_{21} blocks in $\mathcal{B}$, and

(iii) every $(1, 2)$-triple occurs in μ_{12} blocks in $\mathcal{B}$.

It is obvious that the existence of a 3-PGDD$(n, 2, 4; \lambda, \alpha, \beta)$ is equivalent to the existence of a 3-PGDD$(n, 2, 4; \lambda, \beta, \alpha)$, and if $\alpha = \beta$, then the partial group divisible 3-design is a group divisible 3-design.

In this paper, repeated blocks are allowed in any designs; hence, any union of sets of blocks will always be a multiset union. For an integer m and a set B, we denote mB as the union of m copies of B.

Example 1.3. Let $G_1 = \{1, 2, 3, 4\}$, $G_2 = \{a, b, c, d\}$, and $\mathcal{B} = 2\mathcal{B}_1 \cup \mathcal{B}_2 \cup 4\{G_2\}$ where

$$\mathcal{B}_1 = \big\{\{1, 2, 3, x\}, \{1, 2, 4, x\}, \{1, 3, 4, x\}, \{2, 3, 4, x\} \mid x \in G_2\big\},$$
$$\mathcal{B}_2 = \big\{\{a, b, c, i\}, \{a, b, d, i\}, \{a, c, d, i\}, \{b, c, d, i\} \mid i \in G_1\big\}.$$

Then $(G_1 \cup G_2, \mathcal{B})$ is a 3-PGDD$(4, 2, 4; 8, 4, 2)$.

Our main goal in this paper is to determine necessary and sufficient conditions for the existence of a 3-PGDD$(n, 2, 4; \lambda, \mu_{21}, \mu_{12})$ whenever $\mu_{12} = 0$ with a certain property, namely $(n - 2)\lambda = n\mu_{21}$.

As we focus on 3-PGDDs with $\mu_{12} = 0$, we will simply use μ to denote the second index μ_{21} of the design.

2 Preliminary results

In this section, we introduce some designs as well as related results that are required for our construction.

The two triple systems $(X, \mathcal{B})$ and $(X, \mathcal{C})$ are said to be *disjoint* if $\mathcal{B} \cap \mathcal{C} = \emptyset$.

Definition 2.1. A *large set* of TS(n, λ)s of size N is a set

$$\mathcal{N} = \{(X, \mathcal{B}_i) \mid 1 \leq i \leq N\}$$

of N disjoint TS(n, λ)s such that $\{\mathcal{B}_i \mid 1 \le i \le N\}$ is a partition of the set of all triples of X. It follows that $N = (n-2)/\lambda$.

It is well known that there exists a TS$(n, 1)$ for any $n \equiv 1, 3 \pmod 6$. In 1984, Lu [6, 7] successfully constructed a large set of TS$(n, 1)$ except for a few cases. Later on, those missing cases were completely solved by Teirlinck [11] in 1991. Furthermore, Teirlinck [12, 13] had also completely solved the existence problem of a large set of TS(n, λ) for each n and λ. We summarize these results in Theorem 2.3. Additionally, Example 2.2 illustrates a large set of TS$(6, 2)$.

Example 2.2. Table 2.1 illustrates that the set of all triples of $X = \{1, 2, 3, 4, 5, 6\}$ can be partitioned into 2 disjoint TS$(6, 2)$s, say $\mathcal{B}_1$ and $\mathcal{B}_2$. Each column in Table 2.1 represents a triple of X.

Table 2.1: Two disjoint TS$(6, 2)$s that partition the set of all triples of X.

blocks of $\mathcal{B}_1$										blocks of $\mathcal{B}_2$									
1	1	1	1	1	2	2	2	3	3	1	1	1	1	1	2	2	2	3	4
2	2	3	4	5	3	4	5	4	4	2	2	3	3	4	3	3	4	5	5
3	4	5	6	6	6	5	6	5	6	5	6	4	6	5	4	5	6	6	6

Theorem 2.3 (Lu-Teirlinck [6, 7, 11–13])**.**

(i) *There exists a large set of* TS$(n, 1)$ *if and only if* $n \equiv 1, 3 \pmod 6$ *and* $n \neq 7$.

(ii) *If* $n \equiv 0, 4 \pmod 6$, *there exists a large set of* TS$(n, 2)$.

(iii) *If* $n \equiv 2 \pmod 6$, *there exists a large set of* TS$(n, 6)$.

(iv) *If* $n \equiv 5 \pmod 6$, *there exists a large set of* TS$(n, 3)$.

Our main construction also requires the existence of a 3-$(n, 4, \lambda)$ design. Hanani [3] completely solved the existence problem of a 3-$(n, 4, \lambda)$ design as shown in the next theorem.

Theorem 2.4 (Hanani [3])**.** *Necessary and sufficient conditions for the existence of a* 3-$(n, 4, \lambda)$ *design are* $\lambda n \equiv 0 \pmod 2$, $\lambda(n-1)(n-2) \equiv 0 \pmod 3$, *and* $\lambda n(n-1)(n-2) \equiv 0 \pmod 8$. *The conditions of* λ *for each* n *modulo* 12 *such that a* 3-$(n, 4, \lambda)$ *design exists are given in Table* 2.2.

Table 2.2: Conditions on n and λ for the existence of a 3-$(n, 4, \lambda)$ design.

$n \pmod{12}$	λ
$0, 6$	$0 \pmod 3$
$1, 5$	$0 \pmod 2$
$2, 4, 8, 10$	positive integer λ
3	$0 \pmod{12}$
$7, 11$	$0 \pmod 4$
9	$0 \pmod 6$

3 Necessary conditions

Let $(G_1 \cup G_2, \mathcal{B})$ be a 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$. A *replication number* of an element $x \in G_1 \cup G_2$ in the design is the number of blocks in $\mathcal{B}$ containing x. For $i, j \in \{1, 2\}$, we define r_i to be the replication number of elements from G_i. We define r_{ij} to be the number of blocks containing a pair of elements $x \in G_1$ and $y \in G_2$. Obviously, $r_{12} = r_{21}$. Moreover, we let b be the number of blocks in $\mathcal{B}$.

According to the types of blocks contained in $\mathcal{B}$ and the fact that the parameters r_1, r_2, r_{11}, r_{22}, r_{12}, and b are nonnegative integers, we will obtain necessary conditions for the existence of a 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$. First, we determine the parameters we defined in the previous paragraph.

Now we count the number of all triples of $G_1 \cup G_2$ in $\mathcal{B}$ that contain x to determine r_1 and r_2. Let $x \in G_1$. We count the number of all triples containing x in two ways. Since there are r_1 blocks in the design that contain x and each of these blocks gives 3 triples containing x, there are $3r_1$ such triples in total. On the other hand, we can count triples of each type described in Definition 1.2 separately. Therefore, $3r_1 = \binom{n-1}{2}\lambda + n(n-1)\mu$, and hence

$$r_1 = \frac{(n-1)(n-2)\lambda + 2n(n-1)\mu}{6}. \tag{1}$$

We use a similar counting method considering $x \in G_2$ to obtain

$$r_2 = \frac{(n-1)(n-2)\lambda + n(n-1)\mu}{6}. \tag{2}$$

Next, in order to determine r_{11}, r_{22}, and r_{12}, we count the number of triples occurring in the blocks of $\mathcal{B}$ that contain both x and y in two ways. One is to count triples obtained from all blocks of $\mathcal{B}$ that contain both x and y.

Second is to count $(3,0)$-triples, $(0,3)$-triples, and $(2,1)$-triples that contain both x and y. Hence we obtain the following parameters:

$$r_{11} = \frac{(n-2)\lambda + n\mu}{2}, \tag{3}$$

$$r_{22} = \frac{(n-2)\lambda}{2}, \tag{4}$$

$$r_{12} = \frac{(n-1)\mu}{2}. \tag{5}$$

Finally, we determine b by counting the number of all triples in the design. Since each block in $\mathcal{B}$ provides 4 triples, the design provides $4b$ triples. On the other hand, the number of $(3,0)$-triples, $(0,3)$-triples, and $(2,1)$-triples are $\binom{n}{3}$, $\binom{n}{3}$, and $\binom{n}{2}n$, respectively. Therefore, $4b = 2\binom{n}{3}\lambda + \binom{n}{2}n\mu$, and thus

$$b = \frac{2n(n-1)(n-2)\lambda + 3n^2(n-1)\mu}{24}. \tag{6}$$

Now we have everything ready to establish obvious necessary conditions for the existence of a 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$.

Theorem 3.1 (Necessary conditions)**.** *Let $n, \lambda \in \mathbb{N}$ and $\mu \in \mathbb{N} \cup \{0\}$ be such that $n \geq 4$. If there exists a* 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$*, then*

(i) *the parameters in Equations* (1)–(6) *are nonnegative integers,*
(ii) *there exists a* 3-$(n, 4, \lambda)$ *design, and*
(iii) $\lambda \geq \frac{n}{n-2}\mu$.

Proof.
(i) Obviously, the replication numbers as well as the number of blocks are nonnegative integers.
(ii) Since there are no $(1,2)$-triples occurring in a 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$, each $(0,3)$-triple must be contained in some block of a 3-$(n, 4, \lambda)$ design on G_2. Note that the condition of n and λ for the existence of such a design on G_2 is concluded in Theorem 2.4.
(iii) Any $(2,1)$-triple must be contained in a $[3,1]$-block that contains another two $(2,1)$-triples and a triple of G_1. Then the number of triples of G_1 must be at least one-third the number of $(2,1)$-triples, in other words, $\binom{n}{3}\lambda \geq \frac{1}{3}\binom{n}{2}n\mu$. □

Since r_1, r_2, r_{11}, r_{22}, r_{12}, and b are in terms of n, λ, and μ, we can establish another version of Theorem 3.1 as follows.

Corollary 3.2 (Necessary conditions)**.** *Let* $n, \lambda \in \mathbb{N}$ *and* $\mu \in \mathbb{N} \cup \{0\}$ *be such that* $n \geq 4$. *If there exists a* 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$, *then*

(i) μ *is even,*
(ii) *if* n *is odd, then* λ *is even,*
(iii) *if* $n \equiv 0 \pmod 3$, *then* $\lambda \equiv 0 \pmod 3$,
(iv) *if* $n \equiv 2 \pmod 3$, *then* $\mu \equiv 0 \pmod 6$,
(v) *if* $n \equiv 3 \pmod 4$, *then* $\mu \equiv 0 \pmod 4$,
(vi) *there exists a* 3-$(n, 4, \lambda)$ *design, and*
(vii) $\lambda \geq \frac{n}{n-2}\mu$.

Proof.
(i) By Equations (3)–(5), $r_{11} - r_{22} - r_{12} = \frac{\mu}{2}$ is an integer, thus, μ is even.
(ii) By Equation (4), $r_{22} = \frac{(n-2)\lambda}{2}$ is an integer. Hence, if n is odd, λ is even.
(iii) By Equation (1), r_1 is an integer, and since $n \equiv 0 \pmod 3$, we have $(n-1)(n-2)\lambda \equiv 0 \pmod 6$. Hence $\lambda \equiv 0 \pmod 3$.
(iv) By Equations (1)–(2), $r_1 - r_2 = \frac{n(n-1)\mu}{6}$ is an integer, and it follows that $n(n-1)\mu \equiv 0 \pmod 3$. If $n \equiv 2 \pmod 3$, then $\mu \equiv 0 \pmod 3$. Therefore, $\mu \equiv 0 \pmod 6$ as μ is even, by (ii).
(v) By Equation (6), b is an integer, so we have $2n(n-1)(n-2)\lambda + 3n^2(n-1)\mu \equiv 0 \pmod{24}$. When $n \equiv 3 \pmod 4$, it follows that $2n(n-1)(n-2)\lambda \equiv 0 \pmod{24}$, and hence $3n^2(n-1)\mu \equiv 0 \pmod{24}$. Since n is odd, $(n-1)\mu \equiv 0 \pmod 8$, but $n - 1 \equiv 2 \pmod 4$. Therefore, we have $\mu \equiv 0 \pmod 4$.
(vi) and (vii) have been proved in the previous theorem. □

4 Main results

Table 4.1 summarizes the necessary conditions for the existence of a 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$ for each n in modulo 12 from Corollary 3.2 except the condition *(vii)* $\lambda \geq \frac{n}{n-2}\mu$. Note that the condition *(v)* in Corollary 3.2, as concluded in Theorem 2.4, yields the following conditions: $\lambda \equiv 0 \pmod{12}$ when $n \equiv 3 \pmod{12}$ and $\lambda \equiv 0 \pmod 4$ when $n \equiv 7, 11 \pmod{12}$.

Our goal is to study the existence of a 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$ in the extreme case of the bound $\lambda \geq \frac{n}{n-2}\mu$, in other words, when

$$(n-2)\lambda = n\mu. \tag{7}$$

Table 4.1: Conditions on n, λ, and μ obtained from Corollary 3.2(i)–(vi).

$n \pmod{12}$	Conditions for λ and μ
0, 6	$\lambda \equiv 0 \pmod 3$ and $\mu \equiv 0 \pmod 2$
1	$\lambda \equiv \mu \equiv 0 \pmod 2$
2, 8	$\mu \equiv 0 \pmod 6$
3	$\lambda \equiv 0 \pmod{12}$ and $\mu \equiv 0 \pmod 4$
4, 10	$\mu \equiv 0 \pmod 2$
5	$\lambda \equiv 0 \pmod 2$ and $\mu \equiv 0 \pmod 6$
7	$\lambda \equiv 0 \pmod 4$ and $\mu \equiv 0 \pmod 4$
9	$\lambda \equiv 0 \pmod 6$ and $\mu \equiv 0 \pmod 2$
11	$\lambda \equiv 0 \pmod 4$ and $\mu \equiv 0 \pmod{12}$

First, we investigate in Lemma 4.1 that the parameters λ and μ that satisfy the equation $(n-2)\lambda = n\mu$ and all necessary conditions in Corollary 3.2 must be in some specific forms.

Lemma 4.1. *Let $n \geq 4$, $\lambda \geq 1$, $\mu \geq 0$, and $(n-2)\lambda = n\mu$. Then n, λ, and μ satisfy the conditions in Corollary 3.2 if and only if $\lambda = cn$ and $\mu = c(n-2)$ for*

$$(n, c) \in \left\{ (4r, s),\ (4r+1, 2s),\ \left(4r+2, \tfrac{s}{2}\right),\ (4r+3, 4s) \ \middle|\ r, s \in \mathbb{N} \right\}.$$

Proof. Let $n \geq 4$, $\lambda \geq 1$, and $\mu \geq 0$ satisfy all conditions in Corollary 3.2 and $(n-2)\lambda = n\mu$. We separate the proof into two cases depending on the parity of n.

Case 1: n is odd.

Since $\mu = \frac{n-2}{n}\lambda$ is a positive integer and $\gcd(n-2, n) = 1$, it follows that $\lambda = cn$ and $\mu = c(n-2)$ for some $c \in \mathbb{N}$. If $n \equiv 1 \pmod 4$, then $\mu = c(n-2)$ is an even integer by Corollary 3.2(i), and hence c must be an even integer. If $n \equiv 3 \pmod 4$, then $\mu = c(n-2) \equiv 0 \pmod 4$ by Corollary 3.2(v), and hence $c \equiv 0 \pmod 4$.

Case 2: n is even.

Since $\gcd\left(\frac{n-2}{2}, \frac{n}{2}\right) = 1$ and $\mu = \frac{n-2}{n}\lambda = \frac{(n-2)/2}{n/2}\lambda$ is a positive integer, it follows that $\lambda = d\left(\frac{n}{2}\right)$ and $\mu = d\left(\frac{n-2}{2}\right)$ for some $d \in \mathbb{N}$. By Corollary 3.2(i), $\mu = d\left(\frac{n-2}{2}\right)$ must be an even integer. If $n \equiv 0 \pmod 4$, then $\frac{n-2}{2}$ is odd, and hence d must be an even integer. Thus $\frac{d}{2} \in \mathbb{N}$. That is, we can write $\lambda = cn$ and $\mu = c(n-2)$, where $c \in \mathbb{N}$. If $n \equiv 2 \pmod 4$, then $\frac{n-2}{2}$ is even, and hence d can be any integer. That is, we can write $\lambda = cn$ and $\mu = c(n-2)$, where $c = \frac{d}{2}$ and $d \in \mathbb{N}$.

Conversely, let $\lambda = cn$ and $\mu = c(n-2)$ for $(n, c) \in \{(4r, s), (4r+1, 2s), \left(4r+2, \frac{s}{2}\right), (4r+3, 4s) \mid r, s \in \mathbb{N}\}$. Together with the equation $(n-2)\lambda = n\mu$, it can be verified directly by Table 4.1 that n, λ, and μ satisfy all conditions given in Corollary 3.2. □

Now, it remains to construct a 3-PGDD$(n, 2, 4; cn, c(n-2), 0)$ for every $(n, c) \in \left\{(4r, s), (4r+1, 2s), \left(4r+2, \frac{s}{2}\right), (4r+3, 4s) \mid r, s \in \mathbb{N}\right\}$.

First, we deal with the case $n \not\equiv 2 \pmod 4$. We begin with our essential theorem revealing that a 3-PGDD$(n, 2, 4; cn, c(n-2), 0)$ can be constructed from a 3-design with a certain index.

Notation. Given a set of triples $\mathcal{T}$ and $x \notin T$ for every $T \in \mathcal{T}$, we denote $\mathcal{T} * x$ as a collection of blocks of size four as follows:

$$\mathcal{T} * x = \left\{T \cup \{x\} \mid T \in \mathcal{T}\right\}.$$

Lemma 4.2. *Let $n, c \in \mathbb{N}$ be such that $n \geq 4$. If a 3-$(n, 4, cn)$ design exists, then a 3-PGDD$(n, 2, 4; cn, c(n-2), 0)$ also exists.*

Proof. Let G_1 and G_2 be groups of size n, where $G_2 = \{x_i \mid 1 \leq i \leq n\}$. Assume that $c \in \mathbb{N}$ is such that a 3-$(n, 4, cn)$ design exists. We know that the set $\mathcal{T}$ of all triples of G_1 is a TS$(n, n-2)$. Next, let $\mathcal{B}_1 = \bigcup_{i=1}^{n} \mathcal{T} * x_i$. Then the blocks in $\mathcal{B}_1$ cover each $(3, 0)$-triple and each $(2, 1)$-triple for n and $n-2$ times, respectively. Thus, the blocks in $c\mathcal{B}_1$ cover each $(3, 0)$-triple and each $(2, 1)$-triple for cn and $c(n-2)$ times, respectively, as desired. By assumption, let $\mathcal{B}_2$ be a 3-$(n, 4, cn)$ design on G_2. Hence, $(G_1 \cup G_2, c\mathcal{B}_1 \cup \mathcal{B}_2)$ is a 3-PGDD$(n, 2, 4; cn, c(n-2), 0)$. □

Example 4.3 illustrates the construction in Lemma 4.2 when $n = 5$ and $c = 2$.

Example 4.3 (*The construction of a* 3-PGDD$(5, 2, 4; 10, 6, 0)$ *with groups* $G_1 = \{1, 2, 3, 4, 5\}$ *and* $G_2 = \{x_i \mid 1 \leq i \leq 5\}$)**.** It can be observed that the set $\mathcal{T}$ of all triples of G_1 is a TS$(5, 3)$ on G_1. Let $\mathcal{B} = \bigcup_{i=1}^{5} \mathcal{T} * x_i$. We illustrate some of these blocks in Table 4.2. Moreover, by Theorem 2.4, there exists a 3-$(5, 4, 10)$ design on G_2, say $(G_2, \mathcal{C})$. Hence $(G_1 \cup G_2, 2\mathcal{B} \cup \mathcal{C})$ is a 3-PGDD$(5, 2, 4; 10, 6, 0)$ as desired.

Table 4.2: Blocks of $\mathcal{T} * x_1$.

blocks of $\mathcal{T} * x_1$									
1	1	1	1	1	1	2	2	2	3
2	2	2	3	3	4	3	3	4	4
3	4	5	4	5	5	4	5	5	5
x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1

We can verify that each $(n,c) \in \{(4r,s),(4r+1,2s),(4r+3,4s) \mid r,s \in \mathbb{N}\}$ satisfies the conditions in Theorem 2.4, and hence a 3-$(n,4,cn)$ design exists. Consequently, a 3-PGDD$(n,2,4;cn,c(n-2),0)$ exists by Lemma 4.2.

Theorem 4.4. *Let* $(n,c) \in \{(4r,s),(4r+1,2s),(4r+3,4s) \mid r,s \in \mathbb{N}\}$. *There exists a* 3-PGDD$(n,2,4;cn,c(n-2),0)$.

Indeed, the construction of a 3-PGDD$(n,2,4;cn,c(n-2),0)$ in Lemma 4.2 also works for the case $n \equiv 2 \pmod 4$ when $c = \frac{s}{2}$ is a natural number, that is, only when s is even. However, the next theorem presents the construction method that works for every natural number s. Our construction technique utilizes large sets of triple systems as mentioned in Theorem 2.3.

Theorem 4.5. *Let* $n,s \in \mathbb{N}$ *be such that* $n \geq 6$ *and* $n \equiv 2 \pmod 4$. *There exists a* 3-PGDD$(n,2,4;\frac{s}{2}n,\frac{s}{2}(n-2),0)$.

Proof. Let G_1 and G_2 be groups of size n, where $G_2 = \{x_i \mid 1 \leq i \leq n\}$.

Case 1: $n \equiv 2 \pmod{12}$.

We start with the construction of a 3-PGDD$(n,2,4;\frac{n}{2},\frac{n-2}{2},0)$.

By Theorem 2.3(iii), the set of all triples of G_1 can be partitioned into $\frac{n-2}{6}$ disjoint TS$(n,6)$s, say

$$\mathcal{T}_1,\mathcal{T}_2,\ldots,\mathcal{T}_{\frac{n-2}{12}},\mathcal{T}'_1,\mathcal{T}'_2,\ldots,\mathcal{T}'_{\frac{n-2}{12}}.$$

Now, let

$$\mathcal{B}_1 = \bigcup_{i=1}^{\frac{n}{2}} \bigcup_{j=1}^{\frac{n-2}{12}} \left(\mathcal{T}_j * x_{2i-1} \cup \mathcal{T}'_j * x_{2i}\right).$$

Then the blocks in $\mathcal{B}_1$ cover each $(3,0)$-triple and each $(2,1)$-triple for $\frac{n}{2}$ and $\frac{n-2}{2}$ times, respectively. Next, note that $\frac{n}{2} \in \mathbb{N}$, so there exists a 3-$(n, 4, \frac{n}{2})$ design on G_2 by Theorem 2.4, say $(G_2, \mathcal{C}_1)$. Then $(G_1, \mathcal{B}_1 \cup \mathcal{C}_1)$ is a 3-PGDD$(n, 2, 4; \frac{n}{2}, \frac{n-2}{2}, 0)$. Therefore,

$$\big(G_1, s\,(\mathcal{B}_1 \cup \mathcal{C}_1)\big) \text{ is a 3-PGDD } \big(n, 2, 4; \tfrac{s}{2}n, \tfrac{s}{2}(n-2), 0\big).$$

Case 2: $n \equiv 6, 10 \pmod{12}$.
First, we construct a 3-PGDD$(n, 2, 4; \frac{n}{2}, \frac{n-2}{2}, 0)$.

By Theorem 2.3(ii), the set of all triples of G_1 can be partitioned into $\frac{n-2}{2}$ disjoint TS$(n, 2)$s, say

$$\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{\frac{n-2}{4}}, \mathcal{T}'_1, \mathcal{T}'_2, \ldots, \mathcal{T}'_{\frac{n-2}{4}}.$$

Now, let

$$\mathcal{B}_2 = \bigcup_{i=1}^{\frac{n}{2}} \bigcup_{j=1}^{\frac{n-2}{4}} \left(\mathcal{T}_j * x_{2i-1} \cup \mathcal{T}'_j * x_{2i}\right).$$

Then the blocks in $\mathcal{B}_2$ cover each $(3, 0)$-triple and each $(2, 1)$-triple for $\frac{n}{2}$ and $\frac{n-2}{2}$ times, respectively. Next, note that $\frac{n}{2} \equiv 0 \pmod 3$ as $n \equiv 6 \pmod{12}$ and $\frac{n}{2} \in \mathbb{N}$ as $n \equiv 10 \pmod{12}$. Hence, a 3-$(n, 4, \frac{n}{2})$ design on G_2 exists by Theorem 2.4, say $(G_2, \mathcal{C}_2)$. Then $(G_1, \mathcal{B}_2 \cup \mathcal{C}_2)$ is a 3-PGDD$(n, 2, 4; \frac{n}{2}, \frac{n-2}{2}, 0)$. Therefore,

$$\big(G_1, s\,(\mathcal{B}_2 \cup \mathcal{C}_2)\big) \text{ is a 3-PGDD } \big(n, 2, 4; \tfrac{s}{2}n, \tfrac{s}{2}(n-2), 0\big). \qquad \square$$

Example 4.6 illustrates the construction in Theorem 4.5 when $n = 6$ and $s = 1$.

Example 4.6 (*The construction of a* 3-PGDD$(6, 2, 4; 3, 2, 0)$ *with groups* $G_1 = \{1, 2, 3, 4, 5, 6\}$ *and* $G_2 = \{x_i \mid 1 \le i \le 6\}$)**.** As seen in Example 2.2, the set of all triples of G_1 can be partitioned into 2 disjoint TS$(6, 2)$s, $\mathcal{B}_1$ and $\mathcal{B}_2$. Next, let $\mathcal{B} = \bigcup_{i=1}^{3} (\mathcal{B}_1 * x_{2i-1} \cup \mathcal{B}_2 * x_{2i})$. We illustrate some of these blocks in Table 4.3. Besides, by Theorem 2.4, there exists a 3-$(6, 4, 3)$ design on G_2, say $(G_2, \mathcal{C})$. Hence, $(G_1 \cup G_2,\ \mathcal{B} \cup \mathcal{C})$ is a 3-PGDD$(6, 2, 4; 3, 2, 0)$ as desired.

Now the following result can be concluded.

Theorem 4.7. *Let $n \ge 4$, $\lambda \ge 1$, $\mu \ge 0$, and $(n-2)\lambda = n\mu$. There exists a* 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$ *if and only if $\lambda = cn$ and $\mu = c(n-2)$ for*

$$(n, c) \in \left\{(4r, s), (4r+1, 2s), \left(4r+2, \tfrac{s}{2}\right), (4r+3, 4s) \mid r, s \in \mathbb{N}\right\}.$$

Table 4.3: Blocks of $\mathcal{B}_1 * x_1$ and $\mathcal{B}_2 * x_2$.

blocks of $\mathcal{B}_1 * x_1$										blocks of $\mathcal{B}_2 * x_2$									
1	1	1	1	1	2	2	2	3	3	1	1	1	1	1	2	2	2	3	4
2	2	3	4	5	3	4	5	4	4	2	2	3	3	4	3	3	4	5	5
3	4	5	6	6	6	5	6	5	6	5	6	4	6	5	4	5	6	6	6
x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2

Proof. The necessary conditions given in Corollary 3.2 and the equation $(n-2)\lambda = n\mu$ give a restriction of λ and μ as in Lemma 4.1. Conversely, by Theorems 4.4 and 4.5, we can construct a 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$ of all possibly λ and μ. □

Finally, we present the main theorem to conclude the construction of the extreme case of a 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$, which we have successfully done.

Theorem 4.8. *The necessary conditions for the existence of a* 3-PGDD$(n, 2, 4; \lambda, \mu, 0)$ *are sufficient when* $(n-2)\lambda = n\mu$.

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Apiwat Peereeyaphat and Chariya Uiyyasathian
Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand
apiwat.prt@gmail.com, chariya.u@chula.ac.th

Dinesh G. Sarvate
Collge of Charleston, Charleston, SC, USA
SarvateD@cofc.edu