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On longest cut-through paths in plane graphs

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Abstract. A path P in a 4-regular plane graph G is called a cut-through path if no two consecutive edges of P are consecutive in the local rotation of edges around the common vertex. We investigate longest cut-through paths in 4-regular plane graphs of several families, providing the lower and the upper bounds on their lengths as well as discussing the extension of the cut-through property for nonregular graphs.

1 Introduction

Throughout this paper, we consider simple connected plane graphs (that is, the graphs without loops or multiple edges, which are drawn in the plane in such a way that no two edges cross). The used graph terminology and notation is from the book [13].

One of the widely studied topics of graph theory concerns the longest paths in graphs. Denoting by $p(G)$ the number of vertices of the longest path in a graph G, one aims to determine the lower and the upper bounds for $p(G)$ in terms of selected graph parameters of G and to describe the graphs attaining these bounds as well as particular sufficient conditions for G to enforce the existence of reasonably long paths. A graph G with $p(G) = |V(G)|$ is called *traceable*; the current state-of-art on these graphs is well documented in the survey [7]. For the opposite side of the problem—to have $p(G)$ as small as possible—the stars $K_{1,r}$ form an infinite graph family with $p(G)$ above bounded by 3. Motivated by this, a lot of attention is paid to the study of longest paths in plane graphs under additional constraints of higher connectivity. In particular, each 4-connected plane graph is Hamiltonian by [12] while there exist infinitely many 3-connected plane graphs such that $p(G) \leq \frac{7}{2}\left|V(G)\right|^{\ln(2)/\ln(3)}$, see [3]. Nevertheless, in the

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latter paper it is proved that for each 3-connected plane graph G, we have $p(G) \geq \left|V(G)\right|^{\ln(2)/\ln(3)}$. For 2-connected plane graphs, it is easy to see that each longest path in $K_{2,r}$, with $r > 2$, has at most 5 vertices.

The longest path problems can be considered also for other kinds of paths in graphs. Notably, for longest induced paths it was proved in [5] that each 3-connected plane graph G contains an induced path of length at least $\frac{1}{2}\left(\frac{1}{3}\log_2\left|V(G)\right| - \log_2\log_2\left|V(G)\right|\right)$. Others have also studied longest heterochromatic paths in edge-colored graphs—as an example, we can mention the paper [2] where the authors proved that, in each edge-colored graph with color degrees of its vertices being lower bounded by a positive integer $k \geq 7$, there exists a heterochromatic path of length at least $\left\lceil \frac{2k}{3} \right\rceil + 1$. Here, we consider these questions for other kinds of paths, which are defined as follows:

Let G be a graph and let R be a system of edge lists (the rotation system) such that each vertex x of G is assigned with a cyclically clockwise-ordered list $l(x)$ of incident edges (the local rotation around x). A path $P = v_0v_1 \dots v_k$, with $k \geq 1$, is called an *anti-A-path* in G with respect to R, if, for each $i \in \{1, \dots, k-1\}$, the edge v_iv_{i+1} of P neither precedes nor succeeds immediately the edge $v_{i-1}v_i$ in the list $l(v_i)$. Note that in the case when R induces an embedding $\widetilde{G}$ of a 2-connected graph G in a plane or a polyhedral embedding into an orientable surface of the higher genus (that is, each face of $\widetilde{G}$ is bounded by a cycle and every two facial cycles meet properly, having either nothing or a single vertex or else a single edge in common), then no two consecutive edges of P lie on the same face of $\widetilde{G}$ (this is, the "anti-property" for the *A-property* of trails, where it is required that every two consecutive edges lie on a common face; for more details on A-trails, see the monograph [6] of H. Fleischner). However, the above mentioned concept can be also used for graph drawings with crossings.

In the case of 4-regular plane (or embedded) graphs, anti-A-paths correspond to so-called *cut-through paths* (CT-paths for short), used before in various contexts—see, for example, the proof in [8] of Steinitz theorem on the characterization of graphs of convex polyhedra. The paper [11] discusses the concept of *straight-ahead walks* in embedded Eulerian graphs that slightly generalizes the CT-property (the consecutive edges $v_{i-1}v_i, v_iv_{i+1}$ of such a walk are opposite in the list $l(v_i)$). Apart of these results, it seems that the properties of CT-paths were not much studied; therefore, our aim is to explore the bounds for the number $p_{\mathrm{CT}}(G)$ of vertices of longest CT-paths in a 4-regular plane/embedded graph G.

The paper is organized as follows. Section 2 contains auxiliary results used for handling CT-paths in plane graphs. In Section 3, we give a general upper bound of $n-2$ for $p_{\mathrm{CT}}(G)$ in an n-vertex 4-regular plane graph G together with the construction showing its sharpness. We also show that this upper bound can be decreased to $\frac{2}{3}n$ if the graph can be decomposed into CT-cycles. Furthermore, we deal with the question of minimum length of longest CT-paths; we show that there are infinitely many 4-regular plane graphs whose longest CT-paths contain just 8 vertices, but only finitely many graphs for longest CT-paths with fewer vertices. Finally, Section 4 deals with anti-A-paths in plane graphs (not necessarily regular) of minimum degree at least 4 and in 4-regular graphs embedded in higher surfaces.

2 Preliminaries

Given a 4-regular plane graph G (with fixed drawing), two adjacent edges of G are called *CT-adjacent* if they are opposite in local edge rotation around their common vertex; a trail (a cycle) of G is a *CT-trail* (*CT-cycle*) if every two of its consecutive edges are CT-adjacent. Note that for an edge $e = uv$ of G, there exists a unique closed CT-trail with initial vertex u and the first edge e. Consequently, the edge set of G can be uniquely partitioned into subsets $E_1, \dots, E_k$ where, for each $i \in \{1, \dots, k\}$, E_i induces a closed CT-trail in G (the graphs $G[E_i]$ are plane and consist of vertices of degrees 4 and possibly 2). If $k = 1$, we will call G a *knot*; if each E_i induces a cycle, G is called *Grötzsch-Sachs graph* (see [4, 9] for their connections with coloring problems). Note that a Grötzsch-Sachs graph corresponds to an arrangement of several closed Jordan curves in the plane in the way that no two of them touch and no three of them intersect in the same point (the intersection points are turned into the graph vertices and segments of closed curves between those intersections into edges); this approach will be often used in our constructions.

Let $P = u_1 u_2 \dots u_k$ be a maximal (not necessarily the longest one) CT-path in G; the edges of G that are incident with vertices of P and not belonging to $E(P)$ are called *outgoing edges* of P. For $i, j \in \{1, \dots, k\}$, with $i < j$, an edge $e = u_i u_j \notin E(P)$ is a *chord* of P. A vertex $x \notin V(P)$ is *under the chord* e (with respect to P), if x lies in the (topological) interior of the closed curve $u_i u_{i+1} \dots u_j u_i$ (similarly, an edge xy is under e if the interior of the arc xy belongs to the interior of that closed curve). If there exists a vertex u_i of P such that at least two outgoing edges incident with u_i are under e, the chord e is *two-sided* for P; otherwise, it is called *one-sided*. Further, e is called a *minimal chord* of P if there is no chord e' of P that is under e (see Figure 2.1 for illustration of these terms).

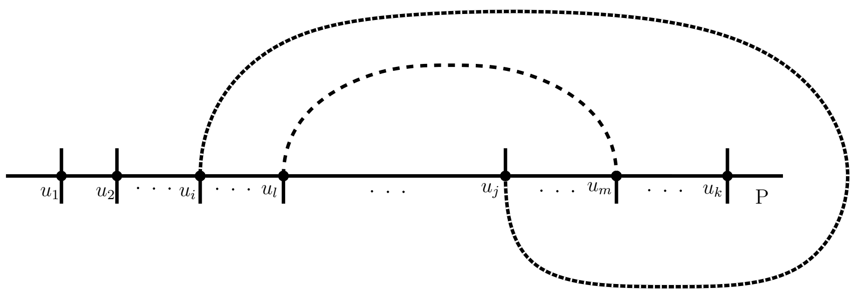

Figure 2.1: One-sided chord $u_l u_m$ and two-sided chord $u_i u_j$ for a CT-path P.

Since G is a simple graph, no CT-path has two distinct chords with the same endvertices, and each maximal CT-path has a chord. In addition, no two-sided chord of a maximal CT-path can be minimal, and, for each CT-path and each chord e, there is a vertex $y \notin V(P)$ that is under e. The latter property implies, in particular, that no 4-regular plane graph contains a Hamiltonian CT-path.

Given a vertex $v \notin V(P)$ that is adjacent to the vertices u_i, u_j of P $(i < j)$, the path $P' = u_i v u_j$ is a *pseudochord* of P if, in the interior of the closed curve $C' = u_i, u_{i+1}, \ldots, u_j, v, u_i$, there is no other neighbor of v. If $j - i = 1$, P' is called a *minimal pseudochord*. With an analogy with chords of CT-paths, we obtain that, if a pseudochord P' of P is not minimal, then there exists a vertex $x \notin V(P)$ that lies under P' (that is, in the interior of C').

Let T be a trail (not necessarily a CT-trail) in G. The subgraph of G induced by edges of T is then a plane graph, and its faces are *regions determined by* T. If $E(T) \subsetneq E(G)$, then $G - E(T)$ consists of subgraphs contained in these regions; we then define a $(d_1, \ldots, d_k)$-*fragment* of T being a connected subgraph $H \subseteq G - E(T)$ such that

- H consists of k vertices of degrees $d_1, \ldots, d_k \in \{1, 2\}$ (the *peripheral* ones; the edges of H incident with these vertices are also called peripheral) and of $|V(H)| - k$ vertices of degree 4 (the *central* ones),
- the peripheral vertices of H are incident with the outerface of H and belong also to T,
- the central vertices of H lie in the same region determined by T.

A (1,1,1,1)-fragment of T with single central vertex is called an *octopus*. An octopus of a CT-path P (with central vertex v and peripheral vertices

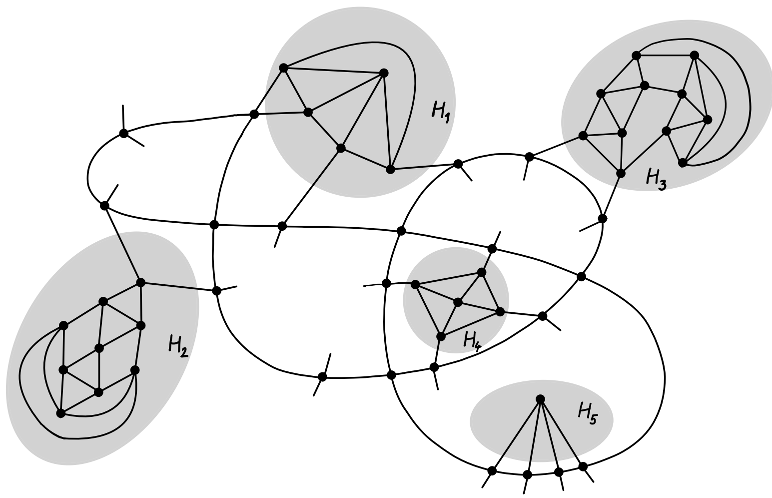

Figure 2.2: An example of a trail and its fragments.

$x_1, \dots, x_4$) is *minimal* if x_1vx_2, x_2vx_3 and x_3vx_4 are minimal pseudochords of P. As an illustrating example, see Figure 2.2: H_1 is a $(2,1,1)$-fragment, H_2, H_3 are $(1,1)$-fragments, H_4 is a $(1,1,1,1)$-fragment that is not an octopus, and H_5 is a minimal octopus.

The following lemmas deal with properties of small fragments of trails:

Lemma 2.1. *Let F be a $(1,1)$-fragment (distinct from K_2) of a trail T of G, and let x, y and xu, yv be its peripheral vertices and edges, respectively.*

- (i) *If $u = v$, then xuy is not a CT-path of G; in addition, G contains CT-paths $xust, yvwz$ (for some vertices s, t, w, z with $s \neq w$).*
- (ii) *If $u \neq v$ and $xuvy$ is not a CT-path of G, then G contains two CT-paths on four vertices such that their first edge is xu (or yv) and the last edge is not vy (or ux).*
- (iii) *If $xuvy$ is a CT-path of G, then G contains a CT-path on at least six vertices whose vertices are from $V(F) \setminus \{x, y\}$.*

Proof.

- (i) Assume that xuy is a CT-path of G. Consider the graph $F - \{x, u, y\}$; as x, y belong to the outerface of F, the obtained graph has exactly two components, each of which contains a single 3-valent vertex, a contradiction (since it is impossible to have an odd number of odd-valent vertices).

Moreover, the outerface of the graph $F - \{x, y\}$ has degree at least 3; hence, there exist vertices s, w such that the path suw is a part of the boundary walk of that outerface and $l(u) = \{yu, xu, wu, su\}$. Since s, w are 4-valent and F is a plane graph, there are vertices t, z such that, in F, the edges us, st (or vw, wz, respectively) are CT-adjacent. Then $xust, yvwz$ are 4-vertex CT-paths of G.

(ii) As u has degree 4 in F, there is a vertex s of F such that xu, us are CT-adjacent. Note that $s \neq v$ and $F - \{x, y\}$ has at least six vertices (otherwise $F - \{x, y\}$ is isomorphic to K_5^-, but then, in plane drawing of F, x, y are separated by a 3-cycle, a contradiction). Then there exists a vertex $t \notin \{u, v\}$ of $F - \{x, y\}$ such that ts, su are CT-adjacent, and so the path $xust$ is a CT-path. The same argument is used for the edge yv.

(iii) By induction on the number of vertices of F. Consider the graph $F - \{x, y, u, v\}$. By assumptions, it has two components F_1, F_2 such that F_1 has vertices p, q (possibly $p = q$) and F_2 has vertices k, l (again, possibly $k = l$) such that up, vq, uk, vl are edges of F. Then F_1 with added edges up, vq can be viewed as a (1,1)-fragment, so, by induction (or by (i), (ii)), it contains a CT-path on four vertices with the first edge up. The same argument can be used on the graph F_2 with added edges uk, vl. Thus, in F, there exist two CT-paths on four vertices with common vertex u (v can be their common vertex as well, but no other vertices), which together form a 6-vertex CT-path in G. □

Lemma 2.2. *Let F be a $(1,1,1,1)$-fragment of T in G. Then either F is an octopus or at least two of its peripheral edges can be extended to a 4-vertex CT-path of F.*

Proof. Let F be a (1,1,1,1)-fragment that is not an octopus. Then there exists a peripheral edge xy and a central vertex z such that xy, yz are CT-adjacent; hence, a maximal CT-path containing the path xyz (note that it ends at another peripheral edge of F) has the desired property. □

3 Longest CT-paths in 4-regular plane graphs

In this section, we derive sharp upper bounds for lengths of the longest CT-paths in 4-regular plane graphs.

Theorem 3.1. *Let G be a simple 4-regular plane graph of order n. Then $p_{CT}(G) \leq n-2$, and the bound is sharp.*

Proof. From the preliminaries of Section 2, we have $p_{\mathrm{CT}}(G) \leq n-1$. Assume that there exists a graph G for which equality holds, and let $P = u_1 \dots u_{n-1}$ be its longest CT-path (hence, there is exactly one vertex u_n of G not lying on P). By maximality of P, there exist vertices $y, z \in V(P)$ such that the edges $u_{n-2}u_{n-1}, u_{n-1}y$ and zu_1, u_1u_2 are CT-adjacent.

If $y \neq u_1$ or $z \neq u_{n-1}$, then the subpath of P determined by vertices u_1, z and all vertices between them contains two vertices u_i, u_j such that the edge u_iu_j is a minimal chord of P. Similarly, such a minimal chord u_ku_l of P exists for the subpath of P determined by y, u_{n-1}. Note that both these chords are one-sided. Then, however, u_n is under u_ku_l with respect to P as well as under u_iu_j, a contradiction.

If $y = u_1$ and $z = u_{n-1}$, then $C = u_1u_2 \dots u_{n-1}u_1$ is a CT-cycle. Using an analogous argument as the above, we obtain that there are two edges of G that are minimal chords of P, one of them being in the interior and another one in the exterior of C. Again, u_n is under both of these chords, a contradiction.

To show the sharpness of the bound $n-2$, note first that the octahedron graph is the only simple 4-regular plane graph on 6 vertices, and its longest CT-path consists of four vertices. In addition, it is known that there is no simple 4-regular plane graph on 7 vertices.

For odd $n \geq 9$, set $t := \frac{n-1}{2}$, take the plane drawing of the path $P = u_1u_2 \dots u_{n-2}$ and two octopuses with central vertices x, y and peripheral edges xu_{t-3}, xu_{t-2}, xu_{t-1}, xu_t and yu_t, yu_{t+1}, yu_{t+2}, yu_{t+3}, respectively. Next, add new edges u_iu_{2t-3-i}, $u_{n-1-i}u_{i+3}$, for $i \in \{1, \dots, t-4\}$, and new edges u_1u_{n-4}, u_1u_{n-3}, $u_{n-2}u_3$, $u_{n-2}u_2$ (all of them are drawn in the way that, in the resulting 4-regular plane graph, they are outgoing with respect to CT-path P). Similarly, for even $n \geq 8$, set $t := \frac{n}{2}$, take $P = u_1u_2 \dots u_{n-2}$ with two octopuses having peripheral edges xu_{t-3}, xu_{t-2}, xu_{t-1}, xu_t and yu_{t-1}, yu_t, yu_{t+1}, yu_{t+2}, respectively; the new edges being added are as before (see Figure 3.1 for illustration). In both cases, P is the desired longest CT-path on $n-2$ vertices. □

Observe that almost all above constructed graphs showing the sharpness of the proved bound (except for the graph on 11 vertices) are actually knots; for Grötzsch-Sachs graphs, we can prove the following smaller upper bound.

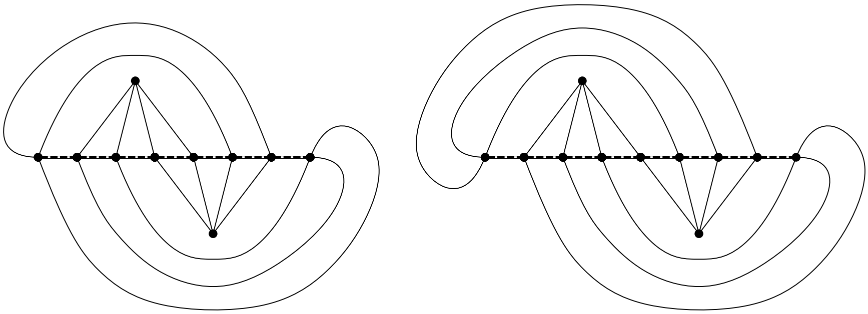

Figure 3.1: Examples of graphs with $p_{\mathrm{CT}}(G) = n - 2$ for $n \in \{10, 11\}$.

Theorem 3.2. *Let G be a Grötzsch-Sachs graph of order n. Then $p_{CT}(G) \leq \frac{2}{3}n$, and the bound is sharp.*

Proof. Let C be the longest CT-cycle in G. Then C contains a longest CT-path of G, so get $p_{\mathrm{CT}}(G) \leq |V(C)|$ and set $|V(C)| := \ell$. Furthermore, the number of all edges which are incident with a vertex of C is at most $\ell + 2\ell = 3\ell$, which is at most the number of all edges of G; since $|E(G)| = 2n$, the result follows. To show the sharpness of the bound, put $n = 6k$ and take a $4k$-cycle $C' = x_1x_2 \ldots x_{4k}x_1$. Now, for every $j \in \{0, \ldots, k-1\}$, add a new vertex y_j into the interior of C and new edges $y_jx_{4j+1}, \ldots, y_jx_{4j+4}$ and, in a similar manner, new vertices z_j with new edges $z_jx_{4j+1}, \ldots, z_jx_{4j+4}$ into the exterior of C. The obtained graph is a plane 4-regular Grötzsch-Sachs graph, and C' is its longest CT-cycle. □

Next, we focus on small values for $p_{\mathrm{CT}}(G)$.

Theorem 3.3. *In the family of 4-regular plane graphs, the following properties hold:*

(a) *$p_{CT}(G) = 4$ if and only if G is the octahedron graph;*

(b) *there is no graph G with $p_{CT}(G) = 5$;*

(c) *for each even $n \geq 10$, there exists G with $p_{CT}(G) = 8$.*

Proof.

(a) Let $P = u_1u_2u_3u_4$ be a longest CT-path in G and let y, z be vertices of G such that u_3u_4, u_4y and u_1z, u_2u_1 are CT-adjacent. By the maximality of P, we have $y, z \in V(P)$.

First, let $y = u_1$ and let u_4u_1, u_1u_2 be not CT-adjacent. Then the interior (as well as the exterior) of the closed trail $u_1u_2u_3u_4u_1$ contains a fragment with an odd number of peripheral 1-valent vertices, a contradiction.

Next, let $y = u_2$. Taking into account the above argument and the fact that G is simple, we can assume that $z = u_3$. Then, for the closed trail $u_1u_2u_4u_3u_1$, the region bounded by the cycle $u_2u_3u_4u_2$ contains a (1,1)-fragment F. By Lemma 2.1, F contains a CT-path u_3uv, with $u, v \notin V(P)$. But then $vuu_3u_1u_2$ is a CT-path on five vertices, a contradiction.

Finally, let $y = u_1$ and $z = u_4$. This yields (as G is simple) that $C = u_1u_2u_3u_4u_1$ is a CT-cycle. If there are two (1,1,1,1)-fragments of C, both being octopuses, then G is the octahedron graph. Otherwise, there is a fragment F_1 of C (say, in the interior of C) that contains a CT-path u_iuv, with $u, v \notin V(P)$; moreover, there is a fragment F_2 in the exterior of C containing a CT-path u_ipq, with $p \notin V(P)$ and $q \neq u_i$. Then vuu_ipq is a CT-path on five vertices, a contradiction.

(b) Let $P = u_1u_2u_3u_4u_5$ be a longest CT-path in G, and let y, z be vertices of G such that u_4u_5, u_5y and u_1z, u_2u_1 are CT-adjacent. Again, by maximality of P, we have $y, z \in V(P)$.

First, let $y = u_1$ and $z = u_5$. This yields (as G is simple) that $C = u_1u_2u_3u_4u_5u_1$ is a CT-cycle; however, C contains a fragment with an odd number of peripheral 1-valent vertices, a contradiction.

Now, let $y = u_1$, but with u_5u_1, u_1u_2 not CT-adjacent. By the fact that G is simple, we have $z \in \{u_3, u_4\}$. If $z = u_4$, then the interior and the exterior of the closed trail $u_4u_1u_2u_3u_4$ contains a fragment with an odd number of 1-valent vertices, a contradiction. Otherwise, for the closed trail $u_1u_5u_4u_3u_2u_1$, there exists a region containing a (1,1)-fragment F with peripheral vertices u_1, u_2. By Lemma 2.1, F contains a CT-path u_1uv such that $u, v \notin \{u_1, \ldots, u_5\}$. Then, however, $vuu_1u_5u_4u_3$ is a CT-path of G on six vertices, a contradiction.

If $y = u_2$, then both the interior and the exterior of the closed trail $u_2u_3u_4u_5u_2$ contain a fragment with an odd number of peripheral 1-valent vertices; hence, we may assume that $y = u_3$. Furthermore, by symmetry and simplicity of G, we can assume that $z = u_3$. Then, for the closed trail $u_3u_4u_5u_3u_1u_2u_3$, there is a region containing a (1,1)-fragment F_1 with peripheral vertices u_1, u_2, a region containing a

(1,1)-fragment F_2 with peripheral vertices u_4, u_5, and another region that contains either a (1,1,1,1)-fragment F or two (1,1)-fragments F_3, F_4 (it may happen that F_3, F_4 are actually two-sided chords of P). By Lemma 2.1, F_2 contains a 4-vertex CT-path with peripheral vertex u_4 that can always be extended (regardless of the nature of F, F_3, and F_4) to a 6-vertex CT-path, a contradiction.

(c) For $n = 10$, take the left graph on Figure 3.1; If $n = 12 + 4k$, take the upper graph on Figure 3.2; and, for $n = 14 + 4k$, take the lower graph on Figure 3.2. These graphs are obtained from arrangements of $k + 5$ closed Jordan curves in "two rows" in the plane. In each of these cases, the longest CT-path has 8 vertices. □

By an extensive computer search on 4-regular plane graphs, we found only four graphs for which $p_{\mathrm{CT}}(G) = 6$ and only one graph with $p_{\mathrm{CT}}(G) = 7$ (see Figure 3.3). Hence, we conjecture that these are the only graphs with the mentioned values for $p_{\mathrm{CT}}(G)$, and we suppose that this might be proved by an analysis of properties of various fragments and their mutual interconnection, which is more complicated than the one in Theorem 3.3(b).

In addition to Theorem 3.3(c) where the constructed graphs were Grötzsch-Sachs graphs, we describe the construction of an infinite family of knots whose longest CT-paths are very short, too. The construction proceeds in the following way: For $k \not\equiv 0 \pmod 3$, take the graph of a k-sided antiprism (which itself is a knot); each of its vertices is then locally replaced with a copy of the configuration in Figure 3.4 (on the right) in such a way that the black/dashed/dotted outgoing edges of the configuration match the style of half-edges in the drawing of k-sided antiprism on the left. It is not hard to see that the obtained graph G_k is also a knot; furthermore, the way of replacement of the antiprism vertices by copies of the configuration yields that, in G_k, each longest CT-path contains at most $5 + 5 + 6 = 16$ vertices.

4 Long anti-A-paths in other plane and embedded graphs

Now we will discuss anti-A-paths in nonregular plane graphs of minimum degree at least 4. Note that, in this case, the maximal anti-A-path with prescribed starting vertex and edge may not be unique, which opens possibilities to have (in contrast with the results of Theorem 5) long anti-A-paths in plane graphs with a (relatively) small number of 4-valent vertices (or with minimum degree 5).

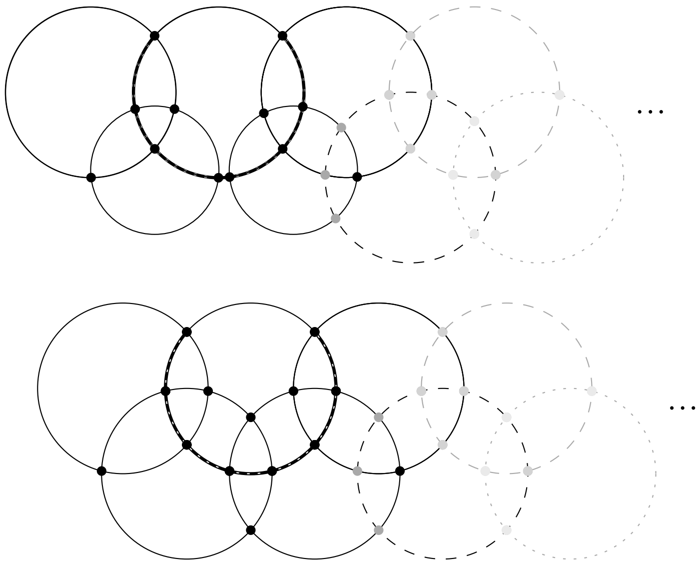

Figure 3.2: 4-regular plane graphs with longest CT-paths on 8 vertices.

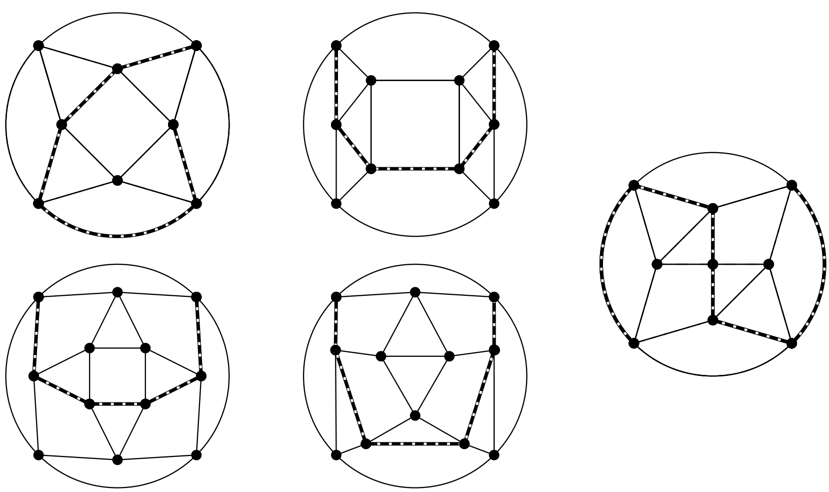

Figure 3.3: All known 4-regular plane graphs with longest CT-paths on 6 and 7 vertices.

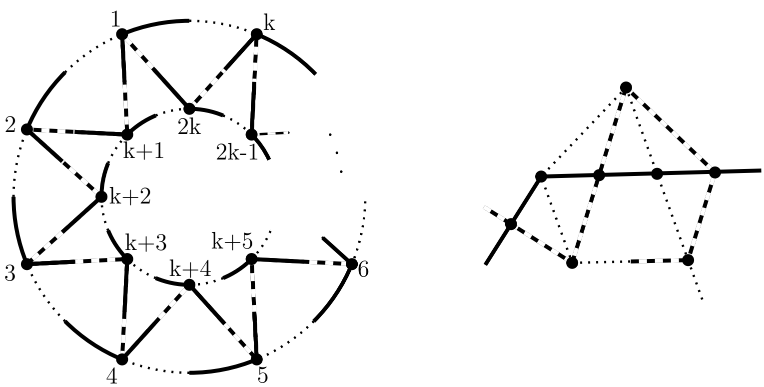

Figure 3.4: A k-sided antiprism graph (left) and the gadget used in the construction of knots with short longest CT-paths.

Lemma 4.1. *Let G be a plane triangulation of minimum degree at least* 4. *Then every induced path in G is also an anti-A-path.*

Proof. By contradiction. Let P be an induced path of G that is not an anti-A-path. Then P contains two consecutive edges uv, vw incident with a common triangular face. This yields that uw is an edge of G, so P is not an induced path, a contradiction. □

Note that the converse of Lemma 4.1 is not true: taking an icosahedron graph and replacing each of its triangular faces by a copy of an octahedron graph, we obtain plane triangulation in which, say, every path consisting of two edges from a triangular face of the original icosahedron graph is an anti-A-path, but is not induced. Note also that the requirement of G being a triangulation is essential because, in a graph of r-sided antiprism with $r \geq 4$, each path of length $\ell \leq r - 2$ that belongs to the boundary of an r-face is an induced path, but not an anti-A-path.

By 3-connectivity of plane triangulations and the result of [5], we obtain that the longest anti-A-paths in plane triangulations of minimum degree at least 4 have lengths at least proportional to the logarithm of their order. Hence, to construct an infinite family of 3-connected plane graphs of minimum degree at least 4 with short longest anti-A-paths, one must consider the graphs with non-triangular faces.

While we have not yet found examples of large 3-connected plane graphs of minimum degree 5 with very short longest anti-A-paths, we can construct an infinite family of nonregular plane graphs of minimum degree at least 4 whose longest anti-A-paths contain at most 20 vertices. Half of their vertices are 4-valent, and 4-valent vertices induce 4-cycles. These graphs can be constructed as follows: take a 4-regular plane multigraph G from Figure 4.1 (on the left) and replace each 4-valent vertex with a gadget (on the right) formed from a copy of the 4-sided antiprism graph. Observe that, in the obtained graph G^+, each anti-A-path, which contains an edge joining two 5-valent vertices from distinct gadgets, either ends inside one of the incident gadgets or passes through it, leaving by the opposite edge of that gadget (see the dashed line in Figure 4.1). In other words, anti-A-paths in G^+ correspond to CT-paths in G. This yields that every longest anti-A-path in G^+ contains at most $2 \cdot 4 + 2 \cdot 6 = 20$ vertices.

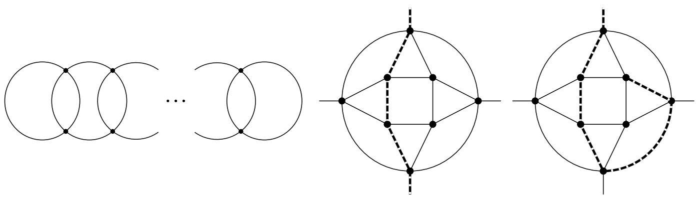

Figure 4.1: A construction of nonregular plane graphs with short longest anti-A-paths.

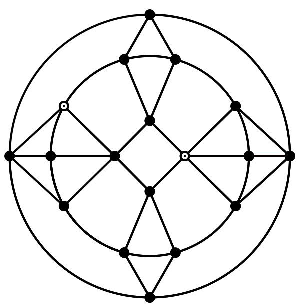

Figure 4.2: An example of a nonregular polyhedral graph of minimum degree 4 with vertices (represented here by white circles) that cannot be connected by an anti-A-path.

The existence of large 3-connected plane graphs of minimum degree 5 with very short longest anti-A-paths would also bring more light into the problem of existence of 3-connected plane graphs of minimum degree 5 in which there are vertices that cannot be connected by any anti-A-path. (So far, such graphs are not known. With 4-valent vertices, nonregular examples are known; see, for example, the graph in Figure 4.2, which is thought to be the smallest 4-connected nonregular with this property.) For plane graphs of minimum degree at least 4 (resp. 4-regular ones), such graphs (as well as their characterization) were described in [10]; the quest for looking for those ones of minimum degree 5 is described in [1].

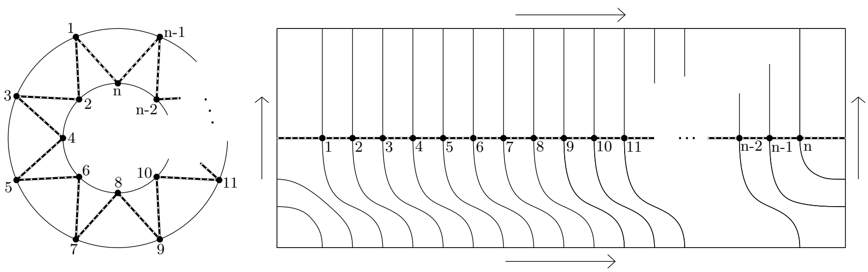

Figure 4.3: A toroidal embedding of a k-sided antiprism with a Hamiltonian CT-path.

Finally, note that, in 4-regular graphs embedded into higher surfaces, there can be Hamiltonian CT-paths—as an example, consider the embedding of a k-sided antiprism in the torus in Figure 4.3. Note also that, for $k \equiv 0 \pmod 3$, the plane drawing of a k-sided antiprism is a Grötzsch-Sachs graph comprised of three CT-cycles of equal lengths, so its longest CT-path has $\frac{4k}{3}$ vertices. Thus, we obtain an infinite family of graphs embeddable in the torus and in the sphere, respectively, such that the difference between lengths of longest CT-paths is linear in terms of the graph order. However, we do not know any example of an infinite family of planar 4-regular graphs whose longest CT-paths have lengths upper bounded by a fixed constant, but their embeddings into non-spheric surfaces of small genus contain CT-paths of length linear in terms of number of vertices.

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