BULLETIN OF THE ICA
Volume **105** (2025), 70–105

Doubly near affine doubly (μ, ν)-resolvable group divisible packing designs

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Abstract. In this paper we introduce and construct doubly (μ, ν)-resolvable packing designs, which are also doubly near affine, using Hadamard and generalized Hadamard matrices. The design is double (μ, ν)-resolvable and double near affine relative to the arrangements of blocks of the design in rows and columns of a rectangle $\mathscr{R}$ (which is an extension of the Kirkman square). The blocks in a row of $\mathscr{R}$ form a μ-resolution class, and those in a column of $\mathscr{R}$ form a ν-resolution class. A design with blocks arranged in the rows and columns of a rectangle are called doubly near affine if two blocks of the design intersect in the same number of treatments if they belong to the same row or to the same column or neither to the same row nor to the same column. In special cases we also obtain a series of doubly near affine doubly (μ, ν)-resolvable optimal transversal packing designs. The transversal design is relative to a partition of the point set into subsets of equal size. Also a new generalization of the Kirkman square, called an extended Kirkman square, is introduced. Examples of extended Kirkman squares nested in a doubly affine doubly resolvable rectangle are given. The advantage of these designs over classical resolvable designs is that each α-resolution class/fractional α-resolution class is itself a useful connected design. The squares as well as rectangles have multiple blocking structure and enable us to estimate not only the effects of a set V of treatments but also those of certain subsets of V. They may also be applied to the construction of 2-factor split-plot designs.

Key words and phrases: transversal design; orthogonal arrays; rectangular design; Hadamard matrix; generalized Hadamard matrix; self-complementary and near self-complementary BIBDs; doubly (μ, ν)-resolvable doubly near affine design; DAR rectangle (rectangle which determines double near affineness and double (μ, ν)-resolution of a block design); extended Kirkman square.
Mathematics Subject Classifications: O5B 15

Received: 26 April 2024
Accepted: 26 November 2024

1 Introduction

The study of resolvable non-symmetric BIBDs started with the Kirkman school girl problem [23]. It took more than 100 years to arrive at the complete solution of the (generalized) problem by Ray-Choudhary and Wilson [34]. Bose [3] started the study of resolvability of designs and enriched it by introducing affine resolvability. To widen the class of affine resolvable designs Shrikhande and Raghavarao [39] introduced affine α-resolvable designs. Calinski and Kageyama [6] introduced $(\alpha_1, \alpha_2, \ldots, \alpha_m)$-resolvability keeping in view its application in statistics. Ozawa et al. [30] constructed split-plot designs by affine μ-resolvable designs. Room [36] initiated the study of a Room square, which is equivalent to a doubly resolvable BIBD with block size $k = 2$ and index $\lambda > 1$. Robinson [35] obtained some doubly resolvable BIBDs with $k > 2$ and $\lambda > 1$, which were studied and generalized by several authors—Fujihara and Vanstone [17], Curran and Vanstone [9], and Vanstone [25, 27], Lamken [24], Colbourn et al. [8], among others. Robinson [35] remarked that the design would be useful in the design of taste panel experiments and in small plot trials when two sources of variations may be removed. Also, doubly resolvable group divisible designs were constructed by Lamken and Vanstone [26, 27], and Dong and Wang [14]. J. Du et al. [16] constructed doubly resolvable group divisible packing designs.

In general, doubly μ-resolvable balanced/partially balanced/packing designs with $\mu > 1$ have received little attention of authors. In this paper we introduce doubly near affine doubly (μ, ν)-resolvable packing designs. Double affineness and (μ, ν)-resolvability of a 1-design D is relative to the arrangements of blocks of D in a rectangle $\mathscr{R}$ (called a DAR rectangle). Blocks of D in the rows of $\mathscr{R}$ form a μ-resolution; each row corresponding to a μ-resolution class and those in the columns of $\mathscr{R}$ form a ν-resolution; each column corresponding to a ν-resolution class. To say D is doubly near affine means two blocks of D have the same number of common treatments if they belong to same row of $\mathscr{R}$ or if they belong to the same column of $\mathscr{R}$ or if they belong neither to the same row nor to the same column of $\mathscr{R}$. A transversal design is defined relative to a partition of the set of treatments into subsets of equal size. The double affineness ensures that a row (column) α-resolution class of each design is a linked design (the dual of a BIBD).

In Section 3 we obtain some 2-parameter and 1-parameter families of such designs from Hadamard matrices. The families, under certain conditions, reduce to a series of transversal designs that are optimal packings when μ, ν are both even or μ is even and $\nu = 1$. When $\mu = \nu = 2s$, where s

is a positive integer, we obtain a series of doubly near affine doubly $2s$-resolvable optimal transversal packing designs. In Section 4 we obtain a 2-parameter family of doubly near affine doubly $(1, \nu)$-resolvable packing designs from generalized Hadamard matrices, which are optimal when $\nu > 2$. In Section 6 we describe optimality of the designs.

In Section 7 we address the application of designs. The notions of an extended Kirkman square (EKS), fractional α-resolution design, and EKS nested in a DAR rectangle are introduced.

For the constructions we first dualize Latin semi-regular rectangular designs (recently introduced by Singh and Saurabh [40]) constructed here by Hadamard matrices, generalized Hadamard matrices, and BIBDs with $v = 2k$. We introduce below some terms.

Definition 1.1 (1-design)**.** A (v, b, r, k)- 1-*design* $D = (V, \mathscr{B})$ is a set V of v treatments together with a collection $\mathscr{B}$ of b subsets (called *blocks*) of V, each subset a k-subset, such that every treatment is contained in r blocks of a 1-design. The parameters satisfy $vr = bk$.

Definition 1.2 (2-design or BIBD)**.** A (v, b, r, k, λ)-BIBD (or (v, k, λ)-BIBD) is a (v, b, r, k)- 1-design $(V, \mathscr{B})$ such that any two distinct treatments of V are contained in λ blocks of $\mathscr{B}$.

For the class of 2-associate partially balanced incomplete block designs (PBIBDs) and its subclasses viz. that of group divisible designs (GDDs), L_2-type designs, triangular designs we refer to Raghavarao [32]. In what follows "treatment" will be called "point".

Definition 1.3 (Transversal design)**.** A *transversal design* (TD) is a triplet $(V, \mathscr{G}, \mathscr{B})$ such that

(i) $\mathscr{G}$ is a partition of a point set V into k point classes each of size n,

(ii) $\mathscr{B}$ is a collection of k-subsets (called blocks) of V,

(iii) every unordered pair of elements from V is contained either in exactly one point class or in exactly λ blocks, but not both.

Here, k, n, and λ are independent parameters of the TD, which is often denoted as $\mathsf{TD}_\lambda(k, n)$. Clearly a $\mathsf{TD}_\lambda(k, n)$ is a semi-regular GDD with parameters $v = kn$, $b = \lambda n^2$, $r = \lambda n$, k, $\lambda_1 = 0$, $\lambda_2 = \lambda$, $m = k$, n. Every block of a TD meets every point class in exactly one point. The dual of a TD is called a (k, n, λ)-net.

Definition 1.4 (Affine $\boldsymbol{\mu}$-resolvable 1-design)**.** A (v, b, r, k)- 1-design D is called *μ-resolvable* if the set of b blocks of D can be partitioned into $m = r/\mu$ classes each of size $n = v\mu/k$ such that in each class of n blocks each point of D is replicated μ times. The classes are called μ-resolution classes. D is called *affine μ-resolvable* if it is μ-resolvable and two distinct blocks of D intersect in Λ_1 or Λ_2 points as they belong to the same or distinct μ-resolution classes. A 1-resolvable 1-design is called a *resolvable 1-design* and 1-resolution class is called a *resolution class*.

We recall the following definitions from Lamken and Vanstone [25] and from Du et al. [16].

Definition 1.5 (Kirkman square)**.** Let v, b, r, k, and λ be the parameters of a BIBD on a set V. Let μ and $m = r/\mu$ be positive integers. A *Kirkman square* $\mathsf{KS}_k(v; \mu, \lambda)$ is an $m \times m$ array S such that

(1) each cell of S is either empty or contains a k-subset of V,

(2) every point of V is contained in precisely μ cells of each row and each column,

(3) the collection of subsets obtained from nonempty cells of S is a (v, b, r, k, λ)-BIBD.

A BIBD D with a Kirkman square $\mathsf{KS}_k(v; \mu, \lambda)$ is called *doubly μ-resolvable*. If the blocks of D are repeated, they are made distinct by labeling them.

A Kirkman square $\mathsf{KS}_2(v; 1, 1)$ is called a Room square.

Definition 1.6 (Generalized Kirkman Square)**.** Let v, b, r, and k be the parameters of a 1-design defined on the set V. Let μ and $m = r/\mu$ be positive integers. A *generalized Kirkman square* $(v; \mu, \lambda, r)$ is an $m \times m$ square that satisfies the conditions (1) and (2) of a Kirkman square but condition (3) is replaced by

(3)′ every 2-subset of V is contained in at most λ non-empty subsets of S.

A 1-design with a generalized Kirkman square is called a *doubly μ-resolvable packing design.*

In the following definition we extend the notion of "doubly μ-resolvable packing design" to "doubly (μ, ν)-resolvable packing design" and incorporate an additional feature of "near double affineness".

Definition 1.7. (Doubly near affine doubly (μ,ν)-resolvable rectangle.) Let v, b, r and k be the parameters of a 1-design defined on the set V. Let μ, ν, $m = r/\mu$, and $n = r/\nu$ be positive integers. A $\mathsf{DAR}_k(v;\mu,\nu,\lambda,r)$ is an $m \times n$ array $\mathscr{R}$ that satisfies the following conditions:

(R_1) each cell of $\mathscr{R}$ is either empty or contains a k-subset of V;

(R_2) every point of V is contained in precisely μ cells of each row and ν cells of each column of $\mathscr{R}$;

(R_3) every 2-subset of V is contained in at most λ non-empty entries of $\mathscr{R}$;

(R_4) (double near affineness) two k-subsets in the same row of $\mathscr{R}$ intersect in Λ_1 points while those in the same column of $\mathscr{R}$ intersect in Λ_2 points; otherwise, they intersect in Λ_3 points.

For examples of DAR rectangles see Sections 3 and 4.

Definition 1.8. A 1-design D with a DAR rectangle will be called a *doubly near affine doubly* (μ,ν)*-resolvable packing* (*design*). Also, v, b, r, k, μ, ν, λ, Λ_1, Λ_2, Λ_3, m, and n will be called parameters of the design. If some 2-subset of V belongs to λ blocks of D, then λ will be called the index of D. Also, Λ_1, Λ_2, and Λ_3 will be called the *intersection numbers* of the design. A design with the array $\mathscr{R}$ satisfying the conditions (R_1), (R_2), and (R_4) only will be called a *doubly near affine doubly* (μ,ν)*-resolvable 1-design.*

Definition 1.9 (Hadamard matrices)**.** An $n \times n$ matrix H_n with entries 1 or -1 and satisfying $H_n H_n^T = nI_n$ is called a *Hadamard matrix* (or *H-matrix*). Hadamard matrices H_n exist for $n \in \{1,2\}$ and for $n = 4t$ for infinitely many values of t. It is conjectured that they exist for order $4t$ for all positive integral values of t.

An H-matrix can always be reduced to a form in which its first row and first column consist of only 1s. Such a form is called *normalized*.

Definition 1.10 (Generalized Hadamard (GH) matrix)**.** Let G by a multiplicative group of order n. A *generalized Hadamard matrix* over G is an $n\nu \times n\nu$ matrix $H_G = [d_{ij}]$ with entries d_{ij} from G so that for $1 \le i < j \le n\nu$, the collection $\{d_{ih}d_{jh}^{-1} : 1 \le h \le n\nu\}$ contains every element of G a total of ν times.

Definition 1.11 (Circulant matrix)**.** An $n \times n$ matrix $\mathsf{circ}(a_1, a_2, \dots a_n)$ is called a circulant matrix if its first row is $a_1, a_2, \dots, a_n$ and subsequent rows are obtained by right shift of each symbol a_i in the preceding row.

In what follows we frequently use the notation $N^c = J - N$ and $I^c = J - I$, where N is the incidence matrix of a 1-design D, J is an all 1s matrix of the size of N as well as I, and I is the identity matrix. Then, N^c is the incidence matrix of the design complementary to D. Also, often a design D will be denoted by its incidence matrix N.

2 (m, n, r, k)-Latin semi-regular rectangular designs (LSR RDs)

The key PBIBD, which is an important source for the construction of our proposed design, is a special type of rectangular design (RD) recently identified and named as a Latin semi-regular (LSR) rectangular design by Singh and Saurabh [40].

2.1 Some basic facts about LSR RDs

We recall the following definitions from Singh and Saurabh [40].

Definition 2.1 (Rectangular design)**.** A *rectangular design* (RD) is an arrangement of $v = mn$ points in b blocks each of size k such that

(i) every point occurs at most once in a block,

(ii) every point occurs in exactly r blocks,

(iii) the mn points can be arranged in an array A of m rows and n columns such that two points in the same row (column) occur together in λ_1 (λ_2) blocks and in λ_3 blocks otherwise.

The integers v, b, r, k, m, n, λ_1, λ_2, and λ_3 are called the *parameters* of the RD, and A will be called the *defining array of the* RD.

Definition 2.2 (Latin semi-regular rectangular design)**.** Let N be the incidence matrix of an RD. Then the RD is called *Latin semi-regular* (LSR) if the eigenvalues θ_1, θ_2, θ_3 of N satisfies $\theta_1 = \theta_2 = 0$, $\theta_3 > 0$.

The eigenvalues of the incidence matrix of an RD$(v, b, r, k, m, n, \lambda_1, \lambda_2, \lambda_3)$ are given (see Raghavarao [32]) by

$$\begin{aligned}\theta_1 &= r - \lambda_1 + (m-1)(\lambda_2 - \lambda_3),\\ \theta_2 &= r - \lambda_2 + (n-1)(\lambda_1 - \lambda_3),\\ \theta_3 &= r - \lambda_1 - \lambda_2 + \lambda_3.\end{aligned}$$

Proposition 2.3 (Reduction of an RD). *An* $\mathsf{RD}(v, b, r, k, m, n, \lambda_1, \lambda_2, \lambda_3)$*, defined on a point set* V *and on an array* A*, reduces with same parameters* v*,* b*,* r*, and* k *to one of the following designs:*

(i) *A group divisible design* (GDD) *if* $\lambda_3 = \lambda_2$*. The other parameters of the* GDD *are* $\lambda_1' = \lambda_1$*,* $\lambda_2' = \lambda_2$*,* $m' = m$*, and* $n' = n$*. The defining partition of* V *is the set of rows of the array* A*.*

(ii) *A* GDD *if* $\lambda_3 = \lambda_1$*. The other parameters of the* GDD *are* $\lambda_1' = \lambda_2$*,* $\lambda_2' = \lambda_1$*,* $m' = n$*, and* $n' = m$*. The defining partition of* V *is the set of columns of* A*.*

(iii) *An* L_2*-type design if* $\lambda_1 = \lambda_2$ *and* $m = n$*. The other parameters of the design are* $\lambda_1' = \lambda_1$*,* $\lambda_2' = \lambda_3$*.*

The proof follows from the definitions of the designs.

Proposition 2.4 (Uniform partition of incidence matrix). *Assume that* D *is a 1-design with parameters* v*,* b*,* r*, and* k *and that* N *is the incidence matrix of* D*. Let* $N = [N_{ij}]$ *for* $i \in \{1, 2, \ldots, m\}$ *and* $j \in \{1, 2, \ldots, m_1\}$ *such that each* N_{ij} *is an* $n \times n_1$ *matrix. Define*

$$R_i^2 = R_i R_i^T = \sum_{k=1}^{m_1} N_{ik} {N_{ik}}^T$$

and

$$R_i R_j = R_i R_j^T = \sum_{k=1}^{m_1} N_{ik} {N_{jk}}^T.$$

If $R_i^2 = rI + \lambda_1 I^c$*, for all* i*,* $R_i R_j = \lambda_2 I + \lambda_3 I^c$*, for all* $i \neq j$*, and* i*,* $j \in \{1, 2, \ldots, m\}$*, then* D *is an* RD *with parameters* $v = mn$*,* $b = mnr/k$*,* r*,* k*,* λ_1*,* λ_2*,* λ_3*,* m*, and* n*.*

Proof. See the proof of Proposition 1 of Singh and Saurabh [40]. □

2.2 Series of LSR RDs

Singh and Saurabh [40] observed that the general family of LSR RDs has four independent parameters. If we take m, n, r, and k as independent parameters of an LSR RD, its other parameters can be expressed as $v = mn$, $b = mnr/k$, $\lambda_1 = r(k - m)/m(n - 1)$, $\lambda_2 = r(k - n)/n(m - 1)$, and $\lambda_3 = r\big((mn - m - n)k + mn)\big)/mn(m - 1)(n - 1)$. An LSR RD with these parameters will be referred as an (m, n, r, k)-LSR RD.

If an (m, n, r, k)-LSR RD D exists, then an (m, n, rx, k)-LSR RD, denoted as $x\#D$, also exists. An easy way to produce an $x\#D$ is to repeat each block of D a total of x times, and then $x\#D$ is called a multiple of D by x. However, we prefer the design $x\#D$ with blocks as distinct as possible. If $x\#D$ is not a multiple of D by x, then it is called a quasi-multiple of D by x.

Here we use two special families of (m, n, r, k)-LSR RDs.

(i) The family $F_1(m, n, x)$ of LSR RDs: when each of m, n is divisible by an integer $u > 1$. The parameters of the family are $v = mn$, $b = u(m-1)(n-1)x$, $r = (m-1)(n-1)x$, $k = mn/u$, $\lambda_1 = (m-1)(n/u-1)x$, $\lambda_2 = (m/u-1)(n-1)x$, and $\lambda_3 = \big((mn-m-n)/u+1\big)x$.

(ii) The family $F_2(m, n, x)$ of LSR RDs: when n divides m. The parameters of the family are $v = mn$, $b = n(m-1)x$, $r = (m-1)x$, $k = m$, $\lambda_1 = 0$, $\lambda_2 = (m/n-1)x$, and $\lambda_3 = mx/n$.

Later we shall see that infinitely many members of the family $F_1(m, n, x)$ and $F_2(m, n, x)$ can be obtained from the Hadamard matrices and generalized Hadamard matrices respectively.

3 Construction of doubly near affine doubly (μ, ν)-resolvable designs from certain BIBDs and Hadamard matrices

In this section first we produce LSR RDs by plugging the incidence matrices of a (v, ℓ, λ)-BIBD with $v = 2\ell$ and its complement into a truncated Hadamard matrix. All such RDs belong to the family $F_1(m, n, x)$, when m is 2 or divisible by 2^2 and n is divisible by 2 or 2^2. Their duals are the required doubly near affine doubly (μ, ν)-resolvable designs. There are two families of $(2k, k, k-1)$-BIBDs (see Mullin and Stinson [29]):

(a) The family of BIBDs with parameters $v = 4s$, $b = 2(4s-1)$, $r = 4s-1$, $\ell = 2s$, and $\lambda = 2s-1$ (i.e., 4 divides v).

(b) The family of BIBDs with parameters $v = 4s+2$, $b = 2(4s+1)$, $r = 4s+1$, $\ell = 2s+1$, and $\lambda = 2s$ (i.e., 2 divides v but 4 does not divide v).

We recall the following definitions and facts from Mullin and Stinson [29].

A $(2\ell, \ell, \ell-1)$-BIBD $D = (V, \mathscr{B})$ is called *self-complementary* (SC) (or truly or strongly SC) if, whenever $B \in \mathscr{B}$, we have $B^c = V - B \in \mathscr{B}$. A $(2\ell, \ell, \ell-1)$-BIBD $(V, \mathscr{B})$ is called *near self-complementary* (NSC) if there is an involuntary mapping $f\colon \mathscr{B} \to \mathscr{B}$ such that, for every $B \in \mathscr{B}$, we have $|B \cap f(B)| = 1$ and $|B \cup f(B)| = v - 1$. In this case $f(B)$ is called the near complement of B. Only family (a) contains self-complementary BIBDs. Family (b) contains simple near self-complementary BIBDs. Also, $\big(2\ell, \ell, x(\ell-1)\big)$-BIBD is the multiple or quasi-multiple of the $(2k, \ell, \ell-1)$-BIBD by x.

Proposition 3.1. *In an* SC $(4s, 2(4s-1), 4s-1, 2s, 2s-1)$-BIBD *any block meets all blocks except itself and its complement in exactly s points.*

Proposition 3.2. *In an* NSC $(4s+2, 8s+2, 4s+1, 2s+1, 2s)$-BIBD *any block meets all blocks except itself and its near complement in exactly s or $s+1$ points.*

We forward below a construction theorem.

Theorem 3.3. *Existence of a Hadamard matrix of order m and a $\big(2\ell, \ell, x(\ell-1)\big)$*-BIBD *implies the existence of an* LSR RD *belonging to the family $F_1(m, 2\ell, x)$ and a doubly near affine doubly $(\ell, m/2)$-resolvable 1-design with the parameters $v = 2(m-1)(2\ell-1)x$, $b = 2m\ell$, $r = m\ell$, $k = (m-1)(2\ell-1)x$, $\mu = \ell$, $\nu = m/2$, $\Lambda_1 = (m-1)(\ell-1)x$, $\Lambda_2 = (m/2-1)(2\ell-1)x$, $\Lambda_3 = (k+x)/2$, m, and n.*

Proof. Let N be the $2\ell \times (4\ell-2)x$ incidence matrix of a $\big(2\ell, \ell, x(\ell-1)\big)$-BIBD. Then,

$$NN^T = x(2\ell-1)I_{2\ell} + x(\ell-1)I_{2\ell}^c. \tag{1}$$

Let H be a normalized Hadamard matrix of order m. Let H' be the $m \times (m-1)$ Hadamard matrix obtained from H by deleting the first column of H. H' will be called a truncated H-matrix. In H' replace 1 by N and -1 by $N^c = J - N$, where J is the $2\ell \times (4t-2)x$ matrix with every entry 1, and denote the resulting matrix by M. Then $M = [N_{ij}]$, for $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, m-1\}$, and $N_{ij} \in \{N, N^c\}$, and M^T (the transpose of M) is the incidence matrix of the required doubly near affine $(\ell, m/2)$-resolvable 1-design.

We claim that M is an LSR RD with parameters

$$v=2m\ell,\quad b=2(m-1)(2\ell-1)x,\quad r=(m-1)(2\ell-1)x,\quad k=m\ell,$$
$$\lambda_1=(\ell-1)(m-1)x,\quad \lambda_2=(m/2-1)(2\ell-1)x,\quad \lambda_3=(r+x)/2,$$
$$m=m,\quad n=2\ell.$$

Clearly m is 2 or divisible by 4. Also, n is divisible by 2 when ℓ is odd and by 4 when ℓ is even. Hence, the RDs belong to the family $F_1(m,n,x)$, where $n=2\ell$, $u=2$ (see Section 2.2(i)). We make the following observations on the matrix M.

- O_1 Every row R_i of sub-matrices N_{ij} of M consists of exactly $m-1$ entries N or N^c.
- O_2 If any two rows R_i and R_j of M are juxtaposed to form a $2\times(m-1)$ array, there are $m/2-1$ columns of the form $\begin{bmatrix}N\\N\end{bmatrix}$ or $\begin{bmatrix}N^c\\N^c\end{bmatrix}$ and $m/2$ columns of the form $\begin{bmatrix}N\\N^c\end{bmatrix}$ or $\begin{bmatrix}N^c\\N\end{bmatrix}$.
- O_3 $R_iR_i^T$ consists of $m-1$ terms of the form $NN^T=(N^c)(N^c)^T$ for all i.
- O_4 $R_iR_j^T$ consists of $m/2-1$ terms of the form $NN^T=(N^c)(N^c)^T$ and $m/2$ terms of the form $N^cN^T=N(N^c)^T=\ell I_v^c$ for $i\neq j$.

The claim follows from Equation (1), Observations O_1 through O_4, Proposition 2.4, and the fact that the eigenvalues of the RD are $\theta_1=\theta_2=0$ and $\theta_3>0$.

Also, from Theorem 1(vi) in Singh and Saurabh [40], it follows that, if $k/m=\mu$ and $k/n=\nu$, then μ and ν are integers, and every block of the RD contains μ points from each row of the defining array A and ν points from each column of A.

Now observe the consequences when M is changed to M^T, the incidence matrix of the dual of the RD. Array A is changed to a rectangle $\mathscr{R}$ (which we have called a DAR rectangle) whose entries are blocks of the design M. Each point of the design M^T belongs to μ blocks of each row of $\mathscr{R}$ and ν blocks of each column of $\mathscr{R}$. The concurrence λ_i of a pair of points is changed to the intersection number Λ_i of the corresponding blocks for $i\in\{1,2,3\}$.

Thus, M^T is a doubly near affine doubly (μ,ν)-resolvable 1-design with parameters mentioned in the theorem. This design will be denoted as $D^1_{\mu,\nu}(x)$ and as $D^1_{\mu,\nu}$ when $x=1$.

If some blocks of the BIBD or LSR RD are repeated, we can obtain the dual by making the blocks distinct by labeling them. □

Corollary 3.4. *If $m = 2$, $\ell = 2s$, $x = 1$, and the* BIBD *is* SC, *then $D^1_{\mu,\nu}$ is an optimal transversal packing design. Its parameters are $v = 2(4s-1)$, $b = 8s$, $r = 4s$, $k = (4s-1)$, $\mu = 2s$, $\nu = 1$, $\Lambda_1 = 2s-1$, $\Lambda_2 = 0$, $\Lambda_3 = 2s$, the index $\lambda = 2s$, $m = 2$, and $n = 4s$. As a transversal design, $D^1_{2s,1}$ has additional parameters $m' = 4s-1$, $n' = 2$, $\lambda'_1 = 0$, and $\lambda'_2 = 2s$. Its dual is an affine resolvable* LSR *rectangular design. Moreover, each row μ-resolution class is an optimal transversal packing with parameters $v' = 2(4s-1)$, $b' = 4s$, $r' = 2s$, $k' = 4s-1$, $m' = 4s-1$, $n' = 2$, $\lambda_1 = 0$, and $\lambda'_2 = s$.*

Proof. Let N_1 be the incidence matrix of a Hadamard 2-design $(4s-1, 2s-1, s-1)$. Then

$$N_1N_1^T = N_1^TN_1 = (2s-1)I + (s-1)I^c, \tag{2}$$

where $I^c = J - I$, and I is the identity matrix, and J is the all 1s matrix of order $4s-1$. Let $e_{1\times m}$, $e_{m\times 1}$, and $J_{m\times n}$ be all 1 matrices of size $1\times m$, $m\times 1$, and $m\times n$, respectively, and let $0_{m\times n}$ denote the null matrix of size $m\times n$. Then it is known (Berardi et al. [2]) that $N = \begin{bmatrix} e_{1\times(4s-1)} & 0_{1\times(4s-1)} \\ N_1 & N_1^c \end{bmatrix}$ is the incidence matrix of an SC BIBD$(4s, 2s, 2s-1)$. Hence from Theorem 3.3 it follows that, for $m = 2$, we have $M = \begin{bmatrix} N \\ J-N \end{bmatrix}$ is an LSR RD D (say), where J is the $4s\times(8s-2)$ all 1 matrix, with parameters $v = 8s$, $b = 2(4s-1)$, $r = 4s-1$, $k = 4s$, $\lambda_1 = 0$, $\lambda_2 = 2s-1$, $\lambda_3 = 2s$, $m = 4s$, and $n = 2$. Also, D^d, the dual of D, has the incidence matrix

$$M^T = \begin{bmatrix} e_{(4s-1)\times 1} & N_1 & 0_{(4s-1)\times 1} & N_1^c \\ 0_{(4s-1)\times 1} & N_1^c & e_{(4s-1)\times 1} & N_1 \end{bmatrix}, \tag{3}$$

which satisfies $M^TM = \begin{bmatrix} 4sI + 2sI^c & 2sI^c \\ 2sI^c & 4s + 2sI^c \end{bmatrix}$, where I is the identity matrix of order $4s-1$. From Propositions 2.3 and 2.4 it follows that M^T is the incidence matrix of a transversal design with parameters mentioned in Corollary 3.4.

Finally, for the last claim, Equation (3) can be written as $M^T = [M_1M_2]$, where $M_1 = \begin{bmatrix} e_{(4s-1)\times 1} & N_1 \\ 0_{(4s-1)\times 1} & N_1^c \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0_{(4s-1)\times 1} & N_1^c \\ e_{(4s-1)\times 1} & N_1 \end{bmatrix}$. It is easy to verify, for $i \in \{1,2\}$, that $M_iM_i^T = \begin{bmatrix} 2sI + sI^c & sI^c \\ sI^c & 2sI + sI^c \end{bmatrix}$. □

We illustrate the construction in Corollary 3.4 with the following example.

Example 3.5 (a $\boldsymbol{D^1_{4,1}}$)**.** Let us start with the Fano plane, i.e., $\mathsf{BIBD}(7,3,1)$ whose incidence matrix is $N_1 = \mathsf{circ}(0,1,1,0,1,0,0)$. Let the designs below be denoted by their incidence matrices N_1, N, M, and M^T as defined in Theorem 3.3. Then the blocks of designs can be written as follows:

$$N_1 = \big\{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}\big\}$$

$$N = \begin{Bmatrix} \{8,1,2,4\}, & \{3,5,6,7\}, \\ \{8,2,3,5\}, & \{4,6,7,1\}, \\ \{8,3,4,6\}, & \{5,7,1,2\}, \\ \{8,4,5,7\}, & \{6,1,2,3\}, \\ \{8,5,6,1\}, & \{7,2,3,4\}, \\ \{8,6,7,2\}, & \{1,3,4,5\}, \\ \{8,7,1,3\}, & \{2,4,5,6\} \end{Bmatrix}$$

$$M = \begin{Bmatrix} B_1 = \{8,1,2,4,11,13,14,15\}, & B_8 = \{3,5,6,7,16,9,10,12\}, \\ B_2 = \{8,2,3,5,12,14,15,9\}, & B_9 = \{4,6,7,1,16,10,11,13\}, \\ B_3 = \{8,3,4,6,13,15,9,10\}, & B_{10} = \{5,7,1,2,16,11,12,14\}, \\ B_4 = \{8,4,5,7,14,9,10,11\}, & B_{11} = \{6,1,2,3,16,12,13,15\}, \\ B_5 = \{8,5,6,1,15,10,11,12\}, & B_{12} = \{7,2,3,4,16,13,14,9\}, \\ B_6 = \{8,6,7,2,9,11,12,13\}, & B_{13} = \{1,3,4,5,16,14,15,10\}, \\ B_7 = \{8,7,1,3,10,12,13,14\}, & B_{14} = \{2,4,5,6,16,15,9,11\} \end{Bmatrix}$$

$$M^T = \begin{Bmatrix} \overline{1} = \{1,5,7,9,10,11,13\}, & \overline{9} = \{2,3,4,6,8,12,14\}, \\ \overline{2} = \{1,2,6,10,11,12,14\}, & \overline{10} = \{3,4,5,7,8,9,13\}, \\ \overline{3} = \{2,3,7,8,11,12,13\}, & \overline{11} = \{1,4,5,6,9,10,14\}, \\ \overline{4} = \{1,3,4,9,12,13,14\}, & \overline{12} = \{2,5,6,7,8,10,11\}, \\ \overline{5} = \{2,4,5,8,10,13,14\}, & \overline{13} = \{1,3,6,7,9,11,12\}, \\ \overline{6} = \{3,5,6,8,9,11,14\}, & \overline{14} = \{1,2,4,7,10,12,13\}, \\ \overline{7} = \{4,6,7,8,9,10,12\}, & \overline{15} = \{1,2,3,5,11,13,14\}, \\ \overline{8} = \{1,2,3,4,5,6,7\}, & \overline{16} = \{8,9,10,11,12,13,14\} \end{Bmatrix}.$$

The defining array of the LSR RD M is

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{bmatrix},$$

and the DAR rectangle $\overline{A}$ of M^T is the 2×8 array

$$\overline{A} = \begin{bmatrix} \overline{1} & \overline{2} & \overline{3} & \overline{4} & \overline{5} & \overline{6} & \overline{7} & \overline{8} \\ \overline{9} & \overline{10} & \overline{11} & \overline{12} & \overline{13} & \overline{14} & \overline{15} & \overline{16} \end{bmatrix}.$$

The parameters of the doubly near affine doubly $(4,1)$-resolvable design M^T (i.e., a $D^1_{4,1}$) are $v = 14$, $b = 16$, $r = 8$, $k = 7$, $\Lambda_1 = 3$, $\Lambda_2 = 0$, $\Lambda_3 = 4$, $m = 2$, and $n = 8$. Also, the $D^1_{4,1}$ is a transversal design with parameters $v = 14$, $b = 16$, $r = 8$, $k = 7$, $m' = 7$, $n' = 2$, $\lambda'_1 = 0$, and $\lambda'_2 = 4$ and with the partition of point set V given by $\big\{\{1,8\}, \{2,9\}, \{3,10\}, \{4,11\}, \{5,12\}, \{6,13\}, \{7,14\}\big\}$. Furthermore, $D^1_{4,1}$ is quasi-double of the GDD SR80 in Clatworthy's table [7]. Each row 4-resolution class is an optimal transversal design with parameters $v = 14$, $b = 8$, $r = 4$, $k = 7$, $m' = 7$, $n' = 2$, $\lambda'_1 = 0$, and $\lambda'_2 = 2$ (same as SR80) and with the same partition of the point set V as that of $D^1_{4,1}$.

Corollary 3.6. *When $\ell = 2s+1$, $x = 1$, and the* BIBD$(2\ell, \ell, \ell-1)$ *is* NSC, *then the packing design $D^1_{2s+1,m/2}$ has parameters $v = 2(m-1)(4s+1)$, $b = 2m(2s+1)$, $r = m(2s+1)$, $k = (m-1)(4s+1)$, $\mu = 2s+1$, $\nu = m/2$, $\Lambda_1 = 2(m-1)s$, $\Lambda_2 = (m/2-1)(4s+1)$, $\Lambda_3 = (k+1)/2$, (index) $\lambda = m(s+1)$, m, and $n = 4s+2$.*

Proof. If M is the incidence matrix of a design D, then the off-diagonal integer entries of M^TM are the sizes of intersections of pair of distinct blocks of D. If N is the incidence matrix of NSC BIBD$(4s+2, 2s+1, 2s)$, then from Proposition 3.2 it follows that the maximum off-diagonal entry of N^TN is $s+1$. Let $M = [N_{ij}]$ be the incidence matrix of the LSR RD as defined in Theorem 3.3. Then each column of m blocks of M contains $m/2$ N's and $m/2$ (N^c)'s. Since $N^TN = (N^c)^TN^c$, each diagonal block entry of M^TM is mN^TN. By Proposition 3.2, the maximum off-diagonal entry of N^TN in $s+1$. Hence the maximum off-diagonal integer entry in each diagonal block of M^TM is $m(s+1)$. It can be verified that each integer entry in each off-diagonal block of M^TM is $m\big(s+\frac{1}{2}\big)$. Hence the maximum size of the intersection of a pair of blocks of the design M is $m(s+1)$, which is the index of the packing design M^T. □

Corollary 3.7. *If $m = 4t$, $\ell = 2s$, $x = 1$, and the* BIBD$(4s, 2s, 2s-1)$ *is* SC, *then the design $D^1_{\mu,\nu}$ has parameters $v = 2(4t-1)(4s-1)$, $b = 16st$, $r = 8st$, $k = (4t-1)(4s-1)$, $\mu = 2s$, $\nu = 2t$, $\Lambda_1 = (4t-1)(2s-1)$, $\Lambda_2 = (2t-1)(4s-1)$, $\Lambda_3 = (k+1)/2$, $m = 4t$, and $n = 4st$. Also, it is an optimal transversal packing design with parameters $\lambda'_1 = 0$, $\lambda'_2 = 4st$, $m' = (4t-1)(4s-1)$, $n' = 2$, and the index $\lambda = 4st$. The dual of $D^1_{2s,2t}$ is an affine resolvable* LSR *rectangular design.*

Proof. Let the incidence matrix N of the BIBD be partitioned as

$$N = [N_1|N_2|\cdots|N_{4s-1}],$$

where each N_i contains exactly two columns corresponding to two disjoint blocks of the BIBD. Then for all $i,j \in \{1,2,\ldots,4s-1\}$, we have $N_i^T N_i = 2sI$ and $N_i^T N_j = sJ$, for $i \neq j$ (by Proposition 3.1). Hence $N^T N = \mathsf{circ}\big(2sI|sJ|\cdots|sJ\big)$ and $N^T N^c = \mathsf{circ}(2sI^c|sJ|\cdots|sJ)$. Let $M = [C_1|C_2|\cdots|C_{4t-1}]$, where C_i are columns of the incidence matrix M of the RD. It is easy to verify that $(N^c)^T N^c = N^T N$ and $N^T N^c = (N^c)^T N$. Hence $C_i^T C_i = 4tN^T N = \mathsf{circ}\big(8stI|4stJ|\cdots|4stJ\big) = P$ (say) and, for $i \neq j$, $C_i^T C_j = 2t(N^T \ldots^T N^c) = \mathsf{circ}\big(4stJ|4stJ|\cdots|4stJ\big) = Q$ (say). Thus $M^T M = \mathsf{circ}\big(P|Q|\cdots|Q\big)$. Now from Proposition 2.3 and 2.4, it follows that M^T is a GDD with parameters mentioned above. Also, from these parameters it is clear that M^T is a transversal design. For optimality of the packing see Section 6. □

Corollary 3.8. *If $s = t$ in Corollary 3.7, then the design $D^1_{2s,2s}$ is a doubly near affine doubly $2s$-resolvable optimal transversal packing with parameters $v = 2(4s-1)^2$, $b = 16s^2$, $r = 8s^2$, $k = (4s-1)^2$, $\Lambda_1 = \Lambda_2 = (4s-1)(2s-1)$, $\Lambda_3 = (k+1)/2$, $\lambda'_1 = 0$, $\lambda'_2 = 4s^2$, $m' = (4s-1)^2$, $n' = 2$, and the index $\lambda = 4s^2$. The dual of $D^1_{2s,2s}$ is an affine resolvable semi-regular L_2-type design (see Theorem 3 ($s = 1$) in Singh and Saurabh* [40]*).*

Remark 3.9. The design $D^1_{2s,2t}$ in Corollary 3.7 can be constructed by Hadamard matrices H_{4t} and H_{4s}. The design $D^1_{2s,1}$ in Corollary 3.7 as well as $D^1_{2s,2s}$ in Corollary 3.8 can be constructed by H_{4s}. These are the consequences of the fact that the SC $(4s, 2s, 2s-1)$-BIBD can be constructed by H_{4s} (see Mullin and Stinson [29]).

Remark 3.10. The family of designs $D^1_{2s,2t}$ in Corollary 3.7 and $D^1_{2s,1}$ in Corollary 3.4 satisfies $b = 4(r-\lambda)$.

Example 3.11 (a $\boldsymbol{D^1_{2,2}}$)**.** Let $m = n = 4$. Consider the following example of a design that appears as a special case in Corollary 3.8 and Theorem 3.3 (the designs used in the construction are denoted by their incidence matrices).

Let

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

be the SC BIBD$(4, 2, 1)$. Assume

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Then

$$N^c = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and

$$M = \begin{bmatrix} N & N & N \\ N & N^c & N^c \\ N^c & N & N^c \\ N^c & N^c & N \end{bmatrix}$$

are designs defined in Theorem 3.3. Blocks of SR L_2-type design M are

$$\begin{aligned}
B_1 &= \{3,4,7,8,9,10,13,14\}, & B_{10} &= \{1,2,5,6,11,12,15,16\}, \\
B_2 &= \{1,4,5,8,10,11,14,15\}, & B_{11} &= \{2,3,6,7,9,12,13,16\}, \\
B_3 &= \{2,4,6,8,9,11,13,15\}, & B_{12} &= \{1,3,5,7,10,12,14,16\}, \\
B_4 &= \{3,4,5,6,11,12,13,14\}, & B_{13} &= \{1,2,7,8,9,10,15,16\}, \\
B_5 &= \{1,4,6,7,9,12,14,15\}, & B_{14} &= \{2,3,5,8,10,11,13,16\}, \\
B_6 &= \{2,4,5,7,10,12,13,15\}, & B_{15} &= \{1,3,6,8,9,11,14,16\}, \\
B_7 &= \{3,4,5,6,9,10,15,16\}, & B_{16} &= \{1,2,7,8,11,12,13,14\}, \\
B_8 &= \{1,4,6,7,10,11,13,16\}, & B_{17} &= \{2,3,5,8,9,12,14,15\}, \\
B_9 &= \{2,4,5,7,9,11,14,16\}, & B_{18} &= \{1,2,6,8,10,12,13,15\}.
\end{aligned}$$

After a permutation of blocks, this affine resolvable design is the same as LS 100 listed in Clatworthy's table [7]. It has an additional property: each block of the design contains exactly two points from each row and from each column of the array

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}.$$

In general, such designs were named as semi-regular L_2-type designs by Singh and Saurabh [40]. Its dual M^T (i.e., $D^1_{2,2}$) has the following blocks:

$$
\begin{array}{ll}
\overline{1} = \{2,5,8,10,12,13,15,16,18\}, & \overline{2} = \{3,6,9,10,11,13,14,16,17\}, \\
\overline{3} = \{1,4,7,11,12,14,15,17,18\}, & \overline{4} = \{1,2,3,4,5,6,7,8,9\}, \\
\hline
\overline{5} = \{2,4,6,7,9,10,12,14,17\}, & \overline{6} = \{3,4,5,7,8,10,11,15,18\}, \\
\overline{7} = \{1,5,6,8,9,11,12,13,16\}, & \overline{8} = \{1,2,3,13,14,15,16,17,18\}, \\
\hline
\overline{9} = \{1,3,5,7,9,11,13,15,17\}, & \overline{10} = \{1,2,6,7,8,12,13,14,18\}, \\
\overline{11} = \{2,3,4,8,9,10,14,15,16\}, & \overline{12} = \{4,5,6,10,11,12,16,17,18\}, \\
\hline
\overline{13} = \{1,3,4,6,8,11,14,16,18\}, & \overline{14} = \{1,2,4,5,9,12,15,16,17\}, \\
\overline{15} = \{2,3,5,6,7,10,13,17,18\}, & \overline{16} = \{7,8,9,10,11,12,13,14,15\}.
\end{array}
$$

$D^1_{2,2}$ has the same parameters as 2-resolvable SR 100 in Clatworthy's table [7]. However, here the blocks are arranged in the cells of a DAR square as

$$
\overline{A} = \begin{bmatrix} \overline{1} & \overline{2} & \overline{3} & \overline{4} \\ \overline{5} & \overline{6} & \overline{7} & \overline{8} \\ \overline{9} & \overline{10} & \overline{11} & \overline{12} \\ \overline{13} & \overline{14} & \overline{15} & \overline{16} \end{bmatrix}.
$$

Clearly $D^1_{2,2}$ is a doubly 2-resolvable GDD. It is doubly near affine, as two blocks in the same row or column intersect in 3 points; otherwise, they intersect in 5 points. Index $\lambda = 4$. Also, $D^1_{2,2}$ is an optimal packing (see Section 6). Each row (column) 2-resolution class of the DAR square is the dual of a $(4, 18, 9, 2, 3)$-BIBD.

Example 3.12 (a $\boldsymbol{D^1_{3,1}}$)**.** Consider the following designs, which appear in Theorem 3.3 and Corollary 3.6 and are denoted by their incidence matrices.

$$
N = \left\{ \begin{array}{ll}
B_1 = \{1,2,3\}, & B_6 = \{3,4,5\}, \\
B_2 = \{1,4,5\}, & B_7 = \{2,3,4\}, \\
B_3 = \{2,4,6\}, & B_8 = \{3,5,6\}, \\
B_4 = \{1,4,6\}, & B_9 = \{1,2,5\}, \\
B_5 = \{2,5,6\}, & B_{10} = \{1,3,6\}.
\end{array} \right\}
$$

$$
M = \left\{ \begin{array}{ll}
B_1 = \{1,2,3,10,11,12\}, & B_6 = \{3,4,5,7,8,12\}, \\
B_2 = \{1,4,5,8,9,12\}, & B_7 = \{2,3,4,7,11,12\}, \\
B_3 = \{2,4,6,7,9,11\}, & B_8 = \{3,5,6,7,8,10\}, \\
B_4 = \{1,4,6,8,9,11\}, & B_9 = \{1,2,5,9,10,12\}, \\
B_5 = \{2,5,6,7,9,10\}, & B_{10} = \{1,3,6,8,10,11\}.
\end{array} \right\}
$$

$$M^T = \left\{ \begin{array}{ll} \overline{1} = \{1,2,4,9,10\}, & \overline{7} = \{3,5,6,7,8\}, \\ \overline{2} = \{1,3,5,7,9\}, & \overline{8} = \{2,4,6,8,10\}, \\ \overline{3} = \{1,6,7,8,10\}, & \overline{9} = \{2,3,4,5,9\}, \\ \overline{4} = \{2,3,4,6,7\}, & \overline{10} = \{1,5,8,9,10\}, \\ \overline{5} = \{2,5,6,8,9\}, & \overline{11} = \{1,3,4,7,10\}, \\ \overline{6} = \{3,4,5,8,10\}, & \overline{12} = \{1,2,6,7,9\}. \end{array} \right\}$$

The starting design N is a near self-complementary $(6, 10, 5, 3, 2)$-BIBD, where a near complement of a block B_i is $f(B_i) = B_{i+5}$, for $i \in \{1, 2, 3, 4, 5\}$ (see Mullin and Stinson [29]). The defining array of LSR RD M is

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{bmatrix},$$

and the DAR rectangle $\overline{A}$ of the doubly near affine doubly $(3, 1)$-resolvable packing design M^T (i.e., $D^1_{3,1}$) is

$$\overline{A} = \begin{bmatrix} \overline{1} & \overline{2} & \overline{3} & \overline{4} & \overline{5} & \overline{6} \\ \overline{7} & \overline{8} & \overline{9} & \overline{10} & \overline{11} & \overline{12} \end{bmatrix}.$$

The parameters of the design $D^1_{3,1}$ are $v = 10$, $b = 12$, $r = 6$, $k = 5$, $\Lambda_1 = 2$, $\Lambda_2 = 0$, $\Lambda_3 = 3$, $\mu = 3$, $\nu = 1$, $\lambda = 4$, $m = 2$, and $n = 6$. Also, it has been verified that $D^1_{3,1}$ is a triangular design based upon the following array:

$$T = \begin{bmatrix} & 1 & 2 & 3 & 8 \\ 1 & & 5 & 6 & 4 \\ 2 & 5 & & 10 & 7 \\ 3 & 6 & 10 & & 9 \\ 8 & 4 & 7 & 9 & \end{bmatrix}.$$

Two distinct points of $D^1_{3,1}$ are 1^{st} associates if they belong to the same row or the same column of the array T; otherwise, they are 2^{nd} associates. $D^1_{3,1}$ has the same parameters as the triangular design listed as T46 in Clatworthy's table [7]. However, the design T46 is given as the double of the design T44 in the table; whereas, the $D^1_{3,1}$ is a quasi-double of T44. Kageyama [22] has proved that an affine μ-resolvable triangular design does not exist for $\mu \in \{1, 2\}$. Example 3.12 shows that a 1-resolvable triangular design $D^1_{3,1}$ exists if we take resolution classes as the columns of $\overline{A}$. Also, $D^1_{3,1}$ is near affine in the sense that two blocks in the same resolution class do not intersect, and those in the distinct resolution classes intersect in three points with exactly one exception. If we take resolution classes row-wise, we can see that $D^1_{3,1}$ is near affine 3-resolvable.

It has been verified that each row 3-resolution class is a triangular design with the same parameters as T44 and is an optimal packing.

4 Doubly near affine doubly $(1,\nu)$-resolvable designs from generalized Hadamard matrices

Theorem 4.1. *Let G be a group of order n. Let $H_G = GH(n\nu, G)$ be a generalized Hadamard matrix of order $n\nu$ over G. Then there exists a doubly near affine doubly $(1,\nu)$-resolvable 1-design with parameters $v = (n\nu-1)x$, $b = \nu n^2$, $r = n\nu$, $k = (n\nu-1)x$, $\Lambda_1 = 0$, $\Lambda_2 = x(\nu-1)$, $\Lambda_3 = \nu x$, $\mu = 1$, ν, $m = n\nu$, n, $\lambda'_1 = 0$, $\lambda'_2 = \nu$, $m' = (n\nu-1)x$, $n' = n$, and index $\lambda = \nu$.*

Proof. First, we prove the theorem for $x = 1$ and reduce H_G into a normalized form, which contains in first row and column only identity element of G. Next delete the first column. Replace the entries of the truncated H_G by $(0,1)$-matrices N_{ij} through a regular permutation matrix representation of G. Denote the resulting $(0,1)$-matrix by N. Then $N = [N_{ij}]$, where $i \in \{1,2,\ldots,n\nu\}$ and $j \in \{1,2,\ldots,n\nu-1\}$. Let R_i be the i^{th} row with entries N_{ij} for $1 \le j \le n\nu - 1$. Then by the definition of H_G it follows that

$$R_i^2 = R_i R_i^T = \sum_{k=1}^{m-1} N_{ik} {N_{ik}}^T = (n\nu-1)I_n$$

and

$$R_i R_j = R_i R_j^T = \sum_{k=1}^{m-1} N_{ik} {N_{jk}}^T = (\nu-1)I_n + \nu I_n^c.$$

Hence by Proposition 2.4, N is the incidence matrix of an RD with parameters $v = \nu n^2$, $b = n(\nu n - 1)$, $r = \nu n - 1$, $k = \nu n$, $\lambda_1 = 0$, $\lambda_2 = \nu - 1$, $\lambda_3 = \nu$, $m = n\nu$, and n. If θ_1, θ_2, θ_3 are eigenvalues of N, then it can be verified that $\theta_1 = \theta_2 = 0$ and $\theta_3 > 0$. Hence the RD is LSR, and N^T is doubly near affine $(1,\nu)$-resolvable with parameters mentioned in the theorem, where $x = 1$.

Consider a collection of $x > 1$ copies of blocks of the above RD, which is an LSR RD with parameters $v = \nu n^2$, $b = n(\nu n - 1)x$, $r = (\nu n - 1)x$, $k = \nu n$, $\lambda_1 = 0$, $\lambda_2 = (\nu-1)x$, $\lambda_3 = \nu x$, $m = n\nu$, and n. If we make the blocks distinct by labeling them, then the dual of the design is the required design. This design will be denoted as $D^2_{1,\nu}(x)$ and as $D^2_{1,\nu}$ when $x = 1$. □

Remark 4.2. The LSR RD $D^2_{1,\nu}$ belongs to the family $F_2(m,n,x)$, where $m = n\nu$ (See Section 2.2(ii)).

From Greig and Colbourn [18] we recall the following definition.

Definition 4.3. An (n, r, ν)-*net* is an affine resolvable 1-design with parameters $v' = \nu n^2$, $b' = rn$, $r' = r$, and $k' = \nu n$ and with r resolution classes, each containing n disjoint blocks. Any two blocks belonging to different resolution classes intersect in ν points.

Remark 4.4. From the above definition of an (n, r, ν)-net, it follows that when $x = 1$ the LSR RD used in the proof of Theorem 4.1 is an $(n, n\nu - 1, \nu)$-net. Thus, we have arrived at some additional combinatorial properties of this subfamily of nets.

Remark 4.5. It is clear that every LSR RD is not a net. The LSR RD in Example 3.12, with incidence matrix M, is not a net. An (n, r, ν)-net with $r \neq n\nu - 1$ is not an LSR RD.

Example 4.6 (a $\boldsymbol{D^2_{1,2}}$)**.** Consider the example of an LSR RD from Singh and Saurabh [40], which is also a $(3, 5, 2)$-net. The dual of the net is a doubly near affine doubly $(1, 2)$-resolvable transversal design $D^2_{1,2}$. The blocks of the RD are

$$\begin{array}{lll} B_1 = \{1,6,8,11,13,18\}, & B_2 = \{2,4,9,12,14,16\}, & B_3 = \{3,5,7,10,15,17\}, \\ B_4 = \{1,5,9,11,15,16\}, & B_5 = \{2,6,7,12,13,17\}, & B_6 = \{3,4,8,10,14,18\}, \\ B_7 = \{1,6,9,10,14,17\}, & B_8 = \{2,4,7,11,15,18\}, & B_9 = \{3,5,8,12,13,16\}, \\ B_{10} = \{1,5,7,12,14,18\}, & B_{11} = \{2,6,8,10,15,16\}, & B_{12} = \{3,4,9,11,13,17\}, \\ B_{13} = \{1,4,8,12,15,17\}, & B_{14} = \{2,5,9,10,13,18\}, & B_{15} = \{3,6,7,11,14,16\}. \end{array}$$

The 6×3 defining array of the RD is given as

$$A = \begin{bmatrix} 1 & 4 & 7 & 10 & 13 & 16 \\ 2 & 5 & 8 & 11 & 14 & 17 \\ 3 & 6 & 9 & 12 & 15 & 18 \end{bmatrix}^T.$$

The dual of the above RD (N^T in Theorem 4.1) is given by the blocks

$$\begin{array}{lll} \overline{1} = \{1,4,7,10,13\}, & \overline{2} = \{2,5,8,11,14\}, & \overline{3} = \{3,6,9,12,15\}, \\ \overline{4} = \{2,6,8,12,13\}, & \overline{5} = \{3,4,9,10,14\}, & \overline{6} = \{1,5,7,11,15\}, \\ \overline{7} = \{3,5,8,10,15\}, & \overline{8} = \{1,6,9,11,13\}, & \overline{9} = \{2,4,7,12,14\}, \\ \overline{10} = \{3,6,7,11,14\}, & \overline{11} = \{1,4,8,12,15\}, & \overline{12} = \{2,5,9,10,13\}, \\ \overline{13} = \{1,5,9,12,14\}, & \overline{14} = \{2,6,7,10,15\}, & \overline{15} = \{3,4,8,11,13\}, \\ \overline{16} = \{2,4,9,11,15\}, & \overline{17} = \{3,5,7,12,13\}, & \overline{18} = \{1,6,8,10,14\}. \end{array}$$

Thus $D^2_{1,2}$ has the parameters $v = 15$, $b = 18$, $r = 6$, $k = 5$, $\Lambda_1 = 0$, $\Lambda_2 = 1$, $\Lambda_3 = 2$, $\lambda = 2$, $\mu = 1$, $\nu = 2$, $m = 6$, $n = 3$, $\lambda'_1 = 0$, $\lambda'_2 = 2$, $m' = 6 - 1 = 5$, and $n' = 3$ (where primed parameters refer to transversal designs with the same v, b, r, k) with the DAR rectangle

$$\overline{A} = \begin{bmatrix} \overline{1} & \overline{4} & \overline{7} & \overline{10} & \overline{13} & \overline{16} \\ \overline{2} & \overline{5} & \overline{8} & \overline{11} & \overline{14} & \overline{17} \\ \overline{3} & \overline{6} & \overline{9} & \overline{12} & \overline{15} & \overline{18} \end{bmatrix}^T .$$

The transversal design is based on the following partition of the point set:

$$\big\{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{10,11,12\},\{13,14,15\}\big\}.$$

This is isomorphic to SR 36 listed in Clatworthy [7] and given by Solution 1. However, Clatworthy's grouping of blocks shows that the design is 1-resolvable only. Here it is shown that the design $D^2_{1,2}$ is 3-resolvable also. Moreover $D^2_{1,2}$ is near affine with intersection numbers 0, 1, and 2. It is also a packing design with index $\lambda = 2$ and packing number 21, which is close to $b = 18$.

Each of the three columns of the array $\overline{A}$, which is a 2-resolution class, is the dual of a $\mathsf{BIBD}(6, 15, 5, 2, 1)$. It has been verified that these designs are triangular designs. The first one is isomorphic to T48 in Clatworthy's table [7], based on the triangular association scheme given by the array

$$\begin{bmatrix} & 13 & 10 & 7 & 1 & 4 \\ 13 & & 8 & 6 & 12 & 2 \\ 10 & 8 & & 3 & 5 & 15 \\ 7 & 6 & 3 & & 14 & 11 \\ 1 & 12 & 5 & 14 & & 9 \\ 4 & 2 & 15 & 11 & 9 & \end{bmatrix}.$$

The other two 2-resolution classes are also triangular designs, but defined upon different arrays. Also, each defining array is the disconnected EKS nested in the corresponding column of DAR rectangle $\overline{A}$ (see EKS I in Section 7).

From this observation the following remark follows:

Remark 4.7. There exist a near optimal transversal packing, which is the block disjoint union of three triangular designs defined on different arrays.

Corollary 4.8. *The design $D^2_{1,\nu}$ in Theorem 4.1 is also a transversal packing with index $\lambda = \nu$. The packing is optimal when $\nu > 2$. Its dual is an affine resolvable* LSR *rectangular design. Moreover, each column ν-resolution class is the dual of a* BIBD *with parameters $v = n\nu$, $b = n(n\nu - 1)$, $r = n\nu - 1$, $k = \nu$, and $\lambda = \nu - 1$.*

Proof. If C_i is the i^{th} column of $N = [N_{ij}]$, i.e., C_i^T is the i^{th} row of N^T, then $C_i^T C_i = n\nu I$ and $C_i^T C_j = \nu J$, for $i \neq j$. Hence by Propositions 2.3 and 2.4, N^T is the incidence matrix of a transversal design with parameters $v = n(\nu n - 1)$, $b = \nu n^2$, $r = \nu n$, $k = \nu n - 1$, $\lambda_1' = 0$, $\lambda_2' = \nu$, $m' = \nu n - 1$, and $n' = n$. Clearly $\lambda = \lambda_2' = \nu$. For optimality see Section 6. The last assertion follows from the definition of a DAR rectangle and Theorem 4.1. □

Corollary 4.9. *There exists a doubly resolvable transversal design $D_{1,1}^2$ with parameters $v = q(q-1)$, $b = q^2$, $r = q$, $k = q-1$, $\Lambda_1 = 0$, $\Lambda_2 = 0$, $\Lambda_3 = 1$, $\lambda_1' = 0$, $\lambda_2' = 1$, $m' = q-1$, $n' = q$, and $\lambda = 1$, where q is prime power.*

Proof. A generalized Hadamard matrix of order q can be obtained by the method of Drake [15]. Using this GH matrix, we deduce the corollary from Theorem 4.1 taking $x = 1$. □

Remark 4.10. Work of Butson [4,5], Drake [15], Jungnickel [21], Street [42], Seberry [37], Dawson [10], de Launey [11–13], Hayden [19], and Zhang et al. [45], among others, yielded some series of generalized Hadamard matrices. Hence, Corollary 4.8 gives infinitely many doubly near affine doubly $(1,\nu)$-resolvable optimal transversal packing designs. For a table of small order generalized Hadamard matrices and theorems on their construction, see Lampio [28].

We recall the following definition from Hedayat et al. [20]:

Definition 4.11 (Orthogonal array.)**.** Let S be a set of n symbols. An $\mathsf{OA}(N,k,n,t)$ is an $N \times k$ array with entries from S such that every $N \times t$ subarray of A contains any t-tuple based on S exactly λ times as a row. In addition, N, k, n, t, and λ are called respectively the run, number of factors, number of levels, strength, and index of the OA.

A transversal design $\mathsf{TD}_\lambda(k,n)$ is equivalent to an $\mathsf{OA}(\lambda n^2, k, n, 2)$, see Hedayat et al. [20, page 242]. The transversal designs in Corollaries 3.4, 3.7 and 4.8 induce the following OAs:

Theorem 4.12. *The following families of* OA*s are equivalent to* TD*s obtained in corollaries:*

from Corollary 3.4(i): $\mathsf{OA}(4s, 4s-1, 2, 2)$,
from Corollary 3.4(ii): $\mathsf{OA}(8s, 4s-1, 2, 2)$,
from Corollary 3.7: $\mathsf{OA}(16st, (4s-1)(4t-1), 2, 2)$,
from Corollary 4.8: $\mathsf{OA}(\nu n^2, \nu n - 1, n, 2)$.

5 Doubly near affine doubly (μ,ν)-resolvable 1-design from BIBDs.

The following theorem yields a doubly near affine doubly (μ,ν)-resolvable 1-design when $\mu,\nu > 1$ and μ,ν are both odd, which is not covered by Theorems 3.3 and 4.1.

Theorem 5.1. *The existence of a* NSC BIBD$(2\ell_i,\ell_i,\ell_i-1)$, *where* $\ell_i > 1$ $(i \in \{1,2\})$ *are odd, implies the existence of a doubly near affine doubly* (μ,ν)*-resolvable 1-design with parameters* $v = 4(2\ell_1-1)(2\ell_2-1)$, $b = 4\ell_1\ell_2$, $r = 2\ell_1\ell_2$, $k = 2(2\ell_1-1)(2\ell_2-1)$, $\Lambda_1 = 2(\ell_1-1)(2\ell_2-1)$, $\Lambda_2 = 2(\ell_2-1)(2\ell_1-1)$, $\Lambda_3 = 2\big((\ell_1-1)(\ell_2-1)+\ell_1\ell_2\big)$, $\mu = \ell_1$, *and* $\nu = \ell_2$.

Proof. Let N_i be the incidence matrix of the $(2\ell_i,\ell_i,\ell_i-1)$-BIBD, for $i \in \{1,2\}$. In the $(0,1)$-matrix N_2 replace 1 by N_1 and 0 by N_1^c and let M be the resulting matrix. Then one can verify that M is the incidence matrix of an LSR RD with parameters $v = 4\ell_1\ell_2$, $b = 4(2\ell_1-1)(2\ell_2-1)$, $r = 2(2\ell_1-1)(2\ell_2-1)$, $k = 2\ell_1\ell_2$, $\lambda_1 = 2(\ell_1-1)(2\ell_2-1)$, $\lambda_2 = 2(\ell_2-1)(2\ell_1-1)$, $\lambda_3 = 2\big((\ell_1-1)(\ell_2-1)+\ell_1\ell_2\big)$, $m = 2\ell_2$, and $n = 2\ell_1$. Also, M^T is the incidence matrix of the required design, which has parameters mentioned in the theorem. □

A design given by Theorem 5.1 will be denoted by $D^3_{2s+1,2t+1}$, where $\mu = 2s+1$ and $\nu = 2t+1$ for $s,t \geq 1$.

Corollary 5.2. *When* $\ell_1 = \ell_2 = 2s+1$, *then the design* $D^3_{2s+1,2s+1}$ *reduces to a doubly near affine doubly* $(2s+1)$*-resolvable 1-design.*

6 Optimality of doubly near affine (μ,ν)-resolvable packings

The packing we consider here is in fact a 2-(v,k,λ) packing (*vide* Stinson et al. [41]).

The first Johnson bound for this packing is

$$U_\lambda(v,k,2) = \left\lfloor \frac{v}{k} \left\lfloor \frac{\lambda(v-1)}{k-1} \right\rfloor \right\rfloor.$$

If b is the number of blocks of the design, then the design is called optimal if $b = U_\lambda(v,k,2)$.

There are infinitely many packing designs constructed in the corollaries, which are optimal or near optimal packings. We tabulate below some series of optimal/near optimal packings. The parameter x has the value 1 for every packing design.

Table 6.1: Series of doubly (μ, ν)-resolvable packings.

	$D^1_{2s,1}$	$D^1_{2s+1,2t}$	$D^1_{2s,2t}$	$D^1_{2s,2s}$	$D^2_{1,\nu}$
v	$2(4s-1)$	$2(4t-1)(4s+1)$	$2(4t-1)(4s-1)$	$2(4s-1)^2$	$n(\nu n-1)$
r	$4s$	$4t(2s+1)$	$8st$	$8s^2$	νn
k	$4s-1$	$(4t-1)(4s+1)$	$(4t-1)(4s-1)$	$(4s-1)^2$	$\nu n-1$
Λ_1	$2s-1$	$(4t-1)(2s-1)$	$(4t-1)(2s-1)$	$(4s-1)(2s-1)$	0
Λ_2	0	$(2t-1)(4s-1)$	$(2t-1)(4s-1)$	$(4s-1)(2s-1)$	$\nu-1$
Λ_3	$2s$	$(k+1)/2$	$(k+1)/2$	$(k+1)/2$	ν
λ	$2s$	$4t(s+1)$	$4st$	$4s^2$	ν
$b-U_\lambda(v,k,2)$	0	$8t$	0	0	$\begin{cases} n \text{ if } \nu \le 2 \\ 0 \text{ if } \nu > 2 \end{cases}$
Optimality	Optimal	Near optimal for small t	Optimal	Optimal	Optimal if $\nu > 2$
Source	Cor. 3.4	Cor. 3.6	Cor. 3.7	Cor. 3.8	Cor. 4.8

7 Applications in the design of experiments

Bailey and Williams [1] remarked that multiple blocking structure play an important role in the control of experimental trend and are used extensively in the design and analysis of experiments in agriculture, horticulture, and forestry.

Each of the designs that we have obtained in the paper has multiple blocking structure. In this section, points will be called treatments.

When $\mu > 1$, each of the m row μ-resolution classes in a DAR rectangle is a linked design (the dual of a BIBD). When $\nu > 1$, each of the n column ν-resolution classes in a DAR rectangle is a linked design. Rao [33] has shown that the dual of a non-symmetric BIBD is a PBIBD under a generic condition. Shah et al. [38] proved (in Corollary 1, Theorem 2) that in the class of (b, v, k, r)- 1-designs containing a linked block design Δ, we have that Δ is A-, D-, and E-optimal for the estimation of treatment effects. Pohl [31] proved that in the larger class of (b, v, k)-designs, in which r varies, the linked design is optimal for the estimation of treatment effects. We expect the following applications:

7.1 Comparing the effects of certain subsets of a treatment set

In general, BIBDs are known to be more efficient than linked designs. We propose below a design whose row as well as column fractional μ-resolution designs are BIBDs under certain conditions, and which can compare the effects of certain subsets of a given treatment set.

Definition 7.1 (Extended Kirkman square)**.** Let V be a set of v points. Then an $m \times m$ array X will be called an *extended Kirkman square* (EKS) if

(1) each cell of X is either empty or contains a k-subset of V,

(2) non-empty sets of every row as well as column of X is a 1-design with the same parameters,

(3) every point belongs to u rows as well as u columns of X, for $1 \leq u \leq m$,

(4) the collection of subsets obtained from nonempty cells of X is a connected (v, b, r, k)- 1-design.

If conditions (1) through (3) hold but (4) does not, then the EKS will be called *disconnected*. An EKS will be called an *extended Kirkman 2-packing* (EKP) if

(5) Every 2-subset of V is contained in at most λ non-empty cells of X.

The independent parameters of an EKS are v, k, m, and u, and so it can be denoted as EKS(v, k, m, u). The parameters of each row (column) 1-design D comprising k-subsets from nonempty cells of the row (column) of X are $v' = uv/m$, $b' = b/m$, $r' = r/u$, and k. Furthermore, D will be called v'/v $(= u/m)$ row (column) r'-resolution design (or u/m row (column) design) of the EKS.

Example 7.2. The extended Kirkman square/packing is interesting when the row (column) fractional α-resolution designs are t-designs for some t (for t-designs see Trung [43]) or PBIBDs or t-packings (see Stinson et al. [41]). When $t = 2$, such examples can be produced easily by any symmetric (v, k, λ)-BIBD D. Let $B_1, B_2, \ldots, B_v$ be the blocks of the symmetric BIBD D. Obtain

$$B_{ij} = \begin{cases} B_i \cap B_j, & \text{if } i \neq j, \\ \emptyset, & \text{if } i = j, \end{cases}$$

for $i, j \in \{1, 2, \ldots, v\}$. Then the $v \times v$ array $X = [B_{ij}]$ is clearly an EKS(v, k, v, k). Each k/v row (column) $(k-1)$-resolution design is a 2-design (derived design of D).

Illustration: An EKS from a complete $(5, 4, 3)$-BIBD

Let the blocks of the BIBD be $B_1 = \{2, 3, 4, 5\}$, $B_2 = \{1, 3, 4, 5\}$, $B_3 = \{1, 2, 4, 5\}$, $B_4 = \{1, 2, 3, 5\}$, and $B_5 = \{1, 2, 3, 4\}$ on the treatment set $V = \{1, 2, 3, 4, 5\}$. The EKS X defined above can be easily obtained as

$$X = \begin{bmatrix} \emptyset & \{3,4,5\} & \{2,4,5\} & \{2,3,5\} & \{2,3,4\} \\ \{3,4,5\} & \emptyset & \{1,4,5\} & \{1,3,5\} & \{1,3,4\} \\ \{2,4,5\} & \{1,4,5\} & \emptyset & \{1,2,5\} & \{1,2,4\} \\ \{2,3,5\} & \{1,3,5\} & \{1,2,5\} & \emptyset & \{1,2,3\} \\ \{2,3,4\} & \{1,3,4\} & \{1,2,4\} & \{1,2,3\} & \emptyset \end{bmatrix}.$$

This EKS is an EKP. The packing is optimal with index $\lambda = 3$.

Each row is a $(4, 3, 2)$-BIBD on the point set containing 4 points (one point is missing). Hence each row is 4/5 a 3-resolution design. The overall design is the BIBD$(5, 3, 2)$. Let X be the space of the experimental design. Treatment effects of every combination of four treatments can be compared through row-wise (column-wise) designs, if the space is homogeneous, while those of all five treatments 1, 2, 3, 4, 5 can be estimated using $(5, 4, 3)$-BIBD. For more examples of EKSs whose fractional row (column) α-resolution designs are BIBDs, we introduce the following:

Definition 7.3 (EKSs nested in row (or column) of a DAR rectangle)**.**

(1) EKS I nested in one row μ-resolution (or column ν-resolution) class. Let $\mu > 1$. Let $B_1, B_2, \ldots, B_n$ be the blocks of a row μ-resolution class (column ν-resolution class, if $\nu > 1$) of a DAR rectangle. Define

$$B_{ij} = \begin{cases} B_i \cap B_j, & \text{if } i \neq j, \\ \emptyset, & \text{if } i = j, \end{cases}$$

for $i, j \in \{1, 2, \ldots, v\}$. If each fractional row (or column) α-resolution design of the $n \times n$ array $X = [B_{ij}]$ is connected, then X is easily seen to be an EKS, which will be called EKS I *nested in the* DAR *rectangle.* If the EKS has index λ, it will be called EKP I.

(2) EKS II nested in two row μ-resolution (or column ν-resolution) classes. Let $\mu > 1$. Let $B'_1, B'_2, \ldots, B'_n$ be the blocks of another row μ-resolution (column ν-resolution) class of the DAR rectangle mentioned in (1). Let

$$C_{ij} = \begin{cases} B'_i \cap B_j, & \text{if } i \neq j, \\ \emptyset, & \text{if } i = j, \end{cases}$$

for $i, j \in \{1, 2, \ldots, v\}$. If each fractional row (or column) α-resolution design of the $n \times n$ array $X = [C_{ij}]$ is connected, then X is seen to be an EKS, which will be called EKS II *nested in the* DAR *rectangle*. If the EKS has the index λ, it will be called EKP II.

Example 7.4. (Nested EKP I from Example 3.5) Here $X = [X_{ij}]_{i,j\in\{1,\ldots,8\}}$ with

$$X_{ij} = \begin{cases} \bar{i} \cap \bar{j}, & \text{if } i \neq j, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\bar{i}$ and $\bar{j}$ are as given in Example 3.5. Then EKP I can be explicitly written as

$$X = \begin{bmatrix} \emptyset & \{1,10,11\} & \{7,11,13\} & \{1,9,13\} & \{5,10,13\} & \{5,9,11\} & \{7,9,10\} & \{1,5,7\} \\ \{1,10,11\} & \emptyset & \{2,11,12\} & \{1,12,14\} & \{2,10,14\} & \{6,11,14\} & \{6,10,12\} & \{1,2,6\} \\ \{7,11,13\} & \{2,11,12\} & \emptyset & \{3,12,13\} & \{2,8,13\} & \{3,8,11\} & \{7,8,12\} & \{2,3,7\} \\ \{1,9,13\} & \{1,12,14\} & \{3,12,13\} & \emptyset & \{4,13,14\} & \{3,9,14\} & \{4,9,12\} & \{1,3,4\} \\ \{5,10,13\} & \{2,10,14\} & \{2,8,13\} & \{4,13,14\} & \emptyset & \{5,8,14\} & \{4,8,10\} & \{2,4,5\} \\ \{5,9,11\} & \{6,11,14\} & \{3,8,11\} & \{3,9,14\} & \{5,8,14\} & \emptyset & \{6,8,9\} & \{3,5,6\} \\ \{7,9,10\} & \{6,10,12\} & \{7,8,12\} & \{4,9,12\} & \{4,8,10\} & \{6,8,9\} & \emptyset & \{4,6,7\} \\ \{1,5,7\} & \{1,2,6\} & \{2,3,7\} & \{1,3,4\} & \{2,4,5\} & \{3,5,6\} & \{4,6,7\} & \emptyset \end{bmatrix}$$

Each row (column) of X is ½ row (column) 3-resolution design, which is a Fano plane. The design whose blocks are taken from all non-empty cells of an EKP is a regular GDD with parameters $v = 14$, $b = 56$, $r = 12$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = \lambda = 2$, $m' = 7$, and $n' = 2$.

The design is the double of R79 listed in Clatworthy's table [7] and is a near optimal packing (2-packing# is 60).

From the above EKP, one can estimate the treatment effects of the eight different combinations of seven treatments out of fourteen treatments with the help of sixteen ½ row (column) 3-resolution designs. Simultaneously, we can also estimate the treatment effects of all the fourteen treatments by one row 4-resolution class of the DAR rectangle of Example 3.5.

Example 7.5 (Nested EKP II from Ex. 3.5)**.** The EKP is $X = [X_{ij}]_{i,j\in\{1,\ldots,8\}}$ with

$$X_{ij} = \begin{cases} \bar{i} \cap \overline{j_1}, & \text{if } i \neq j, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $j_1 = j + 8$ and where $\bar{i}$ and $\overline{j_1}$ are as given in Example 3.5. The EKP II can be explicitly written as

$$X = \begin{bmatrix}
\emptyset & \{5,7,9,13\} & \{1,5,9,10\} & \{5,7,10,11\} & \{1,7,9,11\} & \{1,7,10,13\} & \{1,5,11,13\} & \{9,10,11,13\} \\
\{2,6,12,14\} & \emptyset & \{1,6,10,14\} & \{2,6,10,11\} & \{1,6,11,12\} & \{1,2,10,12\} & \{1,2,11,14\} & \{10,11,12,14\} \\
\{2,3,8,12\} & \{3,7,8,13\} & \emptyset & \{2,7,8,11\} & \{3,7,11,12\} & \{2,7,12,13\} & \{2,3,11,13\} & \{8,11,12,13\} \\
\{3,4,12,14\} & \{3,4,9,13\} & \{1,4,9,14\} & \emptyset & \{1,3,9,12\} & \{1,4,12,13\} & \{1,3,13,14\} & \{9,12,13,14\} \\
\{2,4,8,14\} & \{4,5,8,13\} & \{4,5,10,14\} & \{2,5,8,10\} & \emptyset & \{2,4,10,13\} & \{2,5,13,14\} & \{8,10,13,14\} \\
\{3,6,8,14\} & \{3,5,8,9\} & \{5,6,9,14\} & \{5,6,8,11\} & \{3,6,9,11\} & \emptyset & \{3,5,11,14\} & \{8,9,11,14\} \\
\{4,6,8,12\} & \{4,7,8,9\} & \{4,6,9,10\} & \{6,7,8,10\} & \{6,7,9,12\} & \{4,7,10,12\} & \emptyset & \{8,9,10,12\} \\
\{2,3,4,6\} & \{3,4,5,7\} & \{1,4,5,6\} & \{2,5,6,7\} & \{1,3,6,7\} & \{1,2,4,7\} & \{1,2,3,5\} & \emptyset
\end{bmatrix}.$$

Each row of X is ½ row 4-resolution design, which is the complement of a Fano plane on a treatment-set, that is different for the eight ½ 4-resolution designs. The eight treatment-sets are the blocks of one row 4-resolution class of D^1_{41} in Example 3.5. The same is true for ½ column 4-resolution designs of X, whose eight treatment-sets are the blocks of another row 4-resolution class of D^1_{41}.

It is clear that one can estimate the treatment effects of the sixteen different combinations of seven treatments out of fourteen treatments with the help of sixteen half row/column 4-resolution designs (BIBDs) of EKP II. Simultaneously we can also estimate the treatment effects of all the fourteen treatments with help of two row 4-resolution classes of the DAR rectangle of Example 3.5.

The overall design consisting of all blocks taken from nonempty cells of EKP II is a regular GDD with parameters $v = 14$, $b = 56$, $r = 16$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = \lambda = 4$, $m' = 7$, and $n' = 2$. The GDD is a quasi-double of R113 listed in Clatworthy table [7] and is a near optimal packing (2-packing# is 59).

Remark 7.6. From Example 7.4 it follows that there exists a regular group divisible near optimal packing design with $v = 14$, $b = 56$, $r = 16$, $k = 4$, and $\lambda = 4$, which can be expressed as the block disjoint union of eight $(7, 4, 2)$-BIBDs in two different ways.

7.2 The construction of two-factor split-plot designs

In a split-plot design, blocks are taken as plots (or whole plots), and the experimental units within a block are called split-plots. Ozawa et al. [30] constructed incomplete two-factor split-plot designs by affine α-resolvable 1-design. By the analogous method one can obtain two-factor incomplete split-plot designs by nearly affine μ-resolvable (ν-resolvable) transversal design, which we have obtained here. An additional advantage of these new designs is that all row (column) α-resolution classes are linked designs. Ozawa et al. [30] remarked that the split-plot designs are often used in biological, agricultural, and environmental sciences.

7.3 Fractional factorial designs

The transversal designs obtained in Corollaries 3.4, 3.7, and 4.8 are also orthogonal arrays (see Theorem 4.12) with new properties that may have applications in the theory of fractional factorial designs.

Remark 7.7. The deep application of the designs is expected when the DAR rectangles and EKSs are produced abundantly, especially for blocks of smaller size.

8 Conclusion

The class of doubly resolvable BIBDs has been extended to that of doubly near affine doubly (μ, ν)-resolvable group divisible packings. In the extended class there are some series of designs, each possessing some properties simultaneously, which were studied separately for different designs by design theorists. From Hadamard matrices of order $4s$ and $4t$, we can obtain a two-parameter series of doubly near affine doubly (μ, ν)-resolvable optimal transversal packing designs $D^1_{\mu,\nu}$, where $\mu = 2s$ and $\nu = 2t$. The series reduces to a doubly μ-resolvable transversal packing, when $s = t$, and the series $D^1_{\mu,1}$ reduces to a doubly $(\mu, 1)$-resolvable transversal packing when $\mu = 2s$, both retaining other properties of $D^1_{2s,2t}$. From a generalized Hadamard matrix of order $n\nu$ over a group of order n, we can obtain a doubly near affine doubly $(1, \nu)$-resolvable transversal packing design $D^2_{1,\nu}$ that is also an optimal packing when $\nu > 2$. Some series of packing designs $D^1_{\mu,\nu}$ and $D^2_{1,\nu}$ are given in Table 6.1. When $\mu = 1$ or $\nu = 1$, Table 8.1 contains forty-six $D^i_{\mu,\nu}$ ($i \in \{1, 2\}$) 1-designs/packing designs with $r, k \leq 15$.

The application of the designs in design of experiments follows from the fact that each row resolution class of a DAR rectangle is a linked design. We also introduced EKS and EKS nested in a DAR rectangle so that the associated row (column) fractional α-resolution classes (designs) are BIBDs. These squares yield more efficient experimental designs.

In Table 8.1

- v is number of points;
- b is the number of blocks;
- r is the number of replications;
- k is the block size;
- μ (or ν) is the number of blocks in any row (or column) of DAR rectangle $\mathscr{R}$, to which any point of the design belongs;
- Λ_1, Λ_2, Λ_3 are intersection numbers of pairs of the blocks; and
- λ is the index of the packing design.

Table 8.1: Table of doubly affine doubly (μ, ν)-resolvable designs with $r, k \leq 15$. (Other associated designs are not tabulated.)

	v	b	r	k	μ	ν	Λ_1	Λ_2	Λ_3	λ	Design	Source	Independent Parameters
1.	6	9	3	2	1	1	0	0	1	1	Packing	Cor. 4.9	$(\nu, n) = (1, 3)$
2.	6	8	4	3	2	1	1	0	2	2	Optimal Packing	Cor. 3.4	$s = 1$
3.	12	16	4	3	1	1	0	0	1	1	Packing	Cor. 4.9	$(\nu, n) = (1, 4)$
4.	12	9	3	4	1	1	0	0	2		1-design	Thm. 4.1	$(\nu, n, x) = (1, 3, 2)$
5.	20	25	5	4	1	1	0	0	1	1	Packing	Cor. 4.9	$(\nu, n) = (1, 5)$
6.	10	12	6	5	3	1	2	0	3	4	Packing	Cor. 3.6	$(\ell, m) = (3, 2)$
7.	15	18	6	5	1	2	0	1	2	2	Packing	Cor. 4.8	$(\nu, n, x) = (2, 3, 1)$
8.	18	9	3	6	1	1	0	0	3		1-design	Thm. 4.1	$(\nu, n, x) = (1, 3, 3)$
9.	12	8	4	6	2	1	2	0	4	2	Do	Do	$(\nu, n, x) = (2, 2, 2)$
10.	24	16	4	6	1	1	0	0	2	1	Do	Do	$(\nu, n, x) = (1, 4, 2)$
11.	42	49	7	6	1	1	0	0	1	1	Packing	Cor. 4.9	$(\nu, n) = (1, 7)$
12.	14	16	8	7	4	1	3	0	4	4	Optimal Packing	Cor. 3.4	$s = 2$
13.	28	32	8	7	1	2	0	1	2	2	Packing	Cor. 4.8	$(\nu, n, x) = (2, 4, 1)$
14.	56	64	8	7	1	1	0	0	1	1	Packing	Cor. 4.9	$(\nu, n) = (1, 8)$
15.	24	9	3	8	1	1	0	0	4		1-design	Thm. 4.1	$(\nu, n, x) = (1, 3, 4)$
16.	40	25	5	8	1	1	0	0	2		1-design	Do	$(\nu, n, x) = (1, 5, 2)$
17.	24	27	9	8	1	3	0	2	3	3	Optimal Packing	Cor. 4.8	$(\nu, n) = (3, 3)$
18.	72	81	9	8	1	1	0	0	1	1	Packing	Cor. 4.9	$(\nu, n) = (1, 9)$
19.	18	8	4	9	2	1	3	0	6		1-design	Thm. 3.3	$(m, \ell, x) = (2, 2, 3)$
20.	36	16	4	9	1	1	0	0	3		1-design	Thm. 4.1	$(\nu, n, x) = (1, 4, 3)$
21.	18	16	8	9	2	2	3	3	5	4	Optimal Packing	Cor. 3.8	$(s, t) = (1, 1)$
22.	18	20	10	9	5	1	4	0	5	6	Packing	Cor. 3.6	$(\ell, m) = (5, 2)$
23.	18	20	10	9	1	5	0	4	5	5	Optimal Packing	Cor. 4.8	$(\nu, n, x) = (5, 2, 1)$
24.	45	50	10	9	1	2	0	1	2	2	Packing	Cor. 4.8	$(\nu, n) = (2, 5)$
25.	30	9	3	10	1	1	0	0	5		1-design	Thm. 4.1	$(\nu, n, x) = (1, 3, 5)$
26.	20	12	6	10	3	1	4	0	6		1-design	Thm. 3.3	$(m, \ell, x) = (2, 3, 2)$
27.	30	18	6	10	1	2	0	2	4		1-design	Thm. 4.1	$(\nu, n, x) = (2, 3, 2)$

—*Continued on next page*

	v	b	r	k	μ	ν	Λ_1	Λ_2	Λ_3	λ	Design	Source	Independent Parameters
28.	110	121	11	10	1	1	0	0	1	1	Packing	Cor. 4.9	$(\nu, n) = (1, 11)$
29.	22	24	12	11	6	1	5	0	6	6	Optimal Packing	Cor. 3.4	$s = 3$
30.	33	36	12	11	1	4	0	3	4	4	Do	Cor. 4.8	$(\nu, n) = (4, 3)$
31.	44	48	12	11	1	3	0	2	3	3	Do	Cor. 4.8	$(\nu, n) = (3, 4)$
32.	36	9	3	12	1	1	0	0	6		1-design	Thm. 4.1	$(\nu, n, x) = (1, 3, 6)$
33.	24	8	4	12	2	1	4	0	8		1-design	Thm. 3.3	$(m, \ell, x) = (2, 2, 4)$
34.	48	16	4	12	1	1	0	0	4		Do	Thm. 4.1	$(\nu, n, x) = (1, 4, 4)$
35.	60	25	5	12	1	1	0	0	3		Do	Thm. 4.1	$(\nu, n, x) = (1, 5, 3)$
36.	156	169	13	12	1	1	0	0	1	1	Packing	Cor. 4.9	$(\nu, n) = (1, 13)$
37.	26	28	14	13	1	7	0	6	7	7	Optimal Packing	Cor. 4.8	$(\nu, n) = (7, 2)$
38.	26	28	14	13	7	1	6	0	7	8	Packing	Cor. 3.6	$(\ell, m) = (7, 2)$
39.	91	98	14	13	1	2	0	1	2	2	Packing	Cor. 4.8	$(\nu, n) = (2, 7)$
40.	42	9	3	14	1	1	0	0	7		1-design	Thm. 4.1	$(\nu, n, x) = (1, 3, 7)$
41.	28	16	8	14	4	1	6	0	8		1-design	Thm. 3.3	$(m, \ell, x) = (2, 4, 2)$
42.	56	32	8	14	1	2	0	2	4		1-design	Thm. 4.1	$(\nu, n, x) = (2, 4, 2)$
43.	42	45	15	14	1	5	0	4	5	5	Optimal Packing	Cor. 4.8	$(\nu, n) = (5, 3)$
44.	30	8	4	15	2	1	5	0	10		1-design	Thm. 3.3	$(m, \ell, x) = (2, 2, 5)$
45.	60	16	4	15	1	1	0	0	5		Do	Thm. 4.1	$(\nu, n, x) = (1, 4, 5)$
46.	30	12	6	15	3	1	6	0	9		Do	Thm. 3.3	$(m, \ell, x) = (2, 3, 3)$
47.	45	18	6	15	1	2	0	3	4	2	Do	Thm. 4.1	$(\nu, n, x) = (2, 3, 3)$

9 Open problems

In this paper, not all LSR RDs belonging to the family $F_1(m, n, x)$ or $F_2(m, n, x)$ have been used. For example, one may obtain a family $F_1(m, n, x)$ of LSR RDs when each of m, n is divisible by an integer $u > 2$; this may be obtained by two generalized Hadamard matrices defined on the same group of order u (akin to $D^1_{2s,2t}$ obtained in Corollary 3.7).

So, it is exciting to obtain other such designs/series of designs belonging to these families, which would yield new doubly near affine doubly (μ, ν)-resolvable packings. The designs we have obtained have no cell of the doubly near affine doubly resolvable (DAR) rectangle empty. More such designs can be obtained by allowing empty cells in the DAR rectangle. Also, the existence of a doubly (μ, ν)-resolvable optimal/near optimal packing design when μ and ν are both odd is unknown. Finally, a DAR rectangle and an extended Kirkman square can be further extended to a d-dimensional DAR cuboid and extended Kirkman cube, respectively. In the construction of the former, the starting designs may be higher dimensional Hadamard matrices (see Yang et al. [44]), and in the construction of latter the starting designs may be t-designs. The combinatorial experiments we started here lead to the following pure combinatorial problem: Given the parameters v, b, k, λ of an optimal/near optimal t-packing design D, where b is a composite number, determine how to obtain D and express it as the block disjoint union of maximum number of designs/packing designs satisfying certain conditions and having fixed k.

Acknowledgment

I am obliged to the referee for his valuable comments.

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