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On factorisations of complete multigraphs into line graphs of complete graphs

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Abstract. A connection between residuals of biplanes and factorisations of complete multigraphs into isomorphic copies of line graphs of complete graphs is presented. A biplane with blocks of size $n+1$ can be used to construct a factorisation of $4K_{\binom{n}{2}}$ into $n+1$ copies of the line graph of K_n, thus establishing existence of such factorisations for $n \in \{8, 10, 12\}$. Together with the Hall-Connor Theorem, the connection also gives a new proof of the result that, for $n > 3$, there is no factorisation of $2K_{\binom{n}{2}}$ into copies of the line graph of K_n.

1 Introduction

A *factorisation* of a graph K is a collection of spanning subgraphs whose edge sets partition the edge set of K. We write $K \hookrightarrow G$ to denote a factorisation of K into isomorphic copies of a graph G. The notation λK_m is used to denote the graph of order m that has an edge of multiplicity λ joining each pair of distinct vertices, and K_m may be used when $\lambda = 1$.

The line graph of K_n is denoted by L_n. That is, L_n is the graph of order $\binom{n}{2}$ with a vertex corresponding to each edge of K_n and where two vertices are adjacent if and only if their corresponding edges are adjacent in K_n. Line graphs of complete graphs are sometimes called *triangular graphs*.

This paper is concerned with factorisations $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$. The problem has been considered previously in [6], which deals with the more general problem of factorisations $\lambda K_m \hookrightarrow G$ for an arbitrary graph G of order m. The new result on factorisations $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$ obtained here is that, for each $n \in \{8, 10, 12\}$, there exists a factorisation $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$ if and only

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if $\lambda \equiv 0 \pmod 4$. We also describe a connection between factorisations $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$ and residuals of biplanes. This connection is used both to prove the above-mentioned new result and to give a new proof of the result from [6] that there is no factorisation $2K_{\binom{n}{2}} \hookrightarrow L_n$ for any $n > 3$.

Let P denote the Petersen graph. The complements of the factors in a factorisation $\lambda K_{10} \hookrightarrow P$ form a factorisation $2\lambda K_{10} \hookrightarrow L_5$. Thus, the connection with biplanes gives us a new proof (see Section 2) of the well-known result that $K_{10} \not\hookrightarrow P$. In Sections 2 and 3, we give short constructions for factorisations $2K_{10} \hookrightarrow P$ and $3K_{15} \hookrightarrow \mathcal{K}(6,2)$, where $\mathcal{K}(6,2)$ is the Kneser graph having pairs from $\{1,2,3,4,5,6\}$ as vertices and edges joining disjoint pairs—the complement of L_6. The connection between factorisations $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$ and residuals of biplanes is presented in Section 4, and in Section 5 we summarise what is known about factorisations $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$ for small n.

We need some definitions and results from design theory. A (v,k,λ)-*design* consists of a set of v *points* together with a collection of *blocks* where each block is a k-subset of the point set and each pair of distinct points is contained in exactly λ blocks. A (v,k,λ)-design having v blocks is said to be *symmetric*, and it is known that any two distinct blocks from a symmetric (v,k,λ)-design intersect in exactly λ points.

Symmetric designs with $\lambda = 2$ are known as *biplanes*. The number of points in a biplane with blocks of size k is $\binom{k}{2}+1$, and this is also the number of blocks. Biplanes with block size $k \in \{4,5,6,9,11,13\}$ are known to exist, biplanes with block size $k \in \{7,8,10,12,14,15\}$ are known not to exist, existence of biplanes with block size k is ruled out by the Bruck-Ryser-Chowla Theorem [2,5,13,16] for infinitely many values of k, and existence remains unknown for infinitely many values of k, see [11].

Given any symmetric (v,k,λ)-design, one can obtain a $(v-k,\ k-\lambda,\ \lambda)$-design by choosing any block B, deleting the points of B, deleting the block B itself, and deleting the points of B from each of the remaining blocks. The new $(v-k,\ k-\lambda,\ \lambda)$-design is called the *residual design*, with respect to the block B, of the initial symmetric (v,k,λ)-design.

The residual of a biplane with blocks of size k is a $\left(\binom{k-1}{2},\ k-2,\ 2\right)$-design. The Hall-Connor Theorem [8] states that any $\left(\binom{k-1}{2},\ k-2,\ 2\right)$-design can be extended to a biplane by adding k new points, adding a new block containing all the new points, and adding two of the new points to each block of the $\left(\binom{k-1}{2},\ k-2,\ 2\right)$-design. The initial $\left(\binom{k-1}{2},\ k-2,\ 2\right)$-design is thus a residual design of the resulting biplane. The corresponding result for $\lambda = 1$ is the classic extension of an affine plane to a projective plane where parallel lines are extended to meet at "infinity" and a new "line at infinity" is added.

2 The Petersen graph

Let P denote the Petersen graph. It is well known that $K_{10} \not\hookrightarrow P$, but $\lambda K_{10} \hookrightarrow P$ for all $\lambda \geq 2$ [1]. Several different proofs that $K_{10} \not\hookrightarrow P$ have been published [3, 9, 14, 15]. Here, we describe a connection between factorisations $\lambda K_{10} \hookrightarrow P$ and $(16, 6, 2)$-biplanes and use it to show that $K_{10} \not\hookrightarrow P$ and that $2K_{10} \hookrightarrow P$. We generalise this connection in Section 4. Before proceeding, we present the following quick construction for $2K_{10} \hookrightarrow P$.

Consider a copy of K_6 with vertex set $\{1, 2, 3, 4, 5, 6\}$ and take the 10 triangle factors of this K_6 (10 subgraphs consisting of two vertex disjoint copies of K_3) as the vertices for our factorisation $2K_{10} \hookrightarrow P$. Our copies of P will be $P_1, P_2, \ldots, P_6$. For each $i \in \{1, 2, 3, 4, 5, 6\}$, we join two triangle factors F and F' by an edge in P_i if and only if the triangle of F containing vertex i and the triangle of F' containing vertex i are edge disjoint. It is left as an exercise to show that this is indeed a factorisation $2K_{10} \hookrightarrow P$.

We now proceed to the connection between factorisations $\lambda K_{10} \hookrightarrow P$ and $(16, 6, 2)$-biplanes. First we make the following four observations.

1. If G^{c} denotes the complement of a graph G, then $\{P_1, P_2, \ldots, P_{3\lambda}\}$ is a factorisation $\lambda K_{10} \hookrightarrow P$ if and only if $\{P_1^{\mathrm{c}}, P_2^{\mathrm{c}}, \ldots, P_{3\lambda}^{\mathrm{c}}\}$ is a factorisation $2\lambda K_{10} \hookrightarrow P^{\mathrm{c}}$.
2. The complement P^{c} of P is isomorphic to L_5.
3. The graph L_5 consists of five edge-disjoint copies of K_4, any two of which share exactly one vertex. (For each vertex v of K_5 there is a copy of K_4 on the vertices of L_5 that correspond to the four edges of K_5 incident with v, and the vertex of L_5 corresponding to the edge uv of K_5 is the unique vertex shared by the copies of K_4 arising from u and v.)
4. Any factorisation $\lambda K_{10} \hookrightarrow P^{\mathrm{c}}$ yields a $(10, 4, \lambda)$-design whose blocks can be partitioned to form copies of P^{c}.

Consider an arbitrary $(10, 4, 2)$-design and let $\mathcal{B}$ be its block set. The Hall-Connor Theorem tells us that this design is the residual of a $(16, 6, 2)$-biplane with respect to some block B of the biplane. Because any two blocks of a biplane share exactly two points, for each pair $\{a, b\}$ of distinct points of B, there is a unique block $B_{\{a,b\}} \in \mathcal{B}$ where $B_{\{a,b\}} \cup \{a, b\}$ is a block of the biplane. It follows that

$$\left|B_{\{a,b\}} \cap B_{\{c,d\}}\right| = 1 \quad \text{if and only if} \quad \left|\{a, b\} \cap \{c, d\}\right| = 1$$

and hence that there are precisely 6 ways to choose five blocks from $\mathcal{B}$ to form a copy of P^{c}. Namely, for each point $x \in B$ we can choose the five blocks in the set $S_x = \left\{B_{\{a,b\}} : x \in \{a, b\}\right\}$.

Because S_x and S_y share the block $B_{\{x,y\}}$ it is not possible to form copies of P^{c} from a partition of the blocks of a $(10, 4, 2)$-design. Thus, there is no factorisation $2K_{10} \hookrightarrow P^{\mathrm{c}}$, and so no factorisation $K_{10} \hookrightarrow P$. On the other hand, $\{S_x : x \in B\}$ forms a factorisation $4K_{10} \hookrightarrow P^{\mathrm{c}}$, and from this we obtain a factorisation $2K_{10} \hookrightarrow P$.

3 The Kneser graph $\mathcal{K}(6, 2)$

In [6], a factorisation $4K_{15} \hookrightarrow L_6$ is constructed using the 2-transitive action of the alternating group A_7 on 15 points. This same factorisation is also given in [12]; also see [4]. Because $\mathcal{K}(6, 2)$ is the complement of L_6, the complements of the copies of L_6 in a factorisation $4K_{15} \hookrightarrow L_6$ yield a factorisation $3K_{15} \hookrightarrow \mathcal{K}(6, 2)$. Here, we describe how this factorisation $3K_{15} \hookrightarrow \mathcal{K}(6, 2)$ can be constructed from the Hoffman-Singleton graph [10].

Let H denote the Hoffman-Singleton graph, choose an independent set S of size 15 in H, and let $S_1, S_2, \ldots, S_7$ be the 7 independent sets of size 15 that are disjoint from S. For $i \in \{1, 2, \ldots, 7\}$, let G_i be the graph with vertex set S where vertices x and y from S are adjacent in G_i if and only if the unique common neighbour that x and y have in H is in S_i. Then $G_1, G_2, \ldots, G_7$ is a factorisation $3K_{15} \hookrightarrow \mathcal{K}(6, 2)$.

To see that $G_1, G_2, \ldots, G_7$ is a factorisation $3K_{15} \hookrightarrow \mathcal{K}(6, 2)$, consider the following well-known construction of the Hoffman-Singleton graph H. The 50 vertices of H are taken to be the 35 triples from $\{1, 2, 3, 4, 5, 6, 7\}$ together with the 15 copies of the Fano plane

$$\{124, 235, 346, 457, 156, 267, 137\}$$

in its orbit under A_7. The edges of H are given by joining each pair of disjoint triples and joining each Fano plane to the seven triples it contains.

The 15 Fano planes form our independent set S, the vertex set of our factorisation $3K_{15} \hookrightarrow \mathcal{K}(6,2)$. The 7 independent sets of size 15 that are disjoint from S are $S_1, S_2, \ldots, S_7$ where, for each $i \in \{1,2,3,4,5,6,7\}$, the set S_i consists of the 15 triples from $\{1,2,3,4,5,6,7\}$ that contain i. Because H has girth 5 and diameter 2, each pair of Fano planes has a unique common triple. Thus, a pair of Fano planes with common triple $\{x,y,z\}$ is joined by an edge in G_x, G_y, and G_z, and it follows that $G_1, G_2, \ldots, G_7$ is a factorisation of $3K_{15}$.

It remains to show that $G_1 \cong G_2 \cong \cdots \cong G_7 \cong \mathcal{K}(6,2)$. For this, consider G_1 and the subgraph of H induced by the 15 Fano planes and the 15 triples that contain 1. This induced subgraph is the Tutte-Coxeter graph [7,17,18]. A well-known construction for the Tutte-Coxeter graph is to take the 15 edges of K_6 and the 15 1-factors of K_6 as the vertices, and to join an edge to a 1-factor precisely when the edge appears in the 1-factor. Because we are considering G_1, our K_6 has vertex set $\{2,3,4,5,6,7\}$. If we identify each triple $1ab$ with the edge ab of our K_6 and identify each Fano plane having triples $1ab$, $1cd$, and $1ef$ with the 1-factor $\{ab, cd, ef\}$ of our K_6, then it is clear that the subgraph of H induced by the 15 Fano planes and the 15 triples that contain 1 is indeed the Tutte-Coxeter graph.

It follows that two Fano planes are adjacent in G_1 precisely when they are distance 2 in the above-described Tutte-Coxeter graph, and hence $G_1 \cong \mathcal{K}(6,2)$. To see this, note that two vertices of the Tutte-Coxeter graph that correspond to two edges of K_6 are at distance 2 precisely when the two edges are independent in K_6, and that the Tutte-Coxeter graph has automorphisms that interchange the parts of its bipartition. The same argument shows that $G_2 \cong G_3 \cong \cdots \cong G_7 \cong \mathcal{K}(6,2)$. Thus, $G_1, G_2, \ldots, G_7$ is a factorisation $3K_{15} \hookrightarrow \mathcal{K}(6,2)$.

The above-described factorisation $3K_{15} \hookrightarrow \mathcal{K}(6,2)$ given by $G_1, G_2, \ldots, G_7$ has the property that, for $1 \le i < j \le 7$, the graph $G_i \cap G_j$ consists of 5 vertex-disjoint copies of K_3. The factorisation $4K_{15} \hookrightarrow L_6$ given by $G_1^c, G_2^c, \ldots, G_7^c$ has the property that, for $1 \le i < j \le 7$, the graph $G_i^c \cap G_j^c$ is isomorphic to the line graph of the Petersen graph.

4 Line graphs and biplanes

The following two theorems generalise the connection, presented in Section 2, between factorisations $\lambda K_{10} \hookrightarrow L_5$ and $(16,6,2)$-biplanes.

Theorem 4.1. *If there exists a biplane with block size* $n+1$*, then there exists a factorisation* $4K_{\binom{n}{2}} \hookrightarrow L_n$.

Proof. The residual of a biplane with blocks of size $n+1$ is an $\left(\binom{n}{2}, n-1, 2\right)$-design. Let B be the deleted block of the biplane. For each point $x \in B$, there are n other blocks of the biplane that contain x, and these give rise to a set S_x of n blocks in the residual design. Each block of the residual design forms a copy of K_{n-1}, and the n copies of K_{n-1} corresponding to the blocks in S_x form a copy of L_n. The $n+1$ copies of L_n formed from the $n+1$ deleted points is a factorisation $4K_{\binom{n}{2}} \hookrightarrow L_n$. □

The next theorem was proved in [6] by generalising the linear algebra-based argument that Schwenk [14, 15] used to prove there is no factorisation $K_{10} \hookrightarrow P$. The new proof we give here is based on design theory arguments and uses the Hall-Connor Theorem [8].

Theorem 4.2 (S.M. Cioabă and P.J. Cameron, [6])**.** *For* $n > 3$ *there is no factorisation* $2K_{\binom{n}{2}} \hookrightarrow L_n$.

Proof. First note that $2K_{\binom{n}{2}}$ has valency $(n+1)(n-2)$ and that L_n has valency $2(n-2)$. So if a factorisation $2K_{\binom{n}{2}} \hookrightarrow L_n$ exists, then n is odd and the number of factors is $(n+1)/2$. Because L_n is the union of n pairwise intersecting edge-disjoint copies of K_{n-1}, a factorisation $2K_{\binom{n}{2}} \hookrightarrow L_n$ forms an $\left(\binom{n}{2}, n-1, 2\right)$-design whose block set $\mathcal{B}$ is partitioned into $(n+1)/2$ sets such that any two distinct blocks from the same set of the partition share exactly one point.

By the Hall-Connor Theorem, the $\left(\binom{n}{2}, n-1, 2\right)$-design is the residual, with respect to some deleted block B, of an $\left(\binom{n+1}{2}+1, n+1, 2\right)$-biplane. At this point, it is worth mentioning that the Bruck-Ryser-Chowla Theorem rules out the existence of an $\left(\binom{n+1}{2}+1, n+1, 2\right)$-biplane, and hence also a factorisation $2K_{\binom{n}{2}} \hookrightarrow L_n$, for infinitely many values of n. However, we proceed to show that there can be no factorisation $2K_{\binom{n}{2}} \hookrightarrow L_n$, even if an $\left(\binom{n+1}{2}+1, n+1, 2\right)$-biplane exists.

Because any two blocks of a biplane intersect in exactly two points, for each pair $\{a,b\}$ of distinct points in B there is a unique block $B_{\{a,b\}} \in \mathcal{B}$ such that $B_{\{a,b\}} \cup \{a,b\}$ is a block of the biplane. It follows that

$$\left|B_{\{a,b\}} \cap B_{\{c,d\}}\right| = 1 \quad \text{if and only if} \quad \left|\{a,b\} \cap \{c,d\}\right| = 1.$$

For $n > 3$, any set of n distinct mutually intersecting pairs of points from B consists of all the pairs that contain a common fixed point of B. Thus, there are precisely $n+1$ sets of blocks in $\mathcal{B}$ that form a copy of L_n—namely $S_x = \{B_{\{a,b\}} : x \in \{a,b\}\}$ for each $x \in B$. Because S_x and S_y share the block $B_{\{x,y\}}$, the required partition of $\mathcal{B}$ does not exist. □

5 Factorisations $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$

In this final section we briefly summarise what is known on the existence of factorisations $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$ for small n. It follows from consideration of the valencies of $\lambda K_{\binom{n}{2}}$ and L_n that the number of copies of L_n in a factorisation $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$ is $\lambda(n+1)/4$. Thus, if there exists a factorisation $\lambda K_{\binom{n}{2}} \hookrightarrow L_n$, then

- $n \equiv 3 \pmod 4$;
- $n \equiv 1 \pmod 4$ and λ is even; or
- n is even and $\lambda \equiv 0 \pmod 4$.

Theorem 4.2 tells us that, for $n > 3$, there is no factorisation $2K_{\binom{n}{2}} \hookrightarrow L_n$, and it follows that there is also no factorisation $K_{\binom{n}{2}} \hookrightarrow L_n$. However, as pointed out in [6], it is clear that there is no factorisation $K_{\binom{n}{2}} \hookrightarrow L_n$ for any $n > 3$ because the graph L_n has complete subgraphs of order $n-1$ but its largest independent set has only $\lfloor n/2 \rfloor$ vertices. So L_n is not a subgraph of its complement.

We now discuss small values of n, starting with the smallest non-trivial value of n, namely $n = 4$. For $n \leq 7$, these results can all be found in [6].

The case $n = 4$: The graph L_4 is the graph obtained from K_6 by removing the edges of any 1-factor. Thus, by taking the complements of the 1-factors in any 1-factorisation of K_6, we obtain a factorisation $4K_6 \hookrightarrow L_4$. It follows that there is a factorisation $\lambda K_6 \hookrightarrow L_4$ if and only if $\lambda \equiv 0 \pmod 4$.

The case $n = 5$: When $n = 5$, λ is necessarily even, and the complement of L_5 is the Petersen graph P. Factorisations $2K_{10} \hookrightarrow P$ and $3K_{10} \hookrightarrow P$ are given both in [1] and in [6] (also see Section 2). Taking the complements of the copies of P, we obtain factorisations $4K_{10} \hookrightarrow L_5$ and $6K_{10} \hookrightarrow L_5$. Because there is no factorisation $K_{10} \hookrightarrow P$, and hence no factorisation

$2K_{10} \hookrightarrow L_5$, it follows that there is a factorisation $\lambda K_{10} \hookrightarrow L_5$ if and only if λ is even and at least 4.

The case $n = 6$: A factorisation $4K_{15} \hookrightarrow L_6$ is given in [6] (also see Section 3). It follows that there is a factorisation $\lambda K_{15} \hookrightarrow L_6$ if and only if $\lambda \equiv 0 \pmod 4$. It is worth mentioning that there is no $(22, 7, 2)$-biplane, so the converse of Theorem 4.1 does not hold.

The case $n = 7$: This is the first non-trivial value of n where no value of λ is ruled out solely by valency considerations. We know that, for $\lambda \in \{1, 2\}$, factorisations $\lambda K_{21} \hookrightarrow L_7$ do not exist and neither do $(21, 6, \lambda)$-designs. A factorisation $\lambda K_{21} \hookrightarrow L_7$ with $\lambda = 1440$ is given in [6]. This of course implies the existence of a factorisation $\lambda K_{21} \hookrightarrow L_7$ whenever λ is a multiple of 1440, but this appears to be all that is known in the case $n = 7$. A factorisation $\lambda K_{21} \hookrightarrow L_7$ implies the existence of a $(21, 6, \lambda)$-design, and it is known that such designs exist for all $\lambda \geq 3$. If a factorisation $\lambda K_{21} \hookrightarrow L_7$ exists for $\lambda \in \{3, 4, 5\}$, then a factorisation $\lambda K_{21} \hookrightarrow L_7$ exists for all $\lambda \geq 3$.

The case $n \in \{8, 10, 12\}$: Because biplanes with blocks of size k exist for $k \in \{9, 11, 13\}$ (see [11]), by Theorem 4.1 we have factorisations $4K_{\binom{n}{2}} \hookrightarrow L_n$ for $n \in \{8, 10, 12\}$. This together with the necessary condition that $\lambda \equiv 0 \pmod 4$ when n is even tells us that for $n \in \{8, 10, 12\}$ there exists a factorisation

$$\lambda K_{\binom{n}{2}} \hookrightarrow L_n \quad \text{if and only if} \quad \lambda \equiv 0 \pmod 4.$$

So the problem is completely settled for $n \in \{8, 10, 12\}$.

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