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A survey on 2-factors of regular graphs

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Abstract. A 2-factor of a graph G is a 2-regular spanning subgraph of G. We present a survey summarising results on the structure of 2-factors in regular graphs, as achieved by various researchers in recent years.

1 Introduction

All graphs considered in this survey are finite and simple (without loops or multiple edges). We shall use the term *multigraph* when multiple edges are permitted. For definitions and notations not explicitly stated the reader may refer to Bondy and Murty's book *Graph Theory* [15].

Several authors have considered the number of Hamiltonian circuits in k-regular graphs, and there are interesting and beautiful results and conjectures in the literature. In particular, C. A. B. Smith (1940, cf. Tutte [54]) proved that each edge of a 3-regular multigraph lies in an even number of Hamiltonian circuits. This result was extended to multigraphs in which each vertex has an odd degree by Thomason [51].

A graph with exactly one Hamiltonian circuit is said to be *uniquely Hamiltonian*. Thomason's result implies that there are no regular uniquely Hamiltonian multigraphs of odd degree. In 1975, Sheehan [48] posed the following famous conjecture:

Conjecture 1.1 (Sheehan [48])**.** There are no uniquely Hamiltonian k-regular graphs for all integers $k \geq 3$.

It is well known that it is enough to prove it for $k = 4$. This conjecture was verified by Thomassen for bipartite graphs [52] (under the weaker hypothesis that G has minimum degree 3) and for k-regular graphs [53], when

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$k \geq 300$. Ghandehari and Hatami [25] have improved this value for $k \geq 48$ and, recently, by Haxell, Seamone, and Verstraete [30] for $k > 22$.

In this context, several recent papers addressed the problem of characterising particularly regular graphs with certain conditions imposed on their 2-factors. This survey presents the main results obtained in the recent years. We will also discuss the connections of these problems with the particular class of *odd 2-factored snarks*.

2 Preliminaries

An *r-factor* of a graph G is an r-regular spanning subgraph of G. Thus, a *2-factor* of a graph G is a 2-regular spanning subgraph of G, while a 1-factor is also called a *perfect matching* since it is a matching that covers all the vertices. A *1-factorization* of G is a partition of the edge set of G into edge-disjoint 1-factors.

Let G be a bipartite graph with bipartition (X, Y) such that $|X| = |Y|$ and let A be its adjacency matrix. In general $0 \leq |\det(A)| \leq \operatorname{per}(A)$. We say that G is *det-extremal* if $|\det(A)| = \operatorname{per}(A)$. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of G. For F a 1-factor of G we define the *sign* of F, denoted $\operatorname{sgn}(F)$, to be the sign of the permutation of $\{1, 2, \ldots, n\}$ corresponding to F. (Thus G is det-extremal if and only if all 1-factors of G have the same sign.) The following elementary result is a special case of [39, Lemma 8.3.1].

Lemma 2.1. *Let F_1, F_2 be 1-factors in a bipartite graph G and let t be the number of circuits in $F_1 \cup F_2$ of length congruent to zero modulo four. Then* $\operatorname{sgn}(F_1)\operatorname{sgn}(F_2) = (-1)^t$.

Before proceeding, we recall a standard operation on graphs that will be recurrent in this survey. Let G_1 and G_2 be two graphs each containing a vertex of degree 3, say $y \in V(G_1)$ and $x \in V(G_2)$. Let x_1, x_2, x_3 be the neighbours of y in G_1 and y_1, y_2, y_3 be the neighbours of x in G_2. We say that the graph $G = (G_1 - y) \cup (G_2 - x) \cup \{x_1y_1, x_2y_2, x_3y_3\}$ is a *star product* of G_1 and G_2 and write $G = (G_1, y) * (G_2, x)$. We remark that if G_1 and G_2 are bridgeless and cubic, then the graph obtained is also bridgeless and cubic. In the opposite direction, for a bridgeless cubic graph G having a 3-edge cut X, it is possible to define a *3-edge-reduction* on X as the graph operation on G that creates two new bridgeless cubic graphs by adding a new vertex to each of the components of $G - X$ and joining it to the degree 2 vertices in the respective component. Both operations are shown in Figure 2.1.

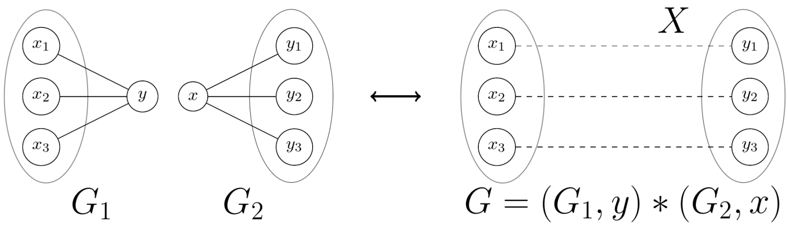


Figure 2.1: From left to right: The Star Product between the two graphs G_1 and G_2. From right to left: The 3-edge-reduction on X in the graph G.

The *Heawood graph* H_0 is the bipartite graph associated with the point/line incidence matrix of the Fano plane $PG(2,2)$. Let $\mathcal{SP}(H_0)$ be the class of graphs obtained from the Heawood graph by repeated star products.

These graphs were used by McCuaig in [42] to characterise the 3-connected cubic det-extremal bipartite graphs:

Theorem 2.2 (McCuaig [42]). *A 3-connected cubic bipartite graph is det-extremal if and only if it belongs to* $\mathcal{SP}(H_0)$.

Note:

(i) Theorem 2.2 has been improved for connectivity 2 graphs by Funk, Jackson, Labbate, and Sheehan in [21];

(ii) Bipartite graphs G with the more general property that some of the entries in the adjacency matrix A of G can be changed from 1 to -1 in such a way that the resulting matrix A^* satisfies $\text{per}(A) = \det(A^*)$ have been characterised in [38, 41, 43].

3 2-factor Hamiltonian graphs

A graph that has a 2-factor is said to be 2-*factor Hamiltonian* if all its 2-factors are Hamiltonian circuits. Examples of such graphs are K_4, K_5, $K_{3,3}$, the Heawood graph H_0, and the cubic graph of girth five obtained from a 9-circuit by adding three vertices, each joined to three vertices of the 9-circuit. This last example is also known as the *Triplex graph* of Robertson, Seymour, and Thomas.

The following property was stated without proof in [22], and it is important for approaching a characterisation of this family of graphs.

Proposition 3.1. *Let G be a bipartite graph represented as a star product $G = (G_1, y) * (G_2, x)$ such that every pair of edges in the 3-edge cut of G belong to a 2-factor. Then G is 2-factor Hamiltonian if and only if G_1 and G_2 are 2-factor Hamiltonian.*

Here, we have stated a slightly different version of the above Proposition 3.1. In fact, we have added here the weaker hypothesis that every pair of edges in the 3-edge cut of G belongs to a 2-factor, which was tacitly assumed in [22].

The bipartite hypothesis in Proposition 3.1 is needed, as the star products of non-bipartite 2-factor Hamiltonian graphs are not necessarily 2-factor Hamiltonian. It happens, for example, when considering $K_4 * K_4$.

In addition to this, it was pointed out by M. Gorsky and T. Johanni in a private communication [28] that, without the hypothesis on the 3-edge cut of G, it is possible to obtain 2-factor Hamiltonian graphs as a star product of two graphs that are not 2-factor Hamiltonian, as shown in Figure 3.1. For these reasons, a proof of the above Proposition 3.1 is given below.

Proof. We start by noticing that since we are assuming that G is bipartite, then so are G_1 and G_2, by the properties of the star product. Let X be the 3-edge cut of the bipartite graph G.

($\Rightarrow$) It follows immediately that the 2-factors in G_1 and G_2 arise from 2-factors of G by contracting the edges of the 2-factor in X. Since G is 2-factor Hamiltonian, then G_1 and G_2 must, at the least, be Hamiltonian graphs. Suppose now, by contradiction, that G_1 or G_2 is not 2-factor Hamiltonian, say G_1. Hence, there exists a 2-factor F_1 of G_1 made of at least two disjoint circuits. Let C be the circuit in F_1 containing the vertex y on which the star product is operating. It follows that the two edges of C incident to y correspond to two edges of X. Based on our assumption, these two edges are part of a 2-factor of G, enabling us to combine C with the Hamiltonian circuit of G_2 to form a 2-factor of G with at least two components, which is a contradiction.

($\Leftarrow$) Suppose now, that G_1 and G_2 are 2-factor Hamiltonian and, by contradiction, suppose that G is not 2-factor Hamiltonian. Note that by hypothesis G admits 2-factors containing edges of X. Let F be a 2-factor of G which is not a Hamiltonian circuit. It follows that $|F \cap X|$ must be equal to two or to zero. In the former case, it follows immediately that G_1 or G_2 are not 2-factor Hamiltonian, which is a contradiction. Suppose now that $|F \cap X| = 0$. Since G is a bipartite graph containing a 2-factor, it has an even number of vertices, and $G - X$ has exactly two components. It follows that both G_1 and G_2 are of odd order, but since they are Hamiltonian, they contain an odd circuit, which is a contradiction. □

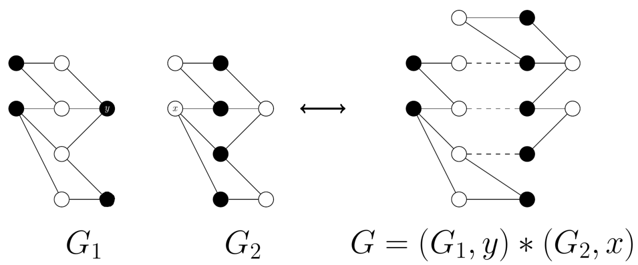


Figure 3.1: The star product $(G_1, y) * (G_2, x)$ between the two non-2-factor Hamiltonian graphs G_1 and G_2 is 2-factor Hamiltonian. However, not every pair of the dashed edges of the 3-edge cut X (dashed) belong to a 2-factor, as shown in [28].

It is worth remarking the additional hypothesis on the 3-edge cut of G is not needed when dealing with bipartite cubic graphs as it is always satisfied, and by using Proposition 3.1, Funk, Jackson, Labbate, and Sheehan constructed an infinite family of 2-factor Hamiltonian cubic bipartite graphs by taking iterated star products of $K_{3,3}$ and H_0 [22]. They conjecture that these are the only non-trivial 2-factor Hamiltonian regular bipartite graphs.

Conjecture 3.2 (Funk et al. [22])**.** Let G be a 2-factor Hamiltonian k-regular bipartite graph. Then either $k = 2$ and G is a circuit or $k = 3$ and G can be obtained from $K_{3,3}$ and H_0 by repeated star products.

We remark here that Conjecture 3.2 is still open, and a positive answer to it will allow us to completely characterise the family of 2-factor Hamiltonian regular bipartite graphs. In the 1980s Sheehan posed the following conjecture [47]:

Conjecture 3.3 (Sheehan [47])**.** There are no 2-factor Hamiltonian k-regular bipartite graphs for all integers $k \geq 4$.

The following properties have been proved by Labbate [34,35] for an equivalent family of cubic graphs (cf. subsection 3.1) and then by Funk, Jackson, Labbate, and Sheehan [22] for 2-factor Hamiltonian graphs:

Lemma 3.4 (Labbate et al. [22, 34, 35])**.** *Let G be a 2-factor Hamiltonian cubic bipartite graph. Then G is 3-connected and $|V(G)| \equiv 2 \pmod 4$.*

A graph H is *maximally 2-factor Hamiltonian* if the multigraph G obtained by adding an edge e with endvertices u, v to H has a disconnected 2-factor containing e.

Lemma 3.5 (Funk et al. [22, Lemma 3.4 (a)(i)])**.** *Graphs obtained by taking star products of H_0 are maximally 2-factor Hamiltonian.*

Funk, Jackson, Labbate, and Sheehan in [22] proved Conjecture 3.3 applying Lemmas 2.1, 3.4, and 3.5 with Theorem 2.2:

Theorem 3.6 (Funk et al. [22])**.** *Let G be a 2-factor Hamiltonian k-regular bipartite graph. Then $k \leq 3$.*

Theorem 3.6 has inspired further results by Faudree, Gould, and Jacobsen [19] who determined the maximum number of edges in both 2-factor Hamiltonian graphs and 2-factor Hamiltonian bipartite graphs. In particular, they proved the following theorems:

Theorem 3.7 (Faudree et al. [19])**.** *If G is a bipartite 2-factor Hamiltonian graph of order n, then*

$$\big|E(G)\big| \leq \begin{cases} n^2/8 + n/2, & \text{if } n \equiv 0 \pmod 4, \\ n^2/8 + n/2 + 1/2, & \text{if } n \equiv 2 \pmod 4, \end{cases}$$

and the bound is sharp.

Theorem 3.8 (Faudree et al. [19])**.** *If G is a 2-factor Hamiltonian graph of order n, then*

$$\big|E(G)\big| \leq \lceil n^2/4 + n/4 \rceil,$$

and the bound is sharp for all $n \geq 6$.

In addition, Diwan [18] has shown the following:

Theorem 3.9 (Diwan [18]). *K_4 is the only 3-regular 2-factor Hamiltonian planar graph.*

Conjecture 3.2 has been partially solved in terms of minimally 1-factorable cubic bipartite graphs, as we explain in the following subsection.

3.1 Minimally 1-factorable graphs

Let G be a k-regular bipartite graph. We say that G is *minimally 1-factorable* if every 1-factor of G is contained in a unique 1-factorization of G.

The results cited above by Funk, Jackson, Labbate, and Sheehan were inspired by results on minimally 1-factorable graphs obtained in [23,34–36].

Proposition 3.10 (Funk et al. [22]). *Let G be a k-regular bipartite graph. If G is minimally 1-factorable, then G is 2-factor Hamiltonian. If $k \in \{2, 3\}$, then G is minimally 1-factorable if and only if G is 2-factor Hamiltonian.*

Theorem 3.6 extends the result of [23] that minimally 1-factorable k-regular bipartite graphs exist only when $k \leq 3$.

Furthermore, Labbate in [36] proved the following characterisation:

Theorem 3.11 (Labbate [36]). *Let G be a minimally 1-factorable k-regular bipartite graph of girth 4. Then either $k = 2$ and G is a circuit or $k = 3$ and G can be obtained from $K_{3,3}$ by repeated star products.*

Hence, it follows from results in [35] that a smallest counterexample to Conjecture 3.2 is cubic and cyclically 4-edge connected, and from Theorem 3.11 it follows that it has girth at least 6. Thus, to prove the conjecture, it would suffice to show that the Heawood graph is the only 2-factor Hamiltonian cyclically 4-edge-connected cubic bipartite graph of girth at least 6.

This seems a challenging task to achieve, at least with the techniques used so far. In [7], partial results were obtained by using *irreducible Levi graphs* (cf. Section 5.1 and Theorem 5.10).

3.2 Perfect Matching Hamiltonian graphs

A graph G admitting a 1-factor is said to have the *Perfect-Matching-Hamiltonian property* (for short, the PMH-property) if every 1-factor M of G can be extended to a Hamiltonian circuit of G, that is, there exists a 1-factor N of G such that $M \cup N$ induces a Hamiltonian circuit of G. This problem was first introduced by Las Vergnas [37] and Häggkvist [29] in the 1970s, and in recent years this concept was studied with particular focus on cubic graphs. The reader may find more details in [5] and [9]. For simplicity, a graph admitting the PMH-property is said to be PMH or a PMH-graph. The class of 2-factor Hamiltonian graphs are of course PMH-graphs. Recently, in [45], in an attempt to look at Conjecture 3.2 from a different point of view, the following statement was proved:

Proposition 3.12 (Romaniello and Zerafa [45]). *Let G be a cubic PMH-graph (not necessarily bipartite). The graph G is not 2-factor Hamiltonian if and only if it admits a 1-factor, which can be extended to a Hamiltonian circuit in exactly one way.*

Proposition 3.12 suggests another way how one can look at Conjecture 3.2. Indeed, a smallest counterexample to this conjecture can be searched for in the class of bipartite cubic PMH-graphs, and hence the Conjecture 3.2 can be equivalently restated in terms of a strictly weaker property than 2-factor Hamiltonicity: the PMH-property.

Conjecture 3.13 (Romaniello and Zerafa [45]). Every bipartite cyclically 4-edge-connected cubic PMH-graph with girth at least 6, except the Heawood graph, admits a 1-factor, which can be extended to a Hamiltonian circuit in exactly one way.

4 2-factor isomorphic graphs

The family of 2-factor Hamiltonian k-regular graphs can be extended to the family of connected k-regular graphs with the more general property that all their 2-factors are isomorphic, i.e., the family of *2-factor isomorphic* k-regular bipartite graphs.

Examples of such graphs are given by all the 2-factor Hamiltonian graphs and the Petersen graph (which is 2-factor isomorphic since all of its 2-factors are of length $(5, 5)$ but it is not 2-factor Hamiltonian). Note that the star product also preserves the property of being 2-factor isomorphic.

In [10] Aldred, Funk, Jackson, Labbate, and Sheehan proved the following existence theorem:

Theorem 4.1 (Aldred et al. [10]). *If G is a 2-factor isomorphic k-regular bipartite graph, then $k \leq 3$.*

They also conjecture that the family of 2-factor isomorphic graphs and the class of 2-factor Hamiltonian k-regular bipartite graphs are, in fact, the same.

Conjecture 4.2 (Aldred et al. [10])**.** Let G be a connected k-regular bipartite graph. Then G is 2-factor isomorphic if and only if G is 2-factor Hamiltonian.

Abreu, Diwan, Jackson, Labbate, and Sheehan proved in [3] that Conjecture 4.2 is false by applying the following construction:

Proposition 4.3 (Abreu et al. [3])**.** *Let G_i be a 2-factor Hamiltonian cubic bipartite graph with k vertices and $e_i = u_i v_i \in E(G_i)$ for $i \in \{1, 2, 3\}$. Let G be the graph obtained from the disjoint union of the graphs $G_i - e_i$ by adding two new vertices w and z and new edges wu_i and zv_i for $i \in \{1, 2, 3\}$. Then G is a non-Hamiltonian connected 2-factor isomorphic cubic bipartite graph of edge-connectivity 2.*

Consider a set $\{G_1, G_2, \ldots, G_k\}$ of 3-edge-connected cubic bipartite graphs and let $\mathcal{SP}(G_1, G_2, \ldots, G_k)$ denote the set of cubic bipartite graphs that can be obtained from $G_1, G_2, \ldots, G_k$ by repeated star products. In Section 3 we have seen that it was shown in [22] that all graphs in $\mathcal{SP}(K_{3,3}, H_0)$ are 2-factor Hamiltonian. Thus we may apply Proposition 4.3 by taking $G_1 = G_2 = G_3$ to be any graph in $\mathcal{SP}(K_{3,3}, H_0)$ to obtain an infinite family of 2-edge-connected non-Hamiltonian 2-factor isomorphic cubic bipartite graphs. This family gives counterexamples to the Conjecture 4.2. Note, however, that Conjecture 4.2 can be modified as follows:

Conjecture 4.4 (Abreu et al. [3])**.** Let G be a 3-edge-connected 2-factor isomorphic cubic bipartite graph. Then G is a 2-factor Hamiltonian cubic bipartite graph.

Recall that a *digraph* is a graph in which the edges have a direction (and they are now ordered pairs of vertices). In [1, 2] Abreu et al. also proved existence theorems for the digraphs and non-bipartite graphs case, as shown below.

For v a vertex of a digraph D, let $d^+(v)$ and $d^-(v)$ denote the out-degree and in-degree of v, respectively. We say that D is k-diregular if for all vertices v of G, we have $d^+(v) = d^-(v) = k$.

Theorem 4.5 (Abreu at al. [1, 2]). *Let D be a digraph with n vertices and let X be a directed 2-factor of D. Suppose that either*

(a) $d^+(v) \geq \lfloor \log_2 n \rfloor + 2$ *for all* $v \in V(D)$ *or*

(b) $d^+(v) = d^-(v) \geq 4$ *for all* $v \in V(D)$.

Then D has a directed 2-factor Y with $Y \ncong X$.

Corollary 4.6 (Abreu at al. [1]). *If G is a k-diregular directed graph, then $k \leq 3$.*

Theorem 4.7 (Abreu at al. [1, 2]). *Let G be a graph with n vertices and let X be a 2-factor of G. Suppose that either*

(a) $d(v) \geq 2\big(\lfloor \log_2 n \rfloor + 2\big)$ *for all* $v \in V(G)$ *or*

(b) *G is a $2k$-regular graph for some $k \geq 4$.*

Then G has a 2-factor Y with $Y \ncong X$.

They have also posed the following open problems and conjecture:

Question 4.8 (Abreu at al. [1]). *Do there exist 2-factor isomorphic bipartite graphs of arbitrarily large minimum degree?*

Question 4.9 (Abreu at al. [2]). *Do there exist 2-factor isomorphic regular graphs of arbitrarily large degree?*

Conjecture 4.10 (Abreu at al. [1]). The graph K_5 is the only 2-factor Hamiltonian 4-regular non-bipartite graph.

5 Pseudo 2-factor isomorphic graphs

In [3] Abreu, Diwan, Jackson, Labbate, and Sheehan extended the aforementioned results on regular 2-factor isomorphic bipartite graphs to the more general family of *pseudo 2-factor isomorphic graphs*, i.e., graphs G with the property that the parity of the number of circuits in a 2-factor is the same for all 2-factors of G.

Examples of such graphs are given by all the 2-factor isomorphic regular graphs and the Pappus graph (i.e., the point/line incidence graph of the Pappus configuration). The family of pseudo 2-factor isomorphic graphs is wider than the one of 2-factor isomorphic regular bipartite graphs:

Proposition 5.1 (Abreu et al. [3]). *The Pappus graph P_0 is pseudo 2-factor isomorphic but not 2-factor isomorphic.*

In [3] Abreu, Diwan, Jackson, Labbate, and Sheehan proved the following existence theorem:

Theorem 5.2 (Abreu et al. [3]). *If G is a pseudo 2-factor isomorphic k-regular bipartite graph, then $k \in \{2, 3\}$.*

They have also shown that there are no planar pseudo 2-factor isomorphic cubic bipartite graphs.

Theorem 5.3 (Abreu et al. [3]). *If G is a pseudo 2-factor isomorphic cubic bipartite graph, then G is non-planar.*

Star products preserve also the property of being pseudo 2-factor isomorphic in the family of cubic bipartite graphs.

Lemma 5.4 (Abreu et al. [3]). *Let G be a star product of two pseudo 2-factor isomorphic cubic bipartite graphs G_1 and G_2. Then G is also pseudo 2-factor isomorphic.*

Thus $K_{3,3}$, H_0 and P_0 can be used to construct an infinite family of 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graphs.

Lemma 5.4 implies that all graphs in $\mathcal{SP}(K_{3,3}, H_0, P_0)$ are pseudo 2-factor isomorphic. In [3] Abreu, Diwan, Jackson, Labbate, and Sheehan conjectured that these are the only 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graphs.

Conjecture 5.5 (Abreu et al. [3]). Let G be a 3-edge-connected cubic bipartite graph. Then G is pseudo 2-factor isomorphic if and only if G belongs to $\mathcal{SP}(K_{3,3}, H_0, P_0)$.

Recall that McCuaig [42] has shown that a 3-edge-connected cubic bipartite graph G is det-extremal if and only if $G \in \mathcal{SP}(H_0)$.

Let G be a graph and let E_1 be an edge cut of G. We say that E_1 is a *non-trivial edge cut* if all components of $G - E_1$ have at least two vertices. The graph G is *essentially 4-edge connected* if G is 3-edge connected and has no non-trivial 3-edge cuts. Let G be a cubic bipartite graph with bipartition

(X, Y) and let K be a non-trivial 3-edge cut of G. Let H_1, H_2 be the components of $G - K$. We have seen that G can be expressed as a star product $G = (G_1, y_K) * (G_2, x_K)$ where $G_1 - y_K = H_1$ and $G_2 - x_K = H_2$. We say that y_K, respectively x_K, is the *marker vertex* of G_1, respectively G_2, *corresponding to the cut* K. Each non-trivial 3-edge cut of G distinct from K is a non-trivial 3-edge cut of G_1 or G_2, and vice versa. If G_i is not essentially 4-edge connected for $i \in \{1, 2\}$, then we may reduce G_i along another non-trivial 3-edge cut. We can continue this process until all the graphs we obtain are essentially 4-edge connected. We call these resulting graphs the *constituents* of G. It is easy to see that the constituents of G are unique, i.e., they are independent of the order we choose to reduce the non-trivial 3-edge cuts of G.

It is also easy to see that Conjecture 5.5 holds if and only if Conjectures 5.6 and 5.7 below are both valid.

Conjecture 5.6 (Abreu et al. [3]). If G is an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph, then $G \in \{K_{3,3}, H_0, P_0\}$.

Conjecture 5.7 (Abreu et al. [3]). If G is a 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graph such that $G = G_1 * G_2$, then G_1 and G_2 are both pseudo 2-factor isomorphic.

In [3] Abreu, Diwan, Jackson, Labbate, and Sheehan obtained partial results on Conjectures 5.6 and 5.7 as follows:

Theorem 5.8 (Abreu et al. [3]). *If G is an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph such that G contains a 4-circuit, then $G = K_{3,3}$.*

They used Theorem 5.8 to deduce some evidence in favour of Conjecture 5.5.

Theorem 5.9 (Abreu et al. [3]). *If G is a 3-edge-connected pseudo 2-factor isomorphic bipartite graph that contains a 4-circuit C, then C is contained in a constituent of G that is isomorphic to $K_{3,3}$.*

Note that Theorem 5.9 generalises Theorem 3.11 obtained by Labbate in [36] for minimally 1-factorable bipartite cubic graphs (or equivalently 2-factor Hamiltonian cubic bipartite graphs) to the family of pseudo 2-factor isomorphic bipartite graphs. Furthermore, Theorem 5.9 leaves the characterisation of pseudo 2-factor isomorphic bipartite graphs open for girth at least 6.

Abreu, Labbate, and Sheehan [8] gave a partial solution to this open case in terms of irreducible configurations of Levi graphs as described in the next subsection.

5.1 Irreducible pseudo 2-factor isomorphic cubic bipartite graphs

An incidence structure is *linear* if two different points are incident with at most one line. A *symmetric configuration* n_k (or n_k *configuration*) is a linear incidence structure consisting of n points and n lines such that each point and line is respectively incident with k lines and points. Let $\mathcal{C}$ be a symmetric configuration n_k. Its *Levi graph* $G(\mathcal{C})$ is a k-regular bipartite graph whose vertex set are the points and the lines of $\mathcal{C}$ and where there is an edge between a point and a line in the graph if and only if they are incident in $\mathcal{C}$. We will indistinctly refer to Levi graphs of configurations as their *incidence graphs*.

It follows from Theorem 5.9 that an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph of girth greater than or equal to 6 is the Levi graph of a symmetric configuration n_3.

In 1886 V. Martinetti [40] characterised symmetric configurations n_3, showing that they can be obtained from an infinite set of so called *irreducible* configurations of which he gave a list. Recently, Boben proved that Martinetti's list of irreducible configurations was incomplete and completed it [11]. Boben's list of irreducible configurations was obtained by characterising their Levi graphs, which he called *irreducible Levi graphs*.

In [8] Abreu, Labbate, and Sheehan characterised *irreducible* pseudo 2-factor isomorphic cubic bipartite graphs (and hence gave a further partial answer to Conjecture 5.5) as follows:

Theorem 5.10 (Abreu et al. [8]). *The Heawood and the Pappus graphs are the only irreducible Levi graphs that are pseudo 2-factor isomorphic.*

This approach is not feasible to prove Conjecture 5.5 and hence the main Conjecture 3.2 by studying the 2-factors of reducible configurations from the set of 2-factors of their underlying irreducible ones as the following discussion shows.

It is well known that the 7_3 configuration, whose Levi graph is the Heawood graph, is not Martinetti extendible and that the Pappus configuration is Martinetti extendible in a unique way; it is easy to show that this extension is not pseudo-2-factor isomorphic. Let $\mathcal{C}$ be a symmetric configuration n_3 and let $\tilde{\mathcal{C}}$ be a symmetric configuration $(n+1)_3$ obtained from $\mathcal{C}$ through a Martinetti extension. It can be easily checked that there are 2-factors in $\tilde{\mathcal{C}}$ that cannot be reduced to a 2-factor in $\mathcal{C}$. On the other hand, all of its Martinetti reductions are no longer pseudo 2-factor isomorphic (for further details cf. [8]). In the next section, we will see that Conjecture 5.5 has been disproved, while Conjecture 3.2 still holds.

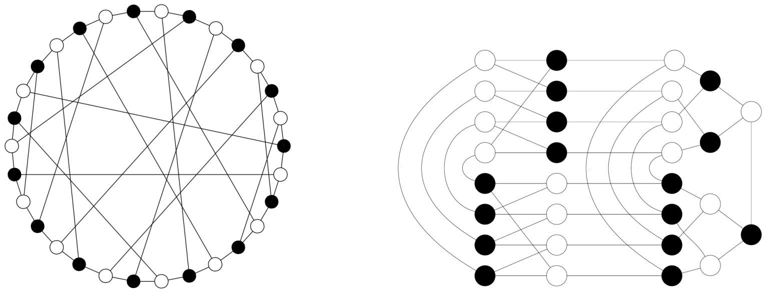
6 A counterexample to the pseudo 2-factor's conjecture

In this section we present the counterexample by Jan Goedgebeur to the pseudo 2-factor isomorphic bipartite graphs obtained from Conjecture 5.5 using exhaustive search via parallel computers (for details refer to [27]). Recently, in [4], it was shown how it could be constructed from the Heawood graph and the generalised Petersen graph $GP(8,3)$, which are the Levi graphs of the Fano 7_3 configuration and the Möbius-Kantor 8_3 configuration, respectively.

Using the program `minibaum` [12], J. Goedgebeur generated all cubic bipartite graphs with girth at least 6 and up to 40 vertices and all cubic bipartite graphs with girth at least 8 and up to 48 vertices. The counts of these graphs can be found in [27, Table 1]. Some of these graphs can be downloaded from the *House of Graphs Database* [16], available online at `https://houseofgraphs.org/`. He then implemented a program that tests if a given graph is pseudo 2-factor isomorphic and applied it to the generated cubic bipartite graphs. This yielded the following results:

Remark 6.1 (Goedgebeur [27]). There is exactly one essentially 4-edge-connected pseudo 2-factor isomorphic graph $\mathcal{G}$ different from the Heawood graph and the Pappus graph among the cubic bipartite graphs with girth at least 6 and with at most 40 vertices.

Remark 6.2 (Goedgebeur [27]). There is no essentially 4-edge-connected pseudo 2-factor isomorphic graph among the cubic bipartite graphs with girth at least 8 and with at most 48 vertices.



(a) The graph $\mathcal{G}$, as it appears in the *House of Graphs Database* [16].

(b) The geometric construction of $\mathcal{G}$, as drawn in [4].

Figure 6.1: Two possible drawings of the graph $\mathcal{G}$.

This implies that Conjecture 5.5 (and consequently also Conjecture 5.6) is false. However, since all 2-factor Hamiltonian graphs are pseudo 2-factor isomorphic and $\mathcal{G}$ is not 2-factor Hamiltonian, this implies the following remark:

Remark 6.3 (Goedgebeur [27]). Conjecture 3.2 holds up to at least 40 vertices and holds for cubic bipartite graphs with girth at least 8 up to at least 48 vertices.

The counterexample found has 30 vertices and there are no additional counterexamples up to at least 40 vertices and also no counterexamples among the cubic bipartite graphs with girth at least 8 up to at least 48 vertices.

The counterexample $\mathcal{G}$ is stored in the *House of Graphs Database* [16] and can be obtained by searching for the keywords

```
pseudo 2-factor isomorphic *counterexample
```

where it can be downloaded and several of its invariants can be inspected (see Figure 6.1a).

In what follows we will briefly describe the geometric construction of the counterexample $\mathcal{G}$ presented in [4]. Consider the classical representation of the Möbius-Kantor configuration as two quadrilaterals simultaneously inscribed and circumscribed (cf. [17, p. 430]). Disregarding the circumscription, i.e., removing the corresponding incidences it defines, we obtain a Möbius-Kantor residue in which the valency of 4 points and 4 lines decreases from three to two. Similarly, removing the incidences of a quadrilateral in the Fano configuration, we obtain a Fano residue with 4 points and 4 lines of valency two. The configuration $\mathcal{C}$, which has $\mathcal{G}$ as its Levi graph, then arises by adding incidences among points and lines of valency two between the Fano residue and the Möbius-Kantor residue in a precise way (see Figure 6.1b), fully described in [4, Section 2].

In detail, $\mathcal{G}$ has girth 6 and cyclic edge-connectivity 6, it is not vertex-transitive, it has 312 total 2-factors, and the circuit sizes of its 2-factors are $(6, 6, 18)$, $(6, 10, 14)$, $(10, 10, 10)$, and (30). Moreover, its automorphism group of order 144 is isomorphic to $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes (D_4 \times \mathbb{Z}_2)$. The authors of [4] pointed out that the construction of joining residues of Levi graphs of n_3 configurations does not preserve, in general, strong properties such as being pseudo 2-factor isomorphic. Intuitively, there is too much room to produce 2-factors of both parities; whereas, the Möbius-Kantor residue and the Fano residue are very compact. Moreover, using several copies of the same residues does not preserve the behaviour of the parity of circuits in a 2-factor.

7 Strongly pseudo 2-factor isomorphic graphs

Abreu, Labbate, and Sheehan in [7] have extended the aforementioned results on regular pseudo 2-factor isomorphic bipartite graphs to the not necessarily bipartite case by introducing the family of strongly pseudo 2-factor isomorphic graphs:

Definition 7.1. Let G be a graph that has a 2-factor. For each 2-factor F of G, let $t_i^*(F)$ be the number of circuits of F of length $2i$ modulo 4. Set t_i to be the function defined on the set of 2-factors F of G by

$$t_i(F) = \begin{cases} 0, & \text{if } t_i^*(F) \text{ is even,} \\ 1, & \text{if } t_i^*(F) \text{ is odd} \end{cases} \qquad \text{for } i \in \{0, 1\}.$$

Then G is said to be *strongly pseudo 2-factor isomorphic* if both t_0 and t_1 are constant functions. Moreover, if in addition $t_0 = t_1$, set $t(G) := t_i(F)$, $i \in \{0, 1\}$.

By definition, if G is strongly pseudo 2-factor isomorphic then G is pseudo 2-factor isomorphic. On the other hand there exist graphs such as the dodecahedron that are pseudo 2-factor isomorphic but not strongly pseudo 2-factor isomorphic. The 2-factors of the dodecahedron consist either of a circuit of length 20 or of three circuits: one of length 10 and the other two of length 5.

In the bipartite case, pseudo 2-factor isomorphic and strongly pseudo 2-factor isomorphic are equivalent.

In what follows we will denote by HU, U, SPU, and PU the sets of 2-factor Hamiltonian, 2-factor isomorphic, strongly pseudo 2-factor isomorphic, and pseudo 2-factor isomorphic graphs, respectively. Similarly, $HU(k)$, $U(k)$, $SPU(k)$, and $PU(k)$ respectively denote the k-regular graphs in HU, U, SPU, and PU.

Theorem 7.2 (Abreu et al. [7]). *Let D be a digraph with n vertices and let X be a directed 2-factor of D. Suppose that either*

(a) $d^+(v) \geq \lfloor \log_2 n \rfloor + 2$ *for all* $v \in V(D)$ *or*

(b) $d^+(v) = d^-(v) \geq 4$ *for all* $v \in V(D)$.

Then D has a directed 2-factor Y with a different parity of number of circuits from X.

Let $DSPU$ and DPU be the sets of digraphs in SPU and PU, i.e., strongly pseudo and pseudo 2-factor isomorphic digraphs, respectively. Similarly, $DSPU(k)$ and $DPU(k)$ respectively denote the k-diregular digraphs in $DSPU$ and DPU.

Corollary 7.3 (Abreu et al. [7]).

(i) $DSPU(k) = DPU(k) = \varnothing$ *for* $k \geq 4$*;*

(ii) *If $D \in DPU$, then D has a vertex of out-degree at most* $\lfloor \log_2 n \rfloor + 1$.

Theorem 7.4 (Abreu et al. [7]). *Let G be a graph with n vertices and let X be a 2-factor of G. Suppose that either*

(a) $d(v) \geq 2\big(\lfloor \log_2 n \rfloor + 2\big)$ *for all* $v \in V(G)$ *or*

(b) *G is a $2k$-regular graph for some $k \geq 4$.*

Then G has a 2-factor Y with a different parity of number of circuits from X.

Corollary 7.5 (Abreu et al. [7]).

(i) *If $G \in PU$, then G contains a vertex of degree at most* $2\lfloor \log_2 n \rfloor + 3$*;*

(ii) $PU(2k) = SPU(2k) = \varnothing$ *for* $k \geq 4$.

We know that there are examples of graphs in $PU(3)$, $SPU(3)$, $PU(4)$, and $SPU(4)$. Hence, they are not empty, and we have seen (cf. Conjecture 4.10) that it has been conjectured in [1] that $HU(4) = \{K_5\}$.

There are many gaps in our knowledge even when we restrict attention to regular graphs. Some questions arise naturally. A few of them are listed below.

Problem 7.6. Is $PU(2k+1) = \varnothing$ for $k \geq 2$?

In particular, we are wondering if $PU(7)$ and $PU(5)$ are empty.

Problem 7.7. Is $PU(6)$ empty?

Problem 7.8. Is K_5 the only 4-edge-connected graph in $PU(4)$?

In [7], relations between pseudo strongly 2-factor isomorphic graphs and a class of graphs called *odd 2-factored snarks* are investigated. The next section is devoted to this class of snarks.

8 Odd 2-factored snarks

A *snark* (cf. e.g. [31]) is a bridgeless cubic graph with chromatic index four. (By Vizing's theorem the chromatic index of every cubic graph is either three or four, so a snark corresponds to the special case of four.) In order to avoid trivial cases, snarks are usually assumed to have girth at least five and not to contain a non-trivial 3-edge cut (i.e., they are cyclically 4-edge connected).

Snarks were named by Martin Gardner in 1976 [24] after the mysterious and elusive creature in Lewis Caroll's famous poem *The Hunting of The Snark*, but it was P. G. Tait in 1880 that initiated the study of snarks when he proved that the four colour theorem is equivalent to the statement that *no snark is planar* [50]. The Petersen graph P_{10} is the smallest snark, and Tutte conjectured that all snarks have Petersen graph minors. This conjecture was proven by Robertson, Seymour, and Thomas (cf. [44]). Necessarily, snarks are non-Hamiltonian.

The importance of the snarks does not only depend on the four colour theorem. Indeed, there are several important open problems such as the classical circuit double cover conjecture [46, 49], Fulkerson's conjecture [20], and Tutte's 5-flow conjecture [55] for which it is sufficient to prove them for snarks. Thus, minimal counterexamples to these and other problems must reside, if they exist at all, among the family of snarks.

At present, there is no uniform theoretical method for studying snarks and their behaviour. In particular, little is known about the structure of 2-factors in a given snark.

Snarks also play an important role in characterising regular graphs with some conditions imposed on their 2-factors. Recall that a 2-factor is a 2-regular spanning subgraph of a graph G.

We say that a graph G is *odd 2-factored* (cf. [7]) if for each 2-factor F of G each circuit of F is odd.

By definition, an odd 2-factored graph G is pseudo 2-factor isomorphic. Note that, odd 2-factoredness is not the same as the *oddness* of a (cubic) graph (cf. e.g. [56]).

Lemma 8.1 (Abreu et al. [7]). *If G is a cubic 3-connected odd 2-factored graph, then G is a snark.*

In [7] Abreu, Labbate, and Sheehan studied which snarks are odd 2-factored and posed the following conjecture:

Conjecture 8.2 (Abreu et al. [7]). A snark is odd 2-factored if and only if G is the Petersen graph, Blanuša 2, or a Flower snark $J(t)$ with $t \geq 5$ and odd.

Below, we report a general construction from [6] of odd 2-factored snarks performing the Isaacs' dot-product [32] on edges with particular properties, called *bold-edges* and *gadget-pairs* respectively, of two snarks L and R.

Construction 8.3 (*Bold-Gadget Dot Product*, Abreu et al. [6]).

1. Take two snarks L and R with bold-edges (cf. Definition 8.4) and gadget-pairs (cf. Definition 8.6), respectively.
2. Choose a bold-edge xy in L.
3. Choose a gadget-pair f, g in R.
4. Perform a dot product $L \cdot R$ using these edges.
5. Obtain a new odd 2-factored snark (cf. Theorem 8.8).

Note that in what follows the existence of a 2-factor in a snark is guaranteed since they are bridgeless by definition.

Definition 8.4 (Abreu et al. [6])**.** Let L be a snark. A *bold-edge* is an edge $e = xy \in L$ such that the following conditions hold:

(i) All 2-factors of $L - x$ and of $L - y$ are odd.

(ii) All 2-factors of L containing xy are odd.

(iii) All 2-factors of L avoiding xy are odd.

Note that not all snarks contain bold-edges (cf. [6, Proposition 4.2] and [6, Lemma 5.1]). Furthermore, conditions (ii) and (iii) are trivially satisfied if L is odd 2-factored.

Lemma 8.5 (Abreu et al. [6])**.** *The edges of the Petersen graph P_{10} are all bold-edges.*

Definition 8.6 (Abreu et al. [6])**.** Let R be a snark. A pair of independent edges $f = ab$ and $g = cd$ is called a *gadget-pair* if the following conditions hold:

(i) There are no 2-factors of R avoiding both f, g.

(ii) All 2-factors of R containing exactly one element of $\{f, g\}$ are odd.

(iii) All 2-factors of R containing both f and g are odd. Moreover, f and g belong to different circuits in each such factor.

(iv) All 2-factors of $\big(R - \{f, g\}\big) \cup \{ac, ad, bc, bd\}$ containing exactly one element of $\{ac, ad, bc, bd\}$ are such that the circuit containing the new edge is even and all other circuits are odd.

Note that finding gadget-pairs in a snark is not an easy task, and in general not all snarks contain gadget-pairs (cf. [6, Lemma 5.2]).

Let $H := \{x_1y_1, x_2y_2, x_3y_3\}$ be the two horizontal edges and the vertical edge respectively (in the pentagon-pentagram representation) of P_{10} (cf. Figure 8.1).

Lemma 8.7 (Abreu et al. [6])**.** *Any pair of distinct edges f, g in the set H of P_{10} is a gadget-pair.*

The following theorem is used to construct new odd 2-factored snarks.

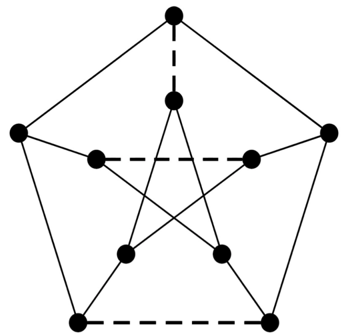


Figure 8.1: Any pair of the dashed edges is a gadget-pair in P_{10}.

Theorem 8.8 (Abreu et al. [6]). *Let xy be a bold-edge in a snark L and let $\{ab, cd\}$ be a gadget-pair in a snark R. Then $L \cdot R$ is an odd 2-factored snark.*

In particular, without going into lengthy details (the interested reader may find those in [6]), this method allowed for the construction of two instances of odd 2-factored snarks of order 26 and 34 isomorphic to those obtained by Brinkmann et al. in [13] through an exhaustive computer search on all snarks of order at most 36 that allowed them to disprove the above conjecture (cf. Conjecture 8.2).

To approach the problem of characterising all odd 2-factored snarks, the possibility of constructing further odd 2-factored snarks with the technique presented above was considered, which relies on finding other snarks with bold-edges and/or gadget-pairs. The results obtained so far give rise to the following partial characterisation:

Theorem 8.9 (Abreu et al. [6]). *Let G be an odd 2-factored snark of cyclic edge-connectivity 4 that can be constructed from the Petersen graph and the Flower snarks using the bold-gadget dot product construction. Then $G \in \{P_{18}, P_{26}, P_{34}\}$.*

Finally, a new conjecture about odd 2-factored snarks was posed in [6].

Conjecture 8.10 (Abreu et al. [6]). If G is a cyclically 5-edge-connected odd 2-factored snark, then G is either the Petersen graph or the Flower snark $J(t)$ for odd $t \geq 5$.

Remark 8.11.

(i) A minimal counterexample to Conjecture 8.10 must be a cyclically 5-edge-connected snark of order at least 36. Moreover, as highlighted in [13], order 34 is a turning point for several properties of snarks.

(ii) It is very likely that, if such counterexample exists, it will arise from the superposition operation by M. Kochol [33] applied to one of the known odd 2-factored snarks.

(iii) J. Goedgebeur [26] checked that none of the snarks (in particular those with girth 6 and of order 38) that G. Brinkmann and he generated in [14] is an odd 2-factored snark.

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