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Cozonal labelings of plane graphs

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Abstract. A cozonal labeling of a plane graph G is a labeling $\ell^*\colon F(G) \to \{1,2\} \subset \mathbb{Z}_3$ such that for X_v (the set of regions with v on their boundary), $\sum_{R\in X_v} \ell^*(R) = 0$. This particular labeling is a dual of zonal labelings, which have an important connection to the Four-Color Theorem. In this paper, we determine relationships between zonal and cozonal labelings, establish some results on cozonal labelings, and characterize cozonal graphs of maximum degree at most 3.

1 Introduction

A graph that can be embedded in the plane (or on the surface of a sphere) such that no two edges cross is a *planar graph*. A specific embedding of a planar graph is a *plane graph*. Let G be a plane graph with vertex set $V(G)$, edge set $E(G)$, and region set $F(G)$. We allow G to have parallel edges and loops. In 2014, Cooroo Egan introduced a new vertex labeling for plane graphs called a zonal labeling (see [4]). A *zonal labeling* of a plane graph G is a labeling $\ell\colon V(G) \to \{1,2\} \subset \mathbb{Z}_3$ such that for every region $R \in F(G)$ with boundary B_R, $\sum_{v\in V(B_R)} \ell(v) = 0$ in $\mathbb{Z}_3$. We call the value $\sum_{v\in V(B_R)} \ell(v)$ the *induced label of* R, and denote this $\ell(R)$. We can then equivalently state that a labeling $\ell\colon V(G) \to \{1,2\}$ is zonal if and only if the induced labeling $\ell\colon F(G) \to \mathbb{Z}_3$ assigns the label 0 to each region. If G has a zonal labeling, we say that G is zonal. A planar graph is said to be zonal if at least one planar embedding is zonal. The interested reader can find additional information on zonal labelings in [1–5].

One motivating reason to study zonal labelings is their connection to the famous Four-Color Theorem. The Four-Color Theorem states that the regions of any plane graph can be colored with four or fewer colors such that no two regions sharing a boundary line have the same color. This is known

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as a *proper region four-coloring*. In studying the Four-Color Theorem, we can restrict our attention to a much smaller family of plane graphs. A connected graph is *k-connected* if removing any set of at most $k-1$ vertices does not disconnect the graph. For the purposes of connectivity, deleting a vertex incident with a loop (given that there is at least one other vertex or edge) or both vertices incident with a pair of parallel edges is said to disconnect the graph. A graph is *k-regular* if each vertex has degree k. We can now define a *cubic map* as a 2-connected cubic (that is, 3-regular) plane graph. It is known that to prove that each plane graph admits a proper region four-coloring, it suffices to prove that each cubic map admits a proper region four-coloring. Turning now to zonality, the following key fact is proven in [4, 5].

Theorem 1.1. *A cubic map M has a zonal labeling if and only if the regions of M admit a proper region four-coloring.*

Therefore by the Four-Color Theorem, we can extend this result further.

Theorem 1.2. *If M is a cubic map, then M has a zonal labeling.*

Moreover, if it could be proven that every cubic map M has a zonal labeling independently of the Four-Color Theorem, this would constitute a new proof of the Four-Color Theorem. As this may be quite difficult, it is also of interest to obtain a more robust understanding of zonal labelings through studying zonality in other graphs.

Here, we will study a related labeling known as a *cozonal labeling*. Cozonal labelings have recently been introduced in [2] where an alternate proof of Theorem 1.1 was presented. We will prove an important connection between zonal and cozonal labelings, establish some results on cozonal labelings, and characterize cozonal graphs of maximum degree at most 3.

2 Preliminaries

All terminology that is not explicitly defined here is standard and can be found in texts such as [6]. A *cozonal labeling* of a plane graph G is a labeling $\ell^*\colon F(G) \to \{1,2\} \subset \mathbb{Z}_3$ such that for X_v the set of regions with v on their boundary, $\sum_{R\in X_v} \ell^*(R) = 0$. We call the value $\sum_{R\in X_v} \ell^*(R)$ the *induced label of* v, and denote this $\ell^*(v)$. We can equivalently state that a labeling $\ell^*\colon F(G) \to \{1,2\}$ is cozonal if and only if the induced labeling $\ell^*\colon V(G) \to \mathbb{Z}_3$ assigns the label 0 to each vertex. A plane graph is said to be cozonal if it admits a cozonal labeling. Similarly, a planar graph is said to be cozonal if at least one planar embedding is cozonal. Examples of cozonal labelings of plane graphs will appear throughout this paper.

We have defined a cozonal labeling in such a way that, given a zonal labeling ℓ of G, there is a natural cozonal labeling ℓ^* of the dual graph G^* and vice versa. The following definition of duality is adapted from [6].

Definition 2.1. Given a plane graph (or multigraph) G, the *dual* G^* of G is a graph such that there are bijections $V(G) \to F(G^*)$, $E(G) \to E(G^*)$, and $F(G) \to V(G^*)$ satisfying the following conditions:

- The region $R \in F(G)$ contains the corresponding vertex $v_R^* \in V(G^*)$ in its interior.
- Each edge $e \in E(G)$ intersects the corresponding edge $e^* \in E(G^*)$ exactly once, and this is the only place where e intersects a vertex or edge of G^*.
- Each edge $e^* \in E(G^*)$ intersects the corresponding edge $e \in E(G)$ exactly once, and this is the only place where e^* intersects a vertex or edge of G.
- The vertex $v \in V(G)$ is contained within the corresponding region $R_v^* \in F(G^*)$.

An example of a graph and its dual is given in Figure 2.1. There are several consequences of this definition, one of which being that a plane multigraph G has a dual if an only if it is connected. In this case, $(G^*)^* = G$. In addition, the following proposition will be useful.

Proposition 2.2. *A vertex $v_1 \in V(G)$ is on the boundary of a region $R_2 \in F(G)$ if and only if the corresponding region $R_1^* \in F(G^*)$ has the corresponding vertex $v_2^* \in V(G^*)$ on its boundary.*

Proof. First, let $v_1 \in V(G)$ be on the boundary of a region $R_2 \in F(G)$. Note that G does not have any edges interior to R_2. Now, consider G^*. Each edge e^* passing through R_2 must have the corresponding vertex $v_2^* \in V(G^*)$ as one of its endpoints (or both, in the case of a loop). Therefore, the region of R_2 is divided into several subregions by segments of edges in G^*, but each edge segment contains v_2^*, and thus v_2^* is on the boundary of each such subregion. Notably, one of these regions also contains v_1 on its boundary, and thus there is a path from v_1 to v_2^* that does not internally cross either G or G^*. Therefore, since v_1 is interior to the region $R_1^* \in F(G)$, this path must be interior to R_1^* as well, and v_2^* is on the boundary of R_1^*. The converse claim follows the same argument. □

With Proposition 2.2, we can justify the aforementioned connection between zonal and cozonal labelings.

Theorem 2.3. *Let G be a connected plane graph (or multigraph). Then G is zonal if and only if G^* is cozonal.*

Proof. Let $v_R^* \in V(G^*)$ denote the corresponding vertex to $R \in F(G)$, and let $R_v^* \in F(G^*)$ denote the corresponding region to $v \in V(G)$. Now let G have a zonal labeling ℓ and let X_R be the set of vertices on the boundary of $R \in F(G)$. Then $\sum_{v \in X_R} \ell(v) = 0$ for all $R \in F(G)$. Now, form the labeling $\ell^*\colon V(G^*) \to \{1, 2\}$ given by $\ell^*(R_v^*) = \ell(v)$. Then we define the set $X_{v_R^*}$ to be the set of regions having v_R^* on their boundary. By Proposition 2.2, the regions having v_R^* on their boundary in G^* correspond directly with the vertices on the boundary of R in G, and $X_{v_R^*}$ correspond exactly to the vertices in X_R. Thus,

$$\ell^*(v_R^*) = \sum_{R^* \in X_{v_R^*}} \ell^*(R^*) = \sum_{v \in X_R} \ell(v) = 0,$$

and ℓ^* is a cozonal labeling. The reverse direction is analagous. □

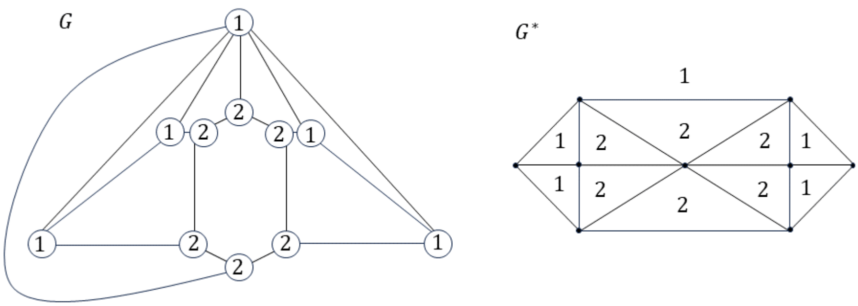

Figure 2.1: A zonal labeling of a plane graph G and the cozonal labeling of its dual.

Cozonal labelings are the dual of zonal labelings. In a sense, the differences between the two are entirely cosmetic, and any claim in zonality has a dual claim in cozonality (and vice versa). However, there are several reasons why we may be interested in studying cozonal labelings. First, while there is a bijection between $E(G)$ and $E(G^*)$, the arrangements of edges are very different in each. In addition, vertices are discrete points, while regions are open sets, allowing for unique explorations in these different settings. Furthermore, a graph that is relatively simple may have a more complicated dual (and vice versa). Lastly, there is the simple reason that labeling regions can sometimes make for a more clear presentation than labeling vertices.

3 Zonality and cozonality of trees and cycles

To illustrate the similarities and differences between these two labelings, we will explore zonal and cozonal labeling of trees and cycles. In [4], it was proven that any nontrivial tree is zonal. The situation for cozonality is quite different; to begin we make a more general observation. We use $\delta(G)$ and $\Delta(G)$ to refer to the minimum and maximum degree of a vertex in G, respectively.

Proposition 3.1. *Let G be a connected plane graph with $\delta(G) = 1$. Then, G is not cozonal.*

Proof. A vertex v of degree 1 can only be on the boundary of one region. The label of a single region cannot be 0, so the sum of labels of regions having v on their boundary is nonzero. □

This has two direct consequences.

Corollary 3.2. *If G is cozonal, then $\delta(G) \geq 2$.*

Corollary 3.3. *No tree T is cozonal.*

As with trees, it can be shown (see [4]) that all cycles are zonal. However, unlike with trees, the same is true with cycles and cozonality.

Proposition 3.4. *Every cycle C is cozonal.*

Proof. Let R_1 be the interior of the cycle C and R_2 be the exterior. Form the labeling $\ell^*(R_i) = i$. Since each vertex is on the boundary of R_1 and R_2, the induced label of each vertex is 0, and ℓ^* is a cozonal labeling. □

4 Cut vertices and bridges in cozonal graphs

A *cut vertex* is a vertex whose removal disconnects a component of a graph. When studying cozonality, 2-connected graphs tend to have nicer properties than connected graphs with cut vertices. The following property indicates a possible reason why. Let $\deg^*(v)$ denote the number of regions having v on their boundary.

Proposition 4.1. *Let G be a nonempty connected plane graph. Then for $v \in V(G)$,*

1. $\deg^*(v) \le \deg(v)$
2. *v is a cut vertex if and only if* $\deg^*(v) < \deg(v)$.

Proof. First, let $e_0, e_1, \ldots, e_{k-1}$ be the k edges incident with v in clockwise order with subscripts in $\mathbb{Z}_k$. Then, for each pair of consecutive clockwise edges e_i, e_{i+1}, the angle in the clockwise direction formed by those edges belongs to some region R_i having v on its boundary. Moreover, these are all the regions having v on their boundary. Since there are k such pairs of consecutive edges, there are at most k regions with v, and $\deg^*(v) \le \deg(v)$. This proves the first claim.

For the second claim, first assume that v is not incident with a loop. If G is a plane graph, then $|V(G)|-|E(G)|+|F(G)| = 2$ if and only if G is connected. Now, let $v \in V(G)$. In $G - v$, there is one fewer vertex, $\deg(v)$ fewer edges (which does not hold if v is incident with a loop), and $\deg^*(v) - 1$ fewer regions (as all regions incident with v are combined into a single region, and all other regions are unaffected). Let $|V(G)| = n, |E(G)| = m, |F(G)| = r$. Given that G is connected,

$$
\begin{aligned}
&v \text{ is not a cut vertex}\\
&\quad\Leftrightarrow G - v \text{ has one component}\\
&\quad\Leftrightarrow (n-1) - (m - \deg(v)) + (r - \deg^*(v) + 1) = 2 = n - m + r\\
&\quad\Leftrightarrow \deg(v) - \deg^*(v) = 0\\
&\quad\Leftrightarrow \deg(v) = \deg^*(v).
\end{aligned}
$$

Therefore, v is not a cut vertex if and only if $\deg^*(v) = \deg(v)$, which is equivalent to the statement that v is a cut vertex if and only if $\deg^*(v) < \deg(v)$.

Now consider the case where v is incident with a loop e. If v is the only vertex and e is the only edge, then $\deg(v) = \deg^*(v) = 2$, and v is not a cut vertex. Otherwise, v is a cut vertex. Therefore, we must show that $\deg^*(v) < \deg(v)$. Let $e_0, e_1, \ldots, e_{k-1}$ be the edges incident with v in clockwise order counted with multiplicity and having subscripts in $\mathbb{Z}_k$, letting $e_i = e_j = e$ for some $i \neq j$. Let R_x be the region between edges e_x and e_{x+1} in the clockwise direction. Then e is on the boundaries of R_{i-1}, R_i, R_{j-1}, and R_j. Since $k \geq 3$, it follows that $|\{i-1, i, j-1, j\}| \geq 3$, but e is only on the boundary of two regions. Therefore, at least one region has been counted more than once, and $\deg^*(v) < \deg(v)$. □

Therefore, we see that cut vertices can have high degree yet be on the boundary of very few regions. Clearly, if a graph G has a vertex on the boundary of only one region, then G cannot be cozonal, as the sum of one nonzero label cannot be zero. With this observation and our previous result, we have the following propositions.

Proposition 4.2. *Let G be a connected plane graph. If G has a vertex v with $\deg^*(v) = 1$, then G is not cozonal.*

Corollary 4.3. *Let G be a connected plane graph. If G has a cut vertex of degree 2, then G is not cozonal.*

A *bridge* is an edge whose removal disconnects a component of a graph. If a connected graph (other than K_2) has a bridge, then at least one vertex incident with that bridge is a cut vertex. However, we shall see that bridges can mostly be ignored without loss of generality. To help us, we will first define the *complementary labeling* to a cozonal labeling $\ell^*\colon F(G) \to \{1, 2\}$ as the labeling $\bar{\ell}^*$ given by $\bar{\ell}^*(R) = 2\ell^*(R) = 3 - \ell^*(R)$. The following is a dualized version of an observation in [4]:

Proposition 4.4. *Let $\ell^*\colon F(G) \to \{1, 2\}$ be a cozonal labeling. Then, $\bar{\ell}^*$ is also a cozonal labeling.*

We will now prove a useful fact about bridges.

Theorem 4.5. *Let G be a connected plane graph with a bridge e. Then G is cozonal if and only if each component of $G - e$ is cozonal.*

Proof. First, suppose G is cozonal with cozonal labeling ℓ^* and, without loss of generality, let e be on the boundary of the exterior region R_E. Let the components of $G - e$ be given by G_1, G_2 and let the exterior region of $G - e$

be given by R'_E. Note that deleting the bridge e does not affect the regions incident with any vertex, except that every vertex initially on the boundary of R_E is now on the boundary of R'_E. Form the labeling $\ell_i^*\colon F(G_i) \to \{1,2\}$ given by $\ell_i^*(R) = \ell(R)$ if R is interior to G_i and $\ell_i^*(R) = \ell^*(R_E)$ if R is the exterior region of G_i. Then $\ell_i^*(v) = \ell^*(v) = 0$ for all $v \in V(G_i), i \in \{1,2\}$, and each component of $G - e$ is cozonal.

Next, let each component G_1, G_2 of $G - e$ have cozonal labelings ℓ_1^*, ℓ_2^*. Again, we assume the deleted edge e is on the boundary of the exterior region of G. Assume, without loss of generality, that the exterior regions R_{E_1}, R_{E_2} are labeled the same for ℓ_1^*, ℓ_2^*, as we may take the complement of one of the labelings if this is not the case. Form the labeling $\ell^*\colon F(G) \to \{1,2\}$ given by $\ell^*(R) = \ell_i^*(R)$ if R is interior to G_i and $\ell^*(R) = \ell_1^*(R_{E_1}) = \ell_2^*(R_{E_2})$ if R is the exterior region of G. Note that for each vertex $v \in G_i$, $\ell^*(v) = \ell_i^*(v) = 0$. Therefore, G is also cozonal. □

Corollary 4.6. *Let G be a plane graph with set B of bridges. Then G is cozonal if and only if each component of $G - B$ is cozonal.*

Proof. First, let G be connected. We show inductively that G is cozonal if and only if after deleting any k bridges of G each component in the resulting graph is cozonal. The case $k = 1$ is handled by Theorem 4.5. Suppose that this is true for all $0 \le k' \le k-1$. Now we successively delete bridges $e_1, \ldots, e_k$, forming graphs $G_1, \ldots, G_k$ where $G_i = G_{i-1} - e_i$. If G is cozonal, then each component of G_{k-1} is cozonal, including the component H_k containing e_k. By Theorem 4.5, each component of $H_k - e_k$ is cozonal; these components, together with the components of G_{k-1} excluding H_k, form all component of G_k, and all components are cozonal. If G is not cozonal, then some component of G_{k-1} is not cozonal. If some component other than the component H_k containing e_k is not cozonal, then this is a component of G_k that is not cozonal. Otherwise H_k is not cozonal, and by Theorem 4.5 a component of $H_k - e_k$ is not cozonal. This is a component of G_k that is not cozonal, completing the proof.

If G is disconnected, append bridges B' to form a connected graph G'. Since the addition or removal of bridges does not affect the regions incident with a vertex, G is cozonal if and only if G' is cozonal. We therefore see by the connected case that G' is cozonal if and only if each component of $G' - (B' \cup B) = G - B$ is cozonal. Thus, G is cozonal if and only if G' is cozonal, which is true if and only if each component of $G - B$ is cozonal. □

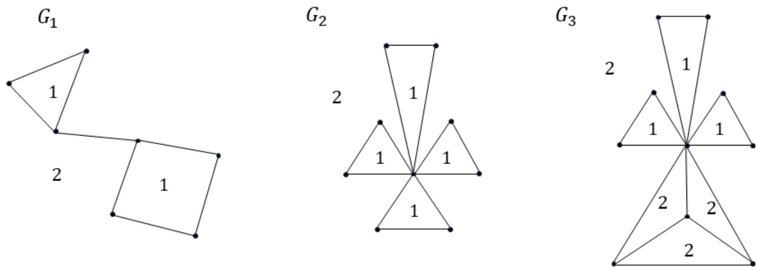

Figure 4.1: Several plane graphs with connectivity 1 and their cozonal labelings.

Corollary 4.7. *Let G be a disconnected plane graph. Then G is cozonal if and only if each component of G is cozonal.*

We end this section with several examples of cozonal graphs with connectivity 1 in Figure 4.1, the first of which contains a bridge.

5 Cozonality in special classes of graphs

We would like to use some results from zonality to state new claims in the context of cozonality. The graphs we will consider in this section are all 2-connected. It is known [6] that G is 2-connected if and only if G^* is 2-connected. We will use this fact and other properties of dual graphs to show that several classes of plane graphs are cozonal.

First, we know from Theorem 1.2 that all cubic maps are zonal. The dual G^* of a cubic map G is a 2-connected map where each region is bounded by a triangle, also known as a *plane triangulation*. Similarly, if G is a plane triangulation, then its dual G^* must be a cubic map. Therefore, we translate Theorem 1.2 to its dual equivalent, which was proven directly in [2] using special features of cozonality.

Theorem 5.1. *If G is a plane triangulation, then G is cozonal.*

An example of a cozonal labeling of a triangulation is given in the first diagram of Figure 5.1. Next, consider the wheel $W_n = C_n \vee K_1$ on $n+1$ vertices. The wheel is not just 2-connected, but also 3-connected. In [3], it was proven that for an integer $n \geq 3$, the wheel W_n is zonal if and only if $n \equiv 0 \pmod 3$. One interesting property of the wheel is that it is self-dual; that is, $W_n \simeq (W_n)^*$. Therefore, we can conclude the following.

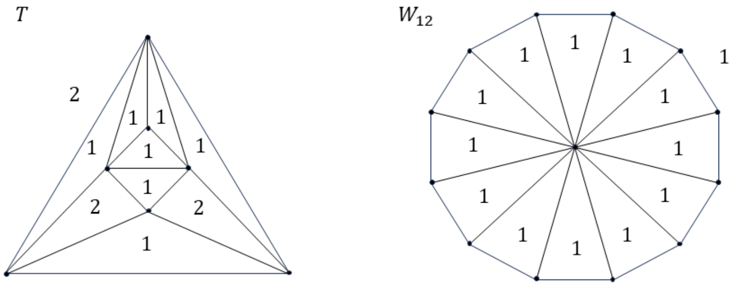

Figure 5.1: Cozonal labelings of a plane triangulation T and the wheel W_{12}.

Theorem 5.2. *For an integer $n \geq 3$, the wheel W_n is cozonal if and only if $n \equiv 0 \pmod 3$.*

An example of a cozonal labeling of the wheel W_{12} is given in the second diagram of Figure 5.1. Examples of zonal labelings are found in [3]. There, in particular, it is shown that if G is a 2-connected bipartite plane graph, then G is zonal. In a bipartite plane graph G, each region is bounded by an even cycle; therefore in the dual G^*, each vertex has even degree. Therefore, G^* is a 2-connected Eulerian graph. Similarly, if G is a 2-connected Eulerian graph, then G^* is a 2-connected graph where each region is bounded by an even cycle. It follows that all cycles of G^* are even, and G^* is a 2-connected bipartite graph. Therefore, G is a 2-connected bipartite plane graph if and only if G^* is a 2-connected Eulerian graph. We can thus conclude the following.

Theorem 5.3. *If G is a 2-connected Eulerian plane graph, then G is cozonal.*

Two examples of cozonal labelings of 2-connected Eulerian plane graphs are given in Figure 5.2. One will notice that the cozonal labelings in this case also happen to be proper 2-colorings of the regions. This is an example of where using the zonal perspective allows us to prove a less than obvious claim in cozonality. The proof that 2-connected bipartite plane graphs are zonal uses the special fact that all boundaries in a 2-connected bipartite graph are even cycles to obtain a zonal labeling. The dual of this fact is that the regions of a 2-connected Eulerian graph can be 2-colored. It is unclear if there is an elegant proof of this latter fact that does not simply reduce to dualizing the proof of the former. Therefore, we see in this case that using the zonal perspective allows us to more easily prove a claim in cozonality.

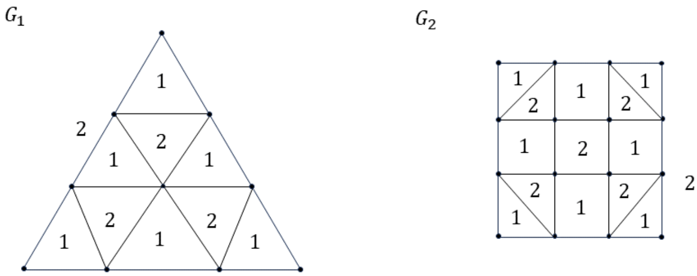

Figure 5.2: 2-connected Eulerian plane graphs G_1, G_2 and their cozonal labelings.

6 Cozonality in connected graphs with $\Delta(G) \leq 3$

Next we turn to characterizing cozonality in graphs with low maximum degree. First, if G is a connected graph with $\Delta(G) = 2$, then G is either a path or a cycle. By Corollary 3.3 and Proposition 3.4, we have the following.

Theorem 6.1. *Let G be a connected plane graph with $\Delta(G) = 2$. Then G is cozonal if and only if G is a cycle C_n for $n \geq 1$.*

By C_2 we refer to the multigraph with two parallel edges between two vertices u, v. By C_1, we refer to a vertex u with a single loop. Both of these are cozonal by the arguments in Proposition 3.4.

Next, we turn to graphs with $\Delta(G) = 3$. We have already seen that all cubic maps are zonal. The following slightly stronger theorem is proven in [4].

Theorem 6.2. *A connected cubic plane graph G is zonal if and only if G is bridgeless.*

The proof of this theorem depends on the Four-Color Theorem, and an independent proof of this would indeed imply the Four-Color Theorem. However, a nearly identical theorem is true for cozonal labelings for entirely different reasons. To begin, we will prove a generally useful fact.

Lemma 6.3. *Let G be a connected plane graph. If G has two adjacent vertices u, v such that $\deg^*(u) \leq 2$, $\deg^*(v) = 3$, and uv is not a bridge, then G is not cozonal.*

Proof. Assume that G is cozonal with cozonal labeling ℓ^*. If $\deg^*(u) = 1$, then G is not cozonal by Proposition 3.1. Therefore, assume $\deg^*(u) = 2$. Since uv is not a bridge, uv is incident with two distinct regions R_1, R_2. These are the only regions incident with u, and since $\ell(u) = 0$, we must have $\ell^*(R_1) + \ell^*(R_2) = 0$. However, v is incident with R_1, R_2, and a third region R_3. It follows that $\ell^*(v) = \ell^*(R_3) \neq 0$, which contradicts the cozonality of G. Therefore, if there are two adjacent vertices u, v such that $\deg^*(u) = 2$, $\deg^*(v) = 3$, and uv is not a bridge, then G is not cozonal. □

Next, we characterize all cozonal connected plane graphs with $\Delta(G) = 3$.

Theorem 6.4. *Let G be a connected plane graph with $\Delta(G) = 3$ with $B \subset E(G)$ the set of all bridges in G (where B is potentially empty). Then G is cozonal if and only if $G - B$ is k-regular for $k \in \{2, 3\}$.*

Proof. By Corollary 4.6, it suffices to prove that $G - B$ is k-regular for $k \in \{2, 3\}$ if and only if each component of $G - B$ is cozonal. First, assume that $G - B$ is either 2- or 3-regular. If $G - B$ is 2-regular, then each component is a cycle, which is cozonal. If $G - B$ is 3-regular, then each component of $G - B$ is 3-regular and bridgeless. Let D be one such component. Then D also has no cut vertices, and $\deg^*(v) = \deg(v) = 3$ for all $v \in V(D)$. Thus, assigning identical labels to all regions in D is a cozonal labeling, and each component of $G - B$ is cozonal.

Next, assume that each component of $G - B$ is cozonal. Note that each component of $G - B$ is bridgeless, and since $\Delta(G) \leq 3$, each component also has no cut vertices. Therefore by Lemma 6.3, no component contains adjacent vertices of degree 2 and 3. Clearly no component can have a vertex of degree 0 or 1, as then the component would not be cozonal. Therefore, each component is either 2- or 3-regular. If there is at least one 3-regular component D, then each vertex already has degree 3, and no bridges of G can be incident with a vertex in $V(D)$. Thus by the connectivity of G, we must have $G = D$, and G is bridgeless and 3-regular. If instead there is no 3-regular component, then each component must be 2-regular. We have thus shown that if each component of $G - B$ is cozonal, then $G - B$ is either 2- or 3-regular, completing the proof. □

It should be noted that the graphs known to be cozonal by Theorem 6.4 include some cubic plane multigraphs with bridges. An example is given in Figure 6.1. We note that in Figure 6.1, every vertex is incident with a bridge. This must always be the case, since if G is cubic with bridges and is cozonal, then $G - B$ must be 2-regular. An end block is only incident with one bridge, and therefore it must have only one vertex. A cycle on one vertex is a loop, and therefore a cubic plane graph with a bridge can only be cozonal if it also has loops. We summarize with the following corollary.

Corollary 6.5. *A loopless connected cubic plane graph G is cozonal if and only if G is bridgeless.*

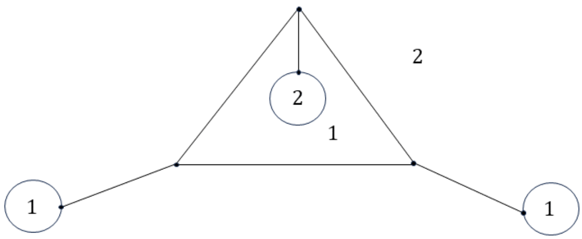

Figure 6.1: A cubic plane multigraph with bridges, which is cozonal.

We can then characterize cozonal graphs of maximum degree up to 3 as follows.

Corollary 6.6. *Let G be a connected graph with $\Delta(G) \leq 3$. Then G is cozonal if and only if one of the following is true:*

- *G is a cycle C.*
- *The graph formed by deleting every bridge of G is 2-regular.*
- *G is a cubic map.*

In the 2-connected case, this can be summarized in the following way.

Corollary 6.7. *Let G be a 2-connected plane graph with $\Delta(G) \leq 3$. Then G is cozonal if and only if G is regular.*

In Figure 6.2 we provide an example of each type of cozonal graph with $\Delta(G) \leq 3$.

7 Concluding remarks

Here we have outlined some basic examples of cozonal graphs, many derived from examples of zonal graphs. In addition, we have completely character-

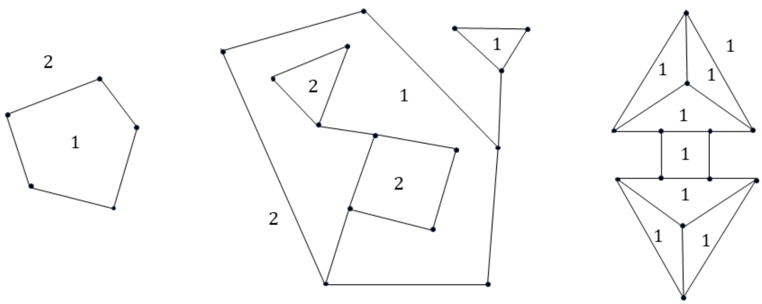

Figure 6.2: Three types of cozonal plane graphs with $\Delta(G) \leq 3$.

ized the cozonal graphs G with $\Delta(G) \leq 3$. We will note that most of our criteria for cozonality were features of the underlying graph, not the embedding; furthermore, the proofs did not refer to any specific embedding (except in the study of wheels and cycles, which are each uniquely embedded in the plane). In the study of zonality, a planar graph G is said to be *absolutely zonal* if every planar embedding of G is zonal (see [3]). We define *absolutely cozonal* analogously. Considering the results in Sections 5 and 6, we obtain the following corollaries.

Corollary 7.1. *If G is a 2-connected Eulerian planar graph, then G is absolutely cozonal.*

Corollary 7.2. *Let G be a connected planar graph with $\Delta(G) \leq 3$. Then G is absolutely cozonal if and only if at least one embedding of G is cozonal.*

Since it is possible for two embeddings of a plane graph to have duals whose underlying graphs are not isomorphic, we see that the study of absolute cozonality is different from that of absolute zonality. Therefore, further research in this direction may be of interest.

Next let us briefly consider the case when $\Delta(G) = 4$. One can immediately note that, unlike when $\Delta(G) \leq 3$, it is possible for a graph with $\Delta(G) = 4$ to have a cut vertex and no bridges. Therefore, we turn to the 2-connected case for the sake of simplicity. First, we see that if G has no vertices of degree 3, then G is 2-connected and Eulerian, and therefore cozonal. On the other hand, by Lemma 6.3 a 2-connected cozonal plane graph cannot have adjacent vertices of degrees 2 and 3. It becomes clear that vertices of degree 3 play a major role in determining cozonality of plane graphs with

$\Delta(G) = 4$, at least in the 2-connected case. This case has been characterized and will appear in a subsequent paper.

Note that the k-regular 2-connected plane graphs for $k \in \{2, 3, 4\}$ are all cozonal. Furthermore, the icosahedron is 2-connected and 5-regular. Since the icosahedron is a plane triangulation, it is also cozonal. One might expect that all 5-regular 2-connected plane graphs are cozonal. Unfortunately, this is not the case, and we present a counterexample in Figure 7.1. Observe that in a cozonal labeling of a 2-connected 5-regular plane graph, each vertex is incident with four regions having one label and a fifth region having the other label. A careful examination of cases shows that such a labeling is not possible for the graph in Figure 7.1.

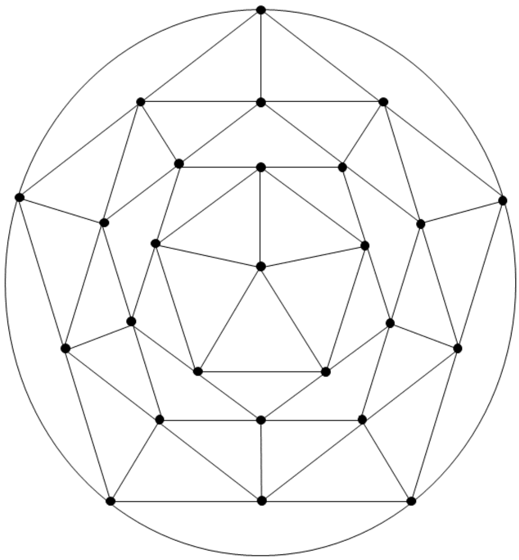

Figure 7.1: A 5-regular 2-connected plane graph that is not cozonal.

We believe that a continued study of cozonality in graphs of low maximum degree, as well as a study of other families of graphs that are or are not cozonal, could be of potential interest moving forward.

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