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# On generalized Turán problems with bounded matching number

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**Abstract.** Given a graph $H$ and a family of graphs $\mathcal{F}$, the generalized Turán number $\mathrm{ex}(n, H, \mathcal{F})$ is the maximum number of copies of $H$ in an $n$-vertex graphs that do not contain any member of $\mathcal{F}$ as a subgraph. Recently there has been interest in studying the case $\mathcal{F} = \{F, M_{s+1}\}$ for arbitrary $F$ and $H = K_r$. We extend these investigations to the case $H$ is arbitrary as well.

## 1 Introduction

Given a positive integer $n$ and a graph $F$, the largest number of edges in an $n$-vertex graph that does not contain $F$ as a subgraph is denoted by $\mathrm{ex}(n, F)$ and is called the *Turán number* of $F$. If every graph from a family $\mathcal{F}$ of graphs is forbidden, then we use the notation $\mathrm{ex}(n, \mathcal{F})$.

The seminal result of this area is due to Turán [17], who showed that $\mathrm{ex}(n, K_{r+1}) = \big|E\big(T_r(n)\big)\big|$, where $T_r(n)$ denotes the Turán graph, i.e., the complete $r$-partite graph with each part of order $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Another early result is due to Erdős and Gallai [6], who showed that for the matching $M_{s+1}$ consisting of $s+1$ independent edges we have

$$\mathrm{ex}(n, M_{s+1}) = \max\big\{\tbinom{2s+1}{2}, \tbinom{s}{2} + s(n-s)\big\}.$$

The above problems were combined by Alon and Frankl [1], who showed that for $n \geq 2s+1$ we have

$$\mathrm{ex}\big(n, \{K_{r+1}, M_{s+1}\}\big) = \max\big\{\tbinom{2s+1}{2}, \big|E\big(T_r(s)\big)\big| + s(n-s)\big\}.$$

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Gerbner [8] considered $\mathrm{ex}(n, \{F, M_{s+1}\})$ in general and determined its value apart from a constant additive term. Here and everywhere in this paper, $n$ goes to infinity while other parameters are considered constant.

Let $\mathcal{H}(F)$ denote the family of graphs obtained by deleting an independent set from $F$ and let $\mathcal{H}^*(F)$ denote the family of graphs obtained by deleting an independent set of order $\alpha(F)$ from $F$ (recall that $\alpha(F)$ is the order of a largest independent set in $F$). If $F$ is bipartite, then let $p = p(F)$ denote the smallest possible order of a color class in a proper two-coloring of $F$.

**Theorem 1.1** (Gerbner [8])**.**

(i) *If $\chi(F) > 2$, $s \geq 1$, and $n$ is large enough, then*

$$\begin{aligned}\mathrm{ex}\big(n, \{F, M_{s+1}\}\big) &= \mathrm{ex}\big(s, \mathcal{H}(F)\big) + s(n-s)\\ &= s(n-s) + O(1).\end{aligned}$$

(ii) *Let $\chi(F) = 2$, $s \geq 1$, and $n$ large enough. If $p(F) > s$, then*

$$\begin{aligned}\mathrm{ex}\big(n, \{F, M_{s+1}\}\big) &= \mathrm{ex}(n, M_{s+1}) = \big|E\big(K_s + \overline{K}_{n-s}\big)\big|\\ &= s(n-s) + \tbinom{s}{2}.\end{aligned}$$

*If $p(F) \leq s$, then*

$$\mathrm{ex}\big(n, \{F, M_{s+1}\}\big) = \big(p(F)-1\big)n + O(1).$$

Given a positive integer $n$ and two graphs $H$ and $F$, the largest number of copies of $H$ in $n$-vertex $F$-free graphs is denoted by $\mathrm{ex}(n, H, F)$. Here a copy of $H$ means a not necessarily induced subgraph that is isomorphic to $H$. If every graph from a family $\mathcal{F}$ is forbidden, we use the notation $\mathrm{ex}(n, H, \mathcal{F})$. After several sporadic results (see, e.g., [14, 21]), the systematic study of these functions was initiated by Alon and Shikhelman [2] (see [13] for a survey).

Let $\mathcal{N}(H, G)$ denote the number of copies of $H$ in $G$. For a family $\mathcal{H}$ of graphs, let $\mathcal{N}(\mathcal{H}, G) = \sum_{H \in \mathcal{H}} \mathcal{N}(H, G)$ and let $\mathrm{ex}(n, \mathcal{H}, \mathcal{F})$ denote the maximum of $\mathcal{N}(\mathcal{H}, G)$, where $G$ is an $n$-vertex $\mathcal{F}$-free graph. We say that an $n$-vertex $\mathcal{F}$-free graph $G$ is an *extremal graph* for $\mathrm{ex}(n, \mathcal{H}, \mathcal{F})$ if $\mathrm{ex}(n, \mathcal{H}, \mathcal{F}) = \mathcal{N}(\mathcal{H}, G)$. Zykov [21] showed that

$$\mathrm{ex}(n, K_k, K_{r+1}) = \mathcal{N}\big(K_k, T_r(n)\big).$$

Wang [18] determined $\mathrm{ex}(n, K_k, M_{s+1})$.

Ma and Hou [15] combined the above problems and studied

$$\mathrm{ex}(n, K_k, \{F, M_{s+1}\}).$$

In the first arXiv version of their paper, they showed the following.

**Theorem 1.2** (Ma and Hou [15])**.**

(i) *If $\chi(F) > 2$ and $s \geq 1$, then*

$$\mathrm{ex}\big(n, K_k, \{F, M_{s+1}\}\big) = \mathrm{ex}\big(s, K_{k-1}, \mathcal{H}(F)\big)n + O(1).$$

(ii) *Let $\chi(F) = 2$, $s \geq 1$ and $n$ be large enough. If $p(F) > s$, then*

$$\mathrm{ex}\big(n, K_k, \{F, M_{s+1}\}\big) = \mathrm{ex}(n, K_k, M_{s+1}) = \mathcal{N}\big(K_k,\, K_s + \overline{K}_{n-s}\big).$$

*If $p(F) \leq s$, then*

$$\mathrm{ex}\big(n, K_k\{F, M_{s+1}\}\big) = \binom{p(F)-1}{k-1}n + O(1).$$

Improvements were obtained in [19,20]. One can say that the above theorem determines $\mathrm{ex}\big(n, K_k, \{F, M_{s+1}\}\big)$ apart from an additive error term $O(1)$. Note, however, that the main term depends on $\mathrm{ex}\big(s, K_{r-1}, \mathcal{H}(F)\big)$, which may be unknown (unlike in Theorem 1.1). Nevertheless, we use the phrase "determine" if the bound only depends on the solution of a generalized Turán problem on $O(1)$ vertices.

In this paper we study $\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big)$ for arbitrary $F$ and $H$. Before turning to that problem, let us discuss what we know about $\mathrm{ex}(n, H, M_{s+1})$ in general.

The order of magnitude of $\mathrm{ex}(n, H, M_{s+1})$ was determined in [9] for every graph $H$. Given a graph $G$ and a set $U \subset V(G)$, the *partial $(m, U)$-blowup of $G$* is obtained the following way. We replace each vertex $u \in U$ with $m$ vertices $u_1, \ldots, u_m$ and replace each vertex $u \notin U$ with one vertex $u_1$. Then we replace each edge $uv$ by the edges $u_i v_j$ for each $i, j$; thus, we replace each edge by 1, $m$, or $m^2$ edges. Let us consider a largest set $U \subset V(H)$ such that the partial $(m, U)$-blowup of $H$ does not contain $M_{s+1}$ for any $m$ and let $b(H) = b(H, s)$ denote the order of $U$. A theorem in [9] shows that $\mathrm{ex}(n, H, M_{s+1}) = \Theta\big(n^{b(H)}\big)$.

It is easy to see that in our case $b(H) \leq \alpha(H)$. Indeed, if we blow up both endpoints of an edge, then the resulting graph contains $M_m$.

We state and prove our results in Section 2. Our main results are the following. We determine the order of magnitude of $\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big)$ for every $H$, $F$, and $s$. For most graphs $F$ (including each non-bipartite graph), we determine $\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big)$ apart from an additive error term $O\big(n^{\alpha(H)-1}\big)$ for every $H$ and $s$. We answer a question of Ma and Hou from the first arXiv version of [15] by showing that in the case $\chi(F) > 2$, the $O(1)$ error term in Theorem 1.2 may be necessary even if

$$\mathrm{ex}\big(s, K_{k-1}, \mathcal{H}(F)\big) > 0.$$

Ma and Hou [15] determined $\mathrm{ex}\big(n, K_k, \{K_{r+1}, M_{s+1}\}\big)$ for each value of $n$, $k$, $r$, and $s$. The extremal graph is either $T_r(2s+1)$ or obtained from $T_{r-1}(s)$ by adding $n-s$ vertices, each joined to each of the $s$ old vertices. We extend their result to counting any complete multipartite graph with at most $r$ parts. The extremal graphs are the same, except the Turán graph is replaced by a not necessarily balanced complete $r$-partite graph in both cases.

## 2 Results and proofs

We use the following theorem of Berge and Tutte [3].

**Theorem 2.1** (Berge-Tutte)**.** *Let $s \geq 1$. A graph $G$ is $M_{s+1}$-free if and only if there is a set $B \subset V(G)$ such that removing $B$ cuts $G$ to connected components $G_1, \dots, G_m$ with each $A_i = V(G_i)$ of odd order such that $|B| + \sum_{i=1}^{m} \frac{|A_i|-1}{2} \leq s$.*

We note that this theorem is usually stated in the equivalent form that the size of the largest matching in a graph $G$ is the minimum of the quantity $\big(|U| + |V(G)| - \ell\big)/2$, where $U \subset V(G)$ and $\ell$ is the number of components of odd order of the graph $G - U$ that we obtain from $G$ by deleting $U$. For the sake of completeness, we show that these statements are equivalent. Observe that if there is a component of even order in $G - U$, then we can add one of the vertices of this component to $U$. This way $|U|$ increases by one and $\ell$ increases by at least one. Therefore, we can assume that there are only components of odd order in $G - U$, and then the two formulas are equal.

We use the following simple facts about the partition in Theorem 2.1.

**Observation 2.2.** Let $s \geq 1$, $H$ be an arbitrary graph, and $G$ be an $n$-vertex $M_{s+1}$-free graph. Let $B$ be a subset of $V(G)$ guaranteed by the Berge-Tutte theorem. Then,

(i) any vertex $v$ outside $B$ is contained in $O\big(n^{\alpha(H)-1}\big)$ copies of $H$;

(ii) there are $O\big(n^{\alpha(H)-1}\big)$ copies of $H$ that contain at least one edge outside $B$.

*Proof.* To show (i), observe that the vertices of $G$ form an independent set with $v$ except for the vertices of $B$ and the vertices of the component $G_i$ that contains $v$. These are at most $3s = O(1)$ vertices, and the rest of the vertices must form an independent set with $v$ in any copy of $H$. Therefore, we can count the copies of $H$ containing $v$ by picking a maximal independent set of $H$ $O(1)$ ways, picking a vertex of it and identifying it with $v$ $O(1)$ ways, identifying the other at most $\alpha - 1$ vertices with vertices of $G$, $O(n^{\alpha(H)-1})$ ways, and then identifying the rest of the vertices of $H$ with vertices of $G$, each at most $3s$ ways.

To show (ii), observe that there are $O(1)$ edges that do not contain any vertex from $B$. Indeed, those are inside some $G_i$ of order more than 1, and the total number of vertices in such components $G_i$ is at most $3s$. There are $O(1)$ ways to pick such an edge $uv$, and there are $O\big(n^{\alpha(H)-1}\big)$ copies of $H$ that contain $v$ by (i). Thus, there are $O\big(n^{\alpha(H)-1}\big)$ copies of $H$ that contain the edge $uv$. □

We use the following simple statement.

**Proposition 2.3.** *If $s \geq 1$ and $|V(H)| \leq s + \alpha(H)$, then*

$$\mathrm{ex}(n, H, M_{s+1}) = \mathcal{N}(H,\, K_s + \overline{K}_{n-s})$$

*for sufficiently large $n$. Otherwise,*

$$\mathrm{ex}(n, H, M_{s+1}) = O\big(n^{\alpha(H)-1}\big).$$

*Proof.* Assume first that $|V(H)| \leq s + \alpha(H)$, let $G$ be an extremal graph for $\mathrm{ex}(n, H, M_{s+1})$, and apply the Berge-Tutte theorem to obtain $B$. If $|B| = s$, then $G$ is a subgraph of $K_s + \overline{K}_{n-s}$. If $|B| < s$, then we compare $\mathcal{N}(H, G)$ to $\mathcal{N}(H,\, K_s + \overline{K}_{n-s})$. We can obtain a copy of $K_s + \overline{K}_{n-s}$ from $G$ by adding $s - |B|$ vertices from components $G_i$ of order 1 to $B$, adding all the edges inside, and removing all the edges from components $G_j$ of

order greater than 1. Then the copies of $H$ that are in $G$ and not in $K_s + \overline{K}_{n-s}$ each contain vertices in some $A_i$ with $|A_i| > 1$. There are $O(1)$ such vertices and each of them is contained in $O\big(n^{\alpha(H)-1}\big)$ copies of $H$ by Observation 2.2. Therefore, there are $O\big(n^{\alpha(H)-1}\big)$ copies of $H$ in $G$ that are not in $K_s + \overline{K}_{n-s}$.

Let $H_0$ denote a graph we obtain from $H$ by deleting $\alpha(H)$ independent vertices. Then $H_0$ has at most $s$ vertices by our assumption on $|V(H)|$, and clearly there are fewer copies of $H_0$ inside $B$ than inside $K_s$.

Let us consider now the copies of $H$ in $G$ that intersect $B$ in a copy of $H_0$ and the copies of $H$ in $K_s + \overline{K}_{n-s}$ that intersect $K_s$ in a copy of $H_0$. We can extend such a copy of $H_0$ in $G$ to $H$ using the vertices outside $B$ at most $\binom{n-s}{\alpha H}$ ways, and similarly, we can extend such a copy of $H_0$ in $K_s + \overline{K}_{n-s}$ to $H$ using the vertices outside $K_s$ exactly $\binom{n-s}{\alpha H}$ ways. Therefore, there are more such copies of $H$ in $K_s + \overline{K}_{n-s}$ by at least $\binom{n-s}{\alpha H}$. In particular, there are at least $\binom{n-s}{\alpha H}$ copies of $H$ in $K_s + \overline{K}_{n-s}$ that are not in $G$. Since this is larger than the number of copies of $H$ in $G$ that are not in $K_s + \overline{K}_{n-s}$, we obtain that $K_s + \overline{K}_{n-s}$ contains more copies of $H$ than $G$, a contradiction.

Assume now that $|V(H)| > s + \alpha(H)$ and recall that by a theorem in [9], $\mathrm{ex}(n, H, M_{s+1}) = \Theta\big(n^{|U|}\big)$ for some $U$ such that the partial $(m, U)$-blowup of $H$ does not contain $M_{s+1}$ for any $m$, in particular for $m > s$. Then $U$ is an independent set. If $|U| = \alpha(H)$, then each of the at least $s + 1$ vertices $v_1, \dots, v_{s+1}$ of $H$ outside $U$ have a neighbor in $U$. We pick an arbitrary such neighbor $u$ for $v_i$. In the partial blow-up, $u$ is replaced by $m$ vertices. If $u$ is picked for multiple vertices $v_i$, we pick a distinct one of the replacing vertices for each $v_i$ and denote it by $u_i$. Then the edges $u_i v_i$ form an $M_{s+1}$, a contradiction. Therefore, $|U| \leq \alpha - 1$, and hence $\mathrm{ex}(n, H, M_{s+1}) = O\big(n^{\alpha-1}\big)$ □

Recall that $b(H) = b(H, s)$ denotes the largest order of a set $U$ such that $M_{s+1}$ is not a subgraph of any partial $(m, U)$-blowup of $H$. Let $b'(H) = b'(H, F, s)$ denote the largest order of a set $U \subset V(H)$ such that neither $F$ nor $M_{s+1}$ is a subgraph of any partial $(m, U)$-blowup of $H$.

**Theorem 2.4.** *For every $H$, $F$, and $s \geq 1$ we have*

$$\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big) = \Theta\big(n^{b'(H)}\big).$$

*Proof.* The lower bound is given by the partial blowup in the definition of $b'(H)$.

Let $G$ be an $n$-vertex $\{F, M_{s+1}\}$-free graph. We apply the Berge-Tutte theorem to $G$, to obtain a set $B$. Let $A$ be the union of components $A_i$ of order more than 1. Observe that $|A \cup B| \leq 2s$. Assume that $G$ contains at least $cn^{b'(H)}$ copies of $H$, where $c$ is sufficiently large. It is more convenient for us to talk about embeddings of $H$: each copy of $H$ is given by an injective function $f\colon V(H) \to V(G)$ such that for each edge $uv$ of $H$, $f(u)f(v)$ is an edge of $G$.

We choose an edge-covering set $S$ of vertices in $H$, i.e., a set such that each edge of $H$ is incident to at least one of its vertices. Observe that the vertex set of each copy of $H$ in $G$ intersects $A \cup B$ in an edge-covering set. There are at most $2^{|V(H)|}$ choices to pick $S$; we pick one that appears as the intersection of $A \cup B$ with at least $cn^{b'(H)}/2^{|V(H)|}$ copies of $H$.

From now on we deal with copies (embeddings) of $H$ where the vertices of $S$ are in $A \cup B$; now we fix those vertices. There are $\binom{|A\cup B|}{|S|}|S|! \leq 2^{2s}|V(H)|!$ ways to embed the vertices of $S$ into $A \cup B$; we fix the embedding that can be extended to at least $cn^{b'(H)}/2^{|V(H)|+2s}|V(H)|!$ embeddings of $H$. It is left to embed the remaining vertices, which form an independent set in $H$. For each vertex $v$ of $H$ outside $S$, the neighborhood of $v$ is a subset $S(v)$ of $S$. Thus we have to pick from $G$ a neighbor of the corresponding vertices in $A \cup B$. Let $S(v) = \{u_1, \dots, u_k\}$. Then we are interested in the order of the common neighborhood of $f(u_1), \dots, f(u_k)$ in $G$, which we denote by $x(v)$.

Recall that the partial $(m, U')$-blowup of $H$ contains $F$ or $M_{s+1}$ for each set $U'$ of vertices of $H$ with order $b'(H) + 1$, for some $m$. For each $U'$, we pick the smallest possible $m$. As there are $O(1)$ ways to choose $U'$, we can pick the largest of those numbers $m$; let us denote it by $m'$.

Let $q$ denote the number of vertices $v$ in $V(H) \setminus S$ such that $x(v) \geq \big(b'(H)+1\big)m' + |V(H)|$. If $q \leq b'(H)$, then we continue by picking the images of the vertices outside $S$. There are at most $n$ ways to pick $q$ of the vertices and less than $\big(b'(H)+1\big)m'+|V(H)|$ ways to pick the other vertices. Therefore, there are at most $\big(\big(b'(H)+1\big)m' + |V(H)|\big)^{|V(H)|}n^{b'(H)}$ embeddings after fixing the vertices of $S$, contradicting the lower bound $cn^{b'(H)}/2^{|V(H)|+2s}|V(H)|!$ established earlier, if $c$ is sufficiently large.

We have obtained that $q \geq b'(H) + 1$. First we pick an embedding of $H$ that extends the earlier embedding of $S$. We fix the embedding of each of the vertices except for $b'(H) + 1$ vertices $v_1, \dots, v_{b'(H)+1}$ with $x(v_i) \geq \big(b'(H) + 1\big)m' + |V(H)|$. We go through the vertices $v_i$ one by one and greedily pick $m'$ vertices outside $A \cup B$ that they can be embedded to,

such that each vertex is picked only once. This is doable, since each time we have to avoid the at most $|V(H)|$ vertices already fixed and the at most $b'(H)m'$ vertices picked earlier. This means we have a way to fix all but $b'(H)+1$ vertices, and there are at least $m'$ common neighbors of $S(v_i)$ for the unfixed vertices $v_i$. In other words, we have in $G$ a partial $(m', U)$-blowup with $U = \{v_1, \dots, v_{b'(H)+1}\}$, which contains $F$ or $M_{s+1}$, a contradiction. □

We are able to generalize Theorem 1.2 for most instances by determining $\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big)$ apart from an additive term $O\big(n^{\alpha(H)-1}\big)$.

**Proposition 2.5.**

(i) *Let $\chi(F) > 2$ and $s \geq 1$. There is an $s$-vertex $\mathcal{H}(F)$-free graph $G_0$ such that $\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big) = \mathcal{N}(H, G_0 + \overline{K}_{n-s}) + O\big(n^{\alpha(H)-1}\big)$.*

(ii) *Let $\chi(F) = 2$ and $s \geq 1$. If $p(F) > s$, then*

$$\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big) = \mathcal{N}(H, K_s + \overline{K}_{n-s}) + O\big(n^{\alpha(H)-1}\big).$$

We remark that $\mathcal{N}(H, G_0 + \overline{K}_{n-s})$ can be determined by determining $\mathcal{N}(H', G_0)$ for every $H' \in \mathcal{H}(H)$ (and for the main term, it is enough to deal with $H' \in \mathcal{H}^*(H)$). Therefore, we want to count multiple graphs in $s$-vertex $\mathcal{H}(F)$-free graphs $G_0$. Such problems were studied in [10]. We also have weights here: each copy of $H'$ is counted $\binom{n-s}{|V(H)|-|V(H')|}$ times, but we are only interested in the main term because of the error term $O\big(n^{\alpha(H)-1}\big)$. Still, this problem seems to be complicated in general.

We can deal with counting multiple graphs at the same time if the same graph is extremal for each of them, in particular if the same graph is extremal for each of the subgraphs of $H$. Two examples of this are counting linear forests if $F$ has a color-critical edge [7] and counting any graph $H$ if $F$ has a color-critical edge and chromatic number at least $300|V(H)|^9 + 1$ [11, 16].

*Proof.* $G_0 + \overline{K}_{n-s}$ is $F$-free by the definition of $\mathcal{H}(F)$ and $M_{s+1}$-free because it contains a set of $s$ vertices such that each edge is incident to at least one of them. This proves the lower bound in (i).

To prove the upper bound in (i), let $G$ be an extremal graph for

$$\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big)$$

and apply the Berge-Tutte theorem to obtain $B$. Let $G_0$ be the graph we obtain from $G[B]$ by adding $s - |B|$ isolated vertices.

Consider two types of copies of $H$ in $G$, those that contain an edge outside $B$ and those that do not contain such an edge. By Observation 2.2, there are $O\big(n^{\alpha(H)-1}\big)$ copies of $H$ that contain an edge outside $B$. The other copies of $H$ intersect $V(G) \setminus B$ in an independent set. There are at most

$$\mathcal{N}\big(H,\, G[B] + \overline{K}_{n-|B|}\big) \le \mathcal{N}\big(H,\, G_0 + \overline{K}_{n-|B|}\big)$$

such copies. We have that

$$\mathcal{N}\big(H,\, G_0 + \overline{K}_{n-|B|}\big) \le \mathcal{N}(H,\, K_s + \overline{K}_{n-s}) + O\big(n^{\alpha(H)-1}\big)$$

by applying Observation 2.2 to $G_0 + \overline{K}_{n-s}$. This completes the proof of (i).

Assume now that $F$ is bipartite and $p > s$. Then $F$ and $M_{s+1}$ are both $K_s + \overline{K}_{n-s}$-free, and Proposition 2.3 implies the upper bound. □

In the remaining case, we have that $p(F) \le s$. It is easy to see using Observation 2.2 that we can obtain a subgraph of $K_s + \overline{K}_{n-s}$ from the extremal graph by deleting some edges such that we delete only $O\big(n^{\alpha(H)-1}\big)$ copies of $H$. In other words, for our purposes it is enough to deal with subgraphs of $K_s + \overline{K}_{n-s}$, and then we can forget about forbidding $M_{s+1}$. However, it is not clear what subgraph of the part $K_s$ is allowed. For example, if $F = K_{s,t}$ with $s \le t$, then we can have the full $K_s$ there. In this case we still have to solve the following unbalanced bipartite version of the generalized Turán problem, with an additive error term $O\big(n^{\alpha(H)-1}\big)$: what is the largest number of copies $H$ in $F$-free subgraphs of $K_{s,n-s}$? Moreover, we also have to solve this for counting subgraphs of $H$.

A natural goal is to get rid of the $O\big(n^{\alpha(H)-1}\big)$ term in Proposition 2.5. Clearly, if $\mathrm{ex}\big(s, \mathcal{H}(H), \mathcal{H}(F)\big) = 0$, then $\mathcal{N}(H,\, G_0 + \overline{K}_{n-s}) = 0$; thus, the $O\big(n^{\alpha(H)-1}\big)$ term can be omitted if and only if $\mathrm{ex}(n, H, F) = 0$, i.e., $H$ contains $F$. In the case $H = K_r$, $\chi(F) > 2$ and $\mathrm{ex}\big(s, K_{r-1}, \mathcal{H}(F)\big) > 0$, Ma and Hou in the first arXiv version of [15] asked whether one can get rid of the $O(1)$ term. We answer this negatively. Given $s$, families $\mathcal{H}_1$ and $\mathcal{H}_2$ of graphs, and an $s$-vertex $\mathcal{H}(F)$-free graph $G_0$, we say a vertex $v$ of $G_0$ is *useless* if it is not contained in any copy of any graph from $\mathcal{H}(H)$. Even with one forbidden graph, there are examples of extremal graphs with useless vertices. The simplest example is when $\mathrm{ex}(s, \mathcal{H}_1, \mathcal{H}_2) = O(1)$ and $s$ is sufficiently large. For example we have $\mathrm{ex}(s, K_3, M_2) \le 1$; thus, there is a useless vertex if $s \ge 4$. See [12] for more on generalized Turán problems

with constant values. Another example is $\mathrm{ex}(s, K_r, S_k)$, where the extremal graph consists of $\lfloor s/k \rfloor$ copies of $K_k$ and a clique on the remaining vertices [4]. If $s - k\lfloor s/k \rfloor < r$, then those additional vertices are useless.

Observe that there is a useless vertex in an extremal graph if and only if $\mathrm{ex}(s, \mathcal{H}_1, \mathcal{H}_2) = \mathrm{ex}(s-1, \mathcal{H}_1, \mathcal{H}_2)$.

**Proposition 2.6.**

(i) *Let $\chi(F) > 2$, $s \geq 2$, and $n$ be sufficiently large. If*

$$\mathrm{ex}\big(s, \mathcal{H}^*(H), \mathcal{H}(F)\big) > \mathrm{ex}\big(s-1, \mathcal{H}^*(H), \mathcal{H}(F)\big),$$

*then there is an $s$-vertex $\mathcal{H}(F)$-free graph $G_0$ such that*

$$\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big) = \mathcal{N}(H, G_0 + \overline{K}_{n-s}).$$

(ii) *Let $\chi(F) > 2$, $s \geq 2$, and $n$ be sufficiently large. If every extremal graph for $\mathrm{ex}\big(s, \mathcal{H}(H), \mathcal{H}(F)\big)$ contains a useless vertex, then for every $s$-vertex $\mathcal{H}(F)$-free graph $G_0$ we have that*

$$\mathrm{ex}\big(n, H, \{F, M_{s+1}\}\big) > \mathcal{N}(H, G_0 + \overline{K}_{n-s}).$$

*Proof.* To prove (i), we follow the proof of Proposition 2.5(i). The additional error term $O\big(n^{\alpha(H)-1}\big)$ came from two parts of the proofs: from the copies of $H$ containing an edge outside $B$ and from the $s - |B|$ additional vertices. If $|B| = s$, then there are no additional vertices and no edge outside

If $|B| < s$, then the proof of Proposition 2.5(i) gives the upper bound $\mathcal{N}(H, G) \leq \mathrm{ex}\big(s-1, \mathcal{H}^*(H), \mathcal{H}(F)\big)\binom{n-|B|}{\alpha(H)} + O\big(n^{\alpha(H)-1}\big)$. This is less than $\mathrm{ex}\big(s, \mathcal{H}^*(H), \mathcal{H}(F)\big)\binom{n-s}{\alpha(H)} = \mathcal{N}(H, G_0 + \overline{K}_{n-s})$ for $n$ sufficiently large, a contradiction.

To prove (ii), consider an $\mathcal{H}(F)$-free subgraph $G_0'$ of $G_0$ on $s-1$ vertices that contains the same copies of graphs in $\mathcal{H}(H)$ as $G_0$. Then $G_0' + \overline{K}_{n-s+1}$ contains more copies of $H$ than $G_0 + \overline{K}_{n-s}$, completing the proof. □

It is a natural question when we have useless vertices, but we are unable to answer this question in general. Given a family $\mathcal{H}$ of graphs, let $\chi(\mathcal{H})$ denote the smallest chromatic number among graphs in $\mathcal{H}$. We can show that there are no useless vertices if $\chi(H) < \chi(F)$ and $n$ is large enough. We also state the result in the form that we use here.

**Proposition 2.7.**

(i) *If $\chi(H) < \chi(F)$ and $n$ is sufficiently large, then*

$$\mathrm{ex}(n, H, F) > \mathrm{ex}(n-1,\, H,\, F).$$

(ii) *If $\chi\big(\mathcal{H}^*(H)\big) < \chi\big(\mathcal{H}(F)\big)$ and $n$ is sufficiently large, then*

$$\mathrm{ex}\big(n, \mathcal{H}(H), \mathcal{H}(F)\big) = \mathrm{ex}\big(n-1,\, \mathcal{H}(H),\, \mathcal{H}(F)\big).$$

*Proof.* We begin by proving (i). Let $G$ be an extremal graph for $\mathrm{ex}(n, H, F)$ and assume indirectly that $v$ is a useless vertex. Let $\chi(H) = k$. Let $H'$ denote the $|V(F)|$-blowup of $H$. If $G$ does not contain $H'$, then

$$\mathcal{N}(H, G) \le \mathrm{ex}(n, H, H') = o\big(n^{|V(H)|}\big).$$

Indeed, Alon and Shikhelman [2] showed that $\mathrm{ex}(n, H, H') = \Theta\big(n^{|V(H)|}\big)$ if and only if $H'$ is not contained in any blow-up of $H$. We have

$$\mathrm{ex}(n, H, F) \ge \mathcal{N}\big(H, T_k(n)\big) = \Theta\big(n^{|V(H)|}\big);$$

thus, for $n$ sufficiently large we obtain a contradiction.

Consider a copy of $H'$ in $G$, and let $U$ be one of the blown-up classes in $H'$. We delete all the edges incident to $v$ from $G$ and add all the edges from $v$ to the common neighbors of vertices of $U$ to obtain a graph $G'$. After this change, $v$ is connected to some of the other blown-up classes in $H'$, and in particular there are some new copies of $H$. Thus

$$\mathcal{N}(H, G') > \mathcal{N}(H, G) = \mathrm{ex}(n, H, F).$$

However, we claim that $G'$ is $F$-free, which gives a contradiction. Indeed, assume that there is a copy of $F$ in $G$ that is not in $G$. Then this copy must contain $v$, and there is a vertex $u \in U$ that is not contained in this copy. Then we can replace $v$ by $u$ to obtain another copy of $F$, since the neighbors of $v$ are all neighbors of $u$. But this new copy of $F$ is in $G$, a contradiction.

The proof of (ii) goes similarly. Let $\mathcal{H}'$ denote the family of graphs that are $|V(F)|$-blowups of some graphs in $\mathcal{H}^*(H)$. If $G$ is $\mathcal{H}'$-free, then the same argument as above completes the proof. If $G$ contains the $|V(F)|$-blowup $H'_0$ for some $H_0 \in \mathcal{H}^*(H)$, then let $U_0$ be one of the blown-up classes. Similarly to the proof of (i), we delete all the edges incident to $v$ from $G$ and add all the edges from $v$ to the common neighbors of vertices of $U_0$

to obtain a graph $G''$. Clearly, the number of copies of graphs in $\mathcal{H}^*(H)$ increases. If there is a member of $\mathcal{H}(F)$ in $G''$, then it contains $v$, and we can replace $v$ by an unused vertex of $U_0$, a contradiction completing the proof. □

Recall that Ma and Hou [15] showed the following.

**Theorem 2.8** (Ma and Hou [15]). *Let $s \geq 2$, $n \geq 2s+1$, and $r \geq k \geq 3$. Then*

$$\begin{aligned}\mathrm{ex}\big(n, K_k, \{K_{r+1}, M_{s+1}\}\big) \\ = \max\big\{\mathcal{N}\big(K_k, T_{r-1}(s) + \overline{K}_{n-s}\big), \mathcal{N}\big(K_k, T_r(2s+1)\big)\big\}.\end{aligned}$$

We determine $\mathrm{ex}\big(n, H, \{K_{r+1}, M_{s+1}\}\big) = \mathcal{N}(H, K)$ where $H$ is a complete multipartite graph with at most $r$ parts. Note that it is known [14] that $\mathrm{ex}(n, H, K_{r+1}) = \mathcal{N}(H, T)$, where $T$ is a complete $r$-partite graph. However, it is a complicated calculation to find which complete $r$-partite graph contains the most copies of $H$. We have the same problem in our theorem below.

The proof of Theorem 2.8 in [15] uses Zykov symmetrization. Given a $K_{r+1}$-free graph $G$ and two non-adjacent vertices $u$ and $v$, we say that we *symmetrize* $u$ to $v$ if we delete all the edges incident to $u$ and, for each edge $vw$, we add the edges $uw$. In other words, we replace the neighborhood of $u$ by the neighborhood of $v$. It is easy to see that such symmetrization steps do not create $K_{r+1}$, and either symmetrizing $u$ to $v$ or symmetrizing $v$ to $u$ does not decrease the number of copies of $K_k$. We apply such steps as long as we can. One has to show that this process terminates. Afterwards, the resulting graph is a complete multipartite graph. This argument was used by Zykov [21] to show that the Turán graph contains the most copies of $K_k$ among $n$-vertex $K_{r+1}$-free graphs.

Let $H$ be a complete $k$-partite graph and $k \leq r$. It was observed in [14] that the Zykov symmetrization does not decrease the number of copies of $H$ either. They used it to show that a complete $r$-partite graph contains the most copies of $H$ among $n$-vertex $K_{r+1}$-free graphs. Analogously, we can use this to extend the proof of Theorem 2.8 in [15] to our more general version. A part of the proof from [15] extends without further ideas. We state the generalization of that part in the next proposition, and we only give a sketch of the proof. The interested reader may find more details in [15].

**Proposition 2.9.** *Let $s \geq 2$ and $H$ be a complete multipartite graph with at most $r$ parts. Then there is an extremal graph $G$ for $\mathrm{ex}\big(n, H, \{K_{r+1}, M_{s+1}\}\big)$ such that, applying the Berge-Tutte theorem, $G[B]$ is a complete multipartite graph and $|A_2| = \cdots = |A_m| = 1$.*

*Sketch of proof.* First we apply Zykov symmetrization for $u, v$ both inside $B$. This does not create $K_{r+1}$ because of the properties of the symmetrization mentioned earlier. This also does not create $M_{s+1}$, because in the resulting graph we still have the same $B$ that cuts the graph into parts as prescribed in the Berge-Tutte theorem, the same parts as originally. We have to show that the process terminates. We start with symmetrizing vertices with different neighborhoods outside $B$. This way we obtain that vertices inside $B$ have the same neighborhood outside $B$, since the number of distinct neighborhoods outside $B$ decreases. After that we symmetrize inside $B$. Whenever we symmetrize $u$ with neighborhood $A$ to $v$, in the next steps we symmetrize each vertex with neighborhood $A$ to $v$. After all these steps, the number of different neighborhoods decreases; thus, this process terminates.

If there are two components in $G - B$, say $A_1$ and $A_2$, that contain edges, then without loss of generality $v_1 \in A_1$ is in the largest number of copies of $K$. Let $v_2, v_3 \in A_2$ such that $A_2 \setminus \{v_2, v_3\}$ is connected and symmetrize $v_2$ and $v_3$ both to $v_1$. Then we do not create $M_{s+1}$, since the same $B$ cuts the graph into parts as prescribed in the Berge-Tutte theorem. Note that the number of vertices in $A_1$ and $A_2$ changed, but neither the parity of $|A_1|$ and $|A_2|$ nor $\sum_{i=1}^{m} \frac{|A_i|-1}{2}$ changed. □

**Proposition 2.10.** *If $s \geq 1$ and a complete $r$-partite $M_{s+1}$-free graph has more than $2s+1$ vertices, then a part has order at least $n - s$.*

*Proof.* Let us pick edges one by one, such that we always pick an edge between the two largest part, and remove its endvertices. We are going to show that, after picking $s$ edges, there are vertices in at least two parts. As we have at least two vertices left, the only possible problem is that we are left with a part $B_0$ of order at least two. Observe that if two parts have the same order at one point of this algorithm, then their order will never differ by more than one. Therefore, $B_0$ was always the largest part, and we removed a vertex from $B_0$ and a vertex outside $B_0$. There are at least $s+1$ vertices avoiding $B_0$, and in each step we remove one. Thus there is an undeleted vertex outside $B_0$. This way we find an $M_{s+1}$, a contradiction completing the proof. □

Now we are ready to state and prove our generalization of Theorem 2.8.

**Theorem 2.11.** *If $s \geq 2$ and $H$ is a complete multipartite graph with at most $r$ parts, then $\mathrm{ex}\big(n, H, \{K_{r+1}, M_{s+1}\}\big) = \mathcal{N}(H, K)$ where either $K$ is a complete multipartite $n$-vertex graph with at most $r$ parts and a part of order at least $n - s$ or $K$ consists of a complete multipartite graph on at most $2s + 1$ vertices with at most $r$ parts and possibly some isolated vertices.*

Note that isolated vertices may come into picture where $\mathrm{ex}(n, H, M_{s+1}) = O(1)$, this is the case for example if $H = K_{s+1}$ by the result of Wang [18] mentioned earlier. If $r + 1 \geq 2s + 2$, then forbidding $K_r$ does not change anything, and the extremal graph is $K_{2s+1}$ with $n - 2s - 1$ isolated vertices.

*Proof.* First we apply Proposition 2.9, and we show that there is an extremal graph $G$ where $A_1$ is also complete multipartite. In the case $|B| = s$, we have $|A_1| = 1$, and we are done. If $|B| < s$, we apply Zykov symmetrization inside $A_1$. After any symmetrization steps inside $A_1$, there are still at most $|B|$ independent edges incident to vertices in $B$ and at most $\lfloor |A_1| / 2 \rfloor$ independent edges inside $A_1$; thus, we do not create $M_{s+1}$. The process terminates as in Proposition 2.9: we can pick the order of the symmetrization steps such that the number of different neighborhoods decreases after some steps.

Assume first that there is an edge $uv$ in $A_1$. Observe that any copy of $H$ containing this edge is inside $A_1 \cup B$. Indeed, a vertex $w$ outside $A_1 \cup B$ is not adjacent to $u$; thus, $w$ has to be in the part of $u$ in the copy of $H$. Similarly $w$ has to be in the part of $v$, but then $uv$ cannot be an edge in the copy of $H$, a contradiction.

Let $G'$ be the graph we obtain by deleting all the edges inside $A_1$. Then $\mathcal{N}(H, G) = \mathcal{N}(H, G') + O(1)$. Let us delete now all the edges incident to $u$ and $v$, connect $u$ to the neighbors of an arbitrary vertex $w \notin A_1 \cup B$, and then connect $v$ to each neighbor of a vertex $w' \in B$. The set $B \cup \{v\}$ cuts the resulting graph $G''$ to parts $A_1, \ldots, A_m, \{u\}$ of odd order, which satisfy the properties in the Berge-Tutte theorem; thus, $G''$ is $M_{s+1}$-free. If there is a copy of $K_{r+1}$ in $G''$, it contains either $u$ or $v$, since each edge of $G''$ that is not present in $G$ is incident to at least one of these two vertices. If $u$ is contained in this copy, the rest of the vertices are in $B$, and $u$ could be replaced by $w$ to obtain a copy of $K_{r+1}$ in $G$. If $v$ is contained in that

copy, then it similarly can be replaced by $w'$. The same argument works if both $u$ and $v$ are in the copy.

We have obtained that $G''$ is an $n$-vertex $\{F, M_{s+1}\}$-free graph. Assume first that each vertex of $G$ is contained in a copy of $H$. Then there is a vertex of $B$ that is contained in $\Omega(n)$ copies of $H$. We pick that as $w'$; hence, $\mathcal{N}(H, G'') = \mathcal{N}(H, G') + \Omega(n) > \mathcal{N}(H, G)$, a contradiction.

This means that if each vertex of $G$ is contained in a copy of $H$, then there are no edges in $A_1$. If $B$ has less than $r$ parts, then we can add all the edges between $B$ and $V(G) \setminus B$ without creating a forbidden graph. The resulting graph is complete multipartite with a part of order at least $n - s$, as requested. If $B$ has $r$ parts, then for each vertex $v$ outside $B$, there is a part such that $v$ is not adjacent to any vertex of that part. Then we can join $v$ to each vertex of $B$ outside that part. Repeating this for each vertex outside $B$, we obtain a complete $r$-partite graph. Proposition 2.10 shows that either there is a part of order at least $n - s$ or there are at most $2s + 1$ vertices.

Assume now that a vertex $v$ of $G$ is not contained in any copy of $H$. Then we can delete all the edges incident to $v$ to obtain $G_0$. If there is a vertex $u$ outside $A_1 \cup B$ that is contained in a copy of $H$, then we can connect $v$ to the neighbors of $u$. The resulting graph $G_0$ is $K_{r+1}$-free and also $M_{s+1}$-free because $B$ still satisfies the properties of the Berge-Tutte theorem. On the other hand $\mathcal{N}(H, G_0) > \mathcal{N}(H, G)$, a contradiction. We have obtained that no vertex outside $A_1 \cup B$ is in any copy of $H$; thus, we can delete the edges that are not inside $A_1 \cup B$ to obtain $G_1$ such that $\mathcal{N}(H, G_1) = \mathcal{N}(H, G)$.

Now we claim that $G_1$ has chromatic number at most $r$. This follows from, e.g., the strong perfect graph theorem [5]. It states that a graph $G$ has chromatic number equal to the order of the largest clique if and only if $G$ contains neither an induced odd cycle of length at least 5 nor the complement of such a cycle.

Assume that $G_1$ contains such a subgraph $C$. Then $C$ does not contain vertices with the same neighborhood; thus, it contains at most one vertex from each blown-up class. Therefore, $C$ contains a clique from $B$ and from $A_1$. If $C$ is a cycle, then the cliques are of order at most 2, while if $C$ is the complement of a cycle then the cliques are of order at most $\big(|V(C)| - 1\big)/2$. In both cases, there are less than $|V(C)|$ vertices in $C$, a contradiction. We remark that with a little bit more effort one could show that $G$ has chromatic number at most $r$ as well, but it is not needed for us.

Consider a proper coloring of $G_1[A_1 \cup B]$ with the fewest possible colors; add all the edges between vertices in different color classes; and then add back the vertices outside $A_1 \cup B$, to obtain $G_2$. We have not added any edge inside $B$; thus, the resulting graph is still $M_{s+1}$-free and $K_{r+1}$-free. The number of copies of $H$ did not decrease; thus, $G_2$ contains $\mathrm{ex}\big(n, H, \{M_{s+1}, K_{r+1}\}\big)$ copies of $H$. If $|A_1 \cup B| \leq 2s+1$, then $G_2$ satisfies the properties listed in the theorem, and we are done. Otherwise, there is a part of order at least $|A \cup B| - s$; in which case, we can add the isolated vertices to that part without creating a forbidden graph. Then the resulting graph satisfies the properties listed in the theorem. □

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