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Generalized plane trees

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Abstract. In this paper, we generalize plane and noncrossing trees by considering trees drawn in the plane such that the vertices are on the boundary of a circle, edges are line segments that do not cross inside the circle, and a butterfly rooted at any vertex can have upto $d \geq 1$ wings. We coin the name d-dimensional plane trees for these trees and enumerate them by root degree, vertices of a certain type, number of vertices, leaves, endpoints, forests, descents, and outdegree sequence. These results unify equivalent results for plane trees and noncrossing trees.

1 Introduction

Trees drawn in the plane have been studied by various researchers. These trees include plane trees, noncrossing trees, t-ary trees, and their generalizations. A *plane tree* is a rooted tree drawn in the plane such that all the subtrees are ordered [1] and a *noncrossing tree* is a tree drawn in the plane such that vertices are on the circumference of a circle and edges are line segments that do not cross inside the circle [11]. A vertex i in a plane tree is at *level* $\ell \geq 0$ if there are ℓ edges on the path from the root to i. So the root is at level 0. A vertex v is a *parent* (resp. *child*) of vertex u if v is adjacent to u and u is at level ℓ when v is at level $\ell - 1$ (resp. $\ell + 1$). The number of children of a given vertex is its *outdegree*. A vertex of outdegree 0 is a *leaf* or *external vertex* and a non-leaf vertex is an *internal vertex*. The *degree* of a vertex is the number of vertices that are adjacent to it. So, in any plane tree, the outdegree and degree of the root vertex are equal. An arrangement of outdegrees (resp. degrees) of vertices of a plane tree is the *outdegree sequence* (resp. *degree sequence*) of that tree. A plane tree in which each internal vertex has at most t children is a *t-ary tree*. A collection of trees is a *forest*. Plane trees have been enumerated according to number of vertices [1], root degree, leaves [2], level of a vertex [2, 5],

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outdegree sequence [20, 21], degreee sequence [4], and forests [21], among other parameters. Quite a number of structures are enumerated by Catalan numbers, which count plane trees [22].

Plane trees have been generalized by giving labels to the vertices such that certain conditions are satisfied [7, 8, 13, 17] or by considering their tree-like counterparts [10, 16, 18]. Gu, Prodinger, and Wagner [8] introduced and studied *k-plane trees*, which are plane trees whose vertices receive labels in the set $\{1, 2, \ldots, k\}$ such that the sum of labels of adjacent vertices is at most $k + 1$. The aforementioned authors enumerated these trees by number of vertices and label of the root. They also related these trees to $(k + 1)$-ary trees. The set of k-plane trees has since been enumerated by root degree, forests [13], and occurrences of vertices of a certain type [17]. Setting $k = 2$, we get 2-plane trees that were introduced and enumerated by Gu and Prodinger [7]. We note that 1-plane trees are just the ordinary plane trees.

Consider a noncrossing tree. As with plane trees, the *degree* of a vertex is the number of edges incident with it. A vertex of degree 1 is called an *endpoint*. If (i, j) is an edge in a noncrossing tree such that i is closest to the root (vertex 1), then the edge is an *ascent* (resp. a *descent*) if $i < j$ (resp. $i > j$). A noncrossing tree is said to be *increasing* if there are no descent edges in the tree. An edge of the form $(i, i + 1)$ where $i = 1, \ldots, n - 1$ or $(1, n)$ is a *boundary edge* in a noncrossing tree on n vertices. Noncrossing trees have been enumerated by number of vertices [11], leaves, outdegree sequence, forests [6], descents [9], boundary edges, endpoints, and maximum degree [3], among other statistics. Tree-like counterparts of noncrossing trees have been considered [14, 16].

Another version of generalizing noncrossing trees is by assigning labels to the vertices of the trees such that certain conditions are satisfied [19, 24]. In 2010, Pang and Lv [19] introduced k-noncrossing trees. A *k-noncrossing tree* is a noncrossing tree whose vertices receive labels in the set $\{1, \ldots, k\}$ such that if (i, j) is an ascent on the path from the root then $i + j \leq k + 1$. Pang and Lv also obtained the number of these trees with a given label of the root and number of vertices. Okoth [15] enumerated k-noncrossing trees by root degree and forests. These trees have also been enumerated by Okoth and Wagner [17] according to occurrences of vertices of a certain type. Nyariaro and Okoth [12] constructed a bijection between the set of $2k$-plane trees and the set of k-noncrossing trees. Therefore, there is a relationship between these combinatorial structures, which are all drawn in the plane. It is worth noting that 2-noncrossing trees were introduced and

enumerated in an earlier paper of Yan and Liu [24] and that 1-noncrossing trees are just noncrossing trees.

One common technique in the enumeration of noncrossing trees is the decomposition of the trees into butterflies, which was introduced by Flajolet and Noy [6]. A *butterfly* is an ordered pair of noncrossing trees that share a root. So, at each vertex we can talk of left and right wings of a butterfly for which each wing is a noncrossing tree. The left wing (resp. right wing) of a butterfly rooted at a given vertex, say k, is a noncrossing tree with labels less than (resp. greater than) k. We now generalize this concept in the following definition.

Definition 1.1. A *d-dimensional plane tree* is a noncrossing tree in which each butterfly rooted at a non-root vertex has one left wing labeled 1 (possibly empty) and $d-1$ right wings labeled in order from left to right as $2, 3, \ldots, d$ (some of which could be possibly empty) and the butterfly rooted at the root is a right wing labeled d.

This name, d-dimensional plane tree, is coined since the tree can be considered as a plane tree in which the d wings are in the d directions of a d-dimensional space. For each butterfly rooted at a given vertex, all the children of that root are labeled using the label of the wing in which they belong. So, each non-root vertex in a d-dimensional plane tree can be considered to belong to exactly one wing. The i^{th} wing of a butterfly rooted at a vertex v in a d-dimensional plane tree comprises of v and all the subtrees rooted at the children of v which are labeled i. In Figure 1.1, we get a 3-dimensional plane tree on 12 vertices. There are three wings of the butterfly rooted at 6: the first, second and third wings have vertex sets $\{5, 6\}$, $\{6, 7, 8, 9\}$ and $\{6, 10\}$ respectively.

Definition 1.2. A *d-dimensional plane forest* is a forest drawn in the plane with vertices on the boundary of a circle so that edges are straight line segments which do not cross inside the circle and each component is a d-dimensional plane tree.

In Figure 1.2, we get a depiction of a 3-dimensional plane forest with three components.

We remark that a 1-dimensional plane tree is a plane tree. Such a tree has no right wings in the butterflies rooted at non-root vertices. Moreover, a 2-dimensional plane tree is a noncrossing tree. This research therefore unifies known results for plane and noncrossing trees by studying a generalization

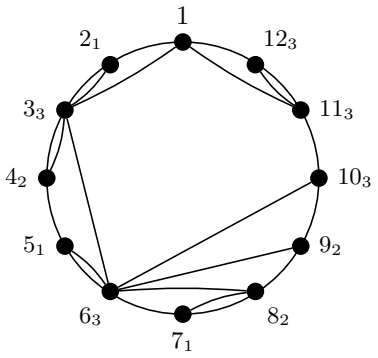

Figure 1.1: A 3-dimensional plane tree on 12 vertices where the label of the wing which a given vertex belongs to is given as subscript to the vertex number.

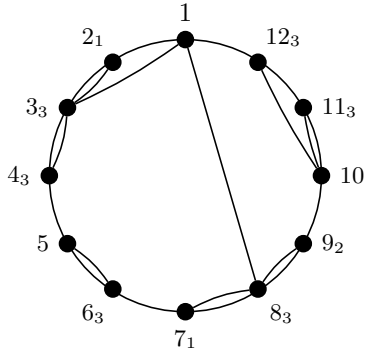

Figure 1.2: A 3-dimensional plane forest on 12 vertices with three components rooted at vertices 1, 5, and 10, respectively.

for which plane and noncrossing trees are special cases. Specifically, we use generating functions to enumerate d-dimensional plane trees according to number of vertices, root degree, and vertices of a given type in Section 2. The enumeration of these trees by number of leaves and endpoints is done in Section 3. In Section 4 and Section 5, we enumerate d-dimensional plane forests and d-dimensional plane trees by number of descents, respectively. The work is extended to obtain counting formulas for these trees with a given outdegree sequence in Section 6. The paper is concluded in Section 7 in which a problem is posed.

2 Root degree and vertices of a given type

We prove our first result.

Theorem 2.1. *There are*

$$\frac{r}{n-1}\binom{n+a_1-2}{a_1}\cdots\binom{n+a_d-2}{a_d} \tag{1}$$

d-dimensional plane trees on n vertices in which the root has degree r, and there are a total of a_i vertices, which are not adjacent to the root, in the i^{th} wing of butterflies rooted at the vertices of the trees, where $i = 1, 2, \ldots, d$.

Proof. Let $T = T(x)$ and $B = B(x)$ be the generating functions for a d-dimensional plane tree and a butterfly in the tree where x marks a vertex. Then

$$T = \frac{x}{1-B} \qquad \text{and} \qquad B = \frac{T^d}{x^{d-1}}. \tag{2}$$

So,

$$B = \frac{T^d}{x^{d-1}} = x\left(\frac{T}{x}\right)^d = \frac{x}{(1-B)^d}.$$

Now, let u_i for $i = 1, 2, \ldots, d$ mark a vertex (not adjacent to the root) in the i^{th} wing of a butterfly in a d-dimensional plane tree. It follows that

$$B = \frac{x}{(1-u_1B)\cdots(1-u_dB)}.$$

Also, $T = x/(1-vB)$ where v marks the root degree. We remain to extract the coefficient of $x^n v^r u_1^{a_1}\cdots u_d^{a_d}$ in T:

$$\begin{aligned}
[x^n v^r u_1^{a_1}\cdots u_d^{a_d}]T &= [x^n v^r u_1^{a_1}\cdots u_d^{a_d}]x(1-vB)^{-1} \\
&= [x^{n-1} v^r u_1^{a_1}\cdots u_d^{a_d}]\sum_{j\geq 0} v^j B^j \\
&= [x^{n-1} u_1^{a_1}\cdots u_d^{a_d}]B^r.
\end{aligned}$$

By the Lagrange inversion formula [21, 23], we obtain

$$\begin{aligned}
[x^n v^r u_1^{a_1} \cdots u_d^{a_d}]T &= \frac{r}{n-1}[b^{n-r-1} u_1^{a_1} \cdots u_d^{a_d}] \prod_{k=1}^{d} (1-u_k b)^{-(n-1)} \\
&= \frac{r}{n-1}[b^{n-r-1} u_1^{a_1} \cdots u_d^{a_d}] \prod_{k=1}^{d} \sum_{s \geq 0} \binom{n+s-2}{s} u_k^s b^s \\
&= \frac{r}{n-1}[b^{n-r-1}] \prod_{k=1}^{d} \binom{n+a_k-2}{a_k} b^{a_1+\cdots+a_d}.
\end{aligned}$$

Since $a_1 + \cdots + a_d = n - r - 1$, then

$$[x^n v^r u_1^{a_1} \cdots u_d^{a_d}]T = \frac{r}{n-1} \prod_{k=1}^{d} \binom{n+a_k-2}{a_k}. \qquad \square$$

Corollary 2.2. *The number of d-dimensional plane trees on n vertices with root degree r is*

$$\frac{dr}{n-r-1} \binom{(d+1)n-r-d-2}{n-r-2}. \tag{3}$$

Proof. We extract the coefficient of $x^n v^r$ in T as follows.

$$\begin{aligned}
[x^n v^r]T &= [x^n v^r] x(1-vB)^{-1} = [x^{n-1}] B^r \\
&= \frac{r}{n-1}[b^{n-r-1}](1-b)^{-d(n-1)} \\
&= \frac{r}{n-1}[b^{n-r-1}] \sum_{i \geq 0} \binom{d(n-1)+i-1}{i} b^i \\
&= \frac{r}{n-1} \binom{(d+1)n-r-d-2}{n-r-1}. \qquad \square
\end{aligned}$$

Setting $d = 1$ in (3), we rediscover the following corollary.

Corollary 2.3 ([2])**.** *There are*

$$\frac{r}{n-r-1} \binom{2n-r-3}{n-r-2}$$

plane trees on n vertices with root degree r.

Also, setting $d = 2$ in (3), we obtain the following.

Corollary 2.4 ([11])**.** *The number of noncrossing trees on n vertices with root degree r is given by*

$$\frac{2r}{n-r-1}\binom{3n-r-4}{n-r-2}.$$

Corollary 2.5. *There are*

$$\frac{a_d}{n(n-1)}\binom{n+a_1-2}{a_1}\cdots\binom{n+a_{d-1}-2}{a_{d-1}}\binom{n+a_d-1}{a_d} \tag{4}$$

d-dimensional plane trees on n vertices such that there are a total of a_i vertices in the i^{th} wing of butterflies rooted at the vertices of the trees, where $i = 1, 2, \ldots, d$.

Proof. Replace a_d by $a_d - r$ in (1) and sum over all values of r. □

In the following corollary, we get a refinement for the number of noncrossing trees.

Corollary 2.6. *The number of noncrossing trees on n vertices with i vertices in the right wings of the butterflies is*

$$\frac{i}{n(n-1)}\binom{2n-i-3}{n-2}\binom{n+i-1}{i}.$$

Proof. If $d = 2$ in (4), then $a_2 = i$ and $a_1 = n - a_2 - 1 = n - i - 1$ so that the number of noncrossing trees on n vertices with i vertices in the right wings of the butterflies is

$$\frac{i}{n(n-1)}\binom{2n-i-3}{n-2}\binom{n+i-1}{i}.$$ □

Theorem 2.7. *The number of d-dimensional plane trees on n vertices is given by*

$$\frac{1}{d(n-1)+1}\binom{(d+1)(n-1)}{n-1}. \tag{5}$$

Proof. We extract the coefficient of x^n in $T = T(x)$. So,

$$\begin{aligned}
[x^n]T &= [x^n]x(1-B)^{-1} = [x^{n-1}]\sum_{j\geq 0} B^j \\
&= \sum_{j\geq 0} \frac{j}{n-1}[b^{n-1-j}](1-b)^{-d(n-1)} \\
&= \sum_{j\geq 0} \frac{j}{n-1}[b^{n-1-j}]\sum_{k\geq 0}\binom{-d(n-1)}{k}(-b)^k \\
&= \sum_{j\geq 0} \frac{j}{n-1}[b^{n-1-j}]\sum_{k\geq 0}\binom{d(n-1)+k-1}{k}b^k \\
&= \sum_{j=0}^{n-1} \frac{j}{n-1}\binom{(d+1)n-d-j-2}{n-j-1} \\
&= \frac{(d+1)n-d-1}{n-1}\sum_{j=0}^{n-1}\binom{(d+1)n-d-j-2}{dn-d-1} \\
&\qquad\qquad - d\sum_{j=0}^{n-1}\binom{(d+1)n-d-j-1}{dn-d}.
\end{aligned}$$

We change the limits of summation and use the hockey stick identity. So,

$$\begin{aligned}
[x^n]T &= \frac{(d+1)n-d-1}{n-1}\sum_{j=dn-d-1}^{(d+1)n-d-2}\binom{j}{dn-d-1} - d\sum_{j=dn-d}^{(d+1)n-d-1}\binom{j}{dn-d} \\
&= \frac{(d+1)n-d-1}{n-1}\binom{(d+1)n-d-1}{dn-d} - d\binom{(d+1)n-d}{dn-d+1} \\
&= \frac{1}{dn-d+1}\binom{(d+1)n-d-1}{dn-d}.
\end{aligned}$$

This completes the proof. □

We remark that we can obtain (5) by summing over all r in (3) or by summing over all a_i's in (4). Now, setting $d = 1$ or $d = 2$ in (5), we find formulas for plane and noncrossing trees, respectively, on n vertices.

3 Leaves and endpoints

Let $T = T(x, u, v)$ and $B = B(x, u, v)$ be the generating functions for d-dimensional trees and butterflies, respectively, where x marks a vertex, u marks a leaf, and v marks root degree. Then,

$$T(x,u,v) = \frac{x}{1-vB} \qquad \text{and} \qquad B(x,u,v) = xu - x + \frac{T^d}{x^{d-1}}.$$

We rewrite B as

$$B(x,u,v) = xu - x + x\left(\frac{T}{x}\right)^d = x\left(u + \frac{1-(1-B)^d}{(1-B)^d}\right),$$

which is in a form we can apply the Lagrange inversion formula. So,

$$\begin{aligned}
[x^n v^r u^\ell]T &= [x^n v^r u^\ell]x(1-vB)^{-1} = [x^{n-1}v^r u^\ell]\sum_{j\geq 0} v^j B^j = [x^{n-1}u^\ell]B^r \\
&= \frac{r}{n-1}[b^{n-r-1}u^\ell]\left(u + \frac{1-(1-b)^d}{(1-b)^d}\right)^{n-1} \\
&= \frac{r}{n-1}[b^{n-r-1}u^\ell]\sum_{i\geq 0}\tbinom{n-1}{i}u^i\left(1-(1-b)^d\right)^{n-i-1}(1-b)^{-d(n-i-1)} \\
&= \frac{r}{n-1}\tbinom{n-1}{\ell}[b^{n-r-1}]\left(1-(1-b)^d\right)^{n-\ell-1}(1-b)^{-d(n-\ell-1)} \\
&= \frac{r}{n-1}\tbinom{n-1}{\ell}[b^{n-r-1}]\sum_{i\geq 0}\tbinom{n-\ell-1}{i}(-1)^i(1-b)^{-d(n-\ell-i-1)} \\
&= \frac{r}{n-1}\tbinom{n-1}{\ell}[b^{n-r-1}]\sum_{i\geq 0}\tbinom{n-\ell-1}{i}(-1)^i\sum_{k\geq 0}\tbinom{-d(n-\ell-i-1)}{k}(-b)^k \\
&= \frac{r}{n-1}\tbinom{n-1}{\ell}[b^{n-r-1}]\sum_{i\geq 0}\tbinom{n-\ell-1}{i}(-1)^i\sum_{k\geq 0}\tbinom{d(n-\ell-i-1)+k-1}{k}b^k \\
&= \frac{r}{n-1}\tbinom{n-1}{\ell}\sum_{i\geq 0}\tbinom{n-\ell-1}{i}(-1)^i\tbinom{d(n-\ell-i-1)+n-r-2}{n-r-1}.
\end{aligned}$$

We formally advertise this result as a theorem.

Theorem 3.1. *The number of d-dimensional plane trees on n vertices with root degree r and ℓ leaves is given by*

$$\frac{r}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{n-\ell-1}\binom{n-\ell-1}{i}\binom{d(n-\ell-i-1)+n-r-2}{n-r-1}(-1)^i. \quad (6)$$

Summing over all r in (6), we obtain the following corollary.

Corollary 3.2. *There are*

$$\frac{1}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{n-\ell-1}\binom{n-\ell-1}{i}\binom{d(n-\ell-i-1)+n-1}{n-2}(-1)^i \quad (7)$$

d-dimensional trees on n vertices with ℓ leaves.

Corollary 3.3 ([2])**.** *The number of plane trees on n vertices with ℓ leaves is the Narayana number*

$$\frac{1}{n-1}\binom{n-1}{\ell}\binom{n-1}{\ell+1}.$$

Proof. Set $d = 1$ in (7) and sum over all i. □

Setting $d = 2$ in (7), we obtain the number of noncrossing trees on n vertices with ℓ leaves as

$$\frac{1}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{n-\ell-1}\binom{n-\ell-1}{i}\binom{2(n-\ell-i-1)+n-1}{n-2}(-1)^i.$$

This result was first obtained by Flajolet and Noy in [6]. We now enumerate d-dimensional plane trees with a given number of endpoints.

Theorem 3.4. *The number of d-dimensional plane trees on n vertices with ℓ endpoints is given by*

$$T_{n,\ell} - S_{n,\ell} + U_{n,\ell} \tag{8}$$

where

$$T_{n,\ell} = \frac{1}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{n-\ell-1}\binom{n-\ell-1}{i}\binom{d(n-\ell-i-1)+n-1}{n-2}(-1)^i,$$

$$S_{n,\ell} = \frac{1}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{n-\ell-1}\binom{n-\ell-1}{i}\binom{d(n-\ell-i-1)+n-3}{n-2}(-1)^i,$$

$$U_{n,\ell} = \frac{1}{n-1}\binom{n-1}{\ell-1}\sum_{i=0}^{n-\ell}\binom{n-\ell}{i}\binom{d(n-\ell-i)+n-3}{n-2}(-1)^i.$$

Proof. The trees on n vertices with ℓ endpoints are either d-dimensional trees on n vertices with root degree greater than 1 and having ℓ leaves or d-dimensional trees on n vertices with root degree 1 and having $\ell-1$ leaves. The number of the former is $T_{n,\ell} - S_{n,\ell}$ since the total number of trees on n vertices with ℓ leaves is given by (7) and the number of trees on n vertices with root degree 1 and ℓ leaves is obtained by setting $r = 1$ in (6). $U_{n,\ell}$ gives the number of d-dimensional trees on n vertices with root degree 1 and having $\ell - 1$ leaves. It is obtained by replacing ℓ by $\ell - 1$ and r by 1 in (6). This completes the proof. □

4 Forests

In this section, we count d-dimensional plane forests.

Theorem 4.1. *The number of d-dimensional plane forests on n vertices with k components is*

$$\frac{1}{d(n-k)+k}\binom{n}{k-1}\binom{d(n-k)+n-1}{n-k}. \tag{9}$$

Proof. Let $T(x)$ and $B(x)$ be the generating functions for d-dimensional plane trees and butterflies, respectively, where x marks a vertex. Then

$$T(x) = \frac{x}{1-B(x)} \qquad \text{and} \qquad B(x) = \frac{x}{(1-B(x))^d}. \tag{10}$$

Let f be a forest and v the number of vertices in the component c containing vertex 1. To obtain f, there are v additional forests (possibly empty) in the spaces between consecutive vertices of c, one to the left of each vertex of c. So a family of forests is obtained from a family of trees by substituting a vertex in a tree by a pair consisting of a vertex and a forest. Let $F = F(x, w)$ be the bivariate generating function for d-dimensional plane forests where x marks a vertex and w marks a component. Then the construction gives,

$$F = 1 + wT(xF). \tag{11}$$

Let $T = T(xF)$ and $B = B(xF)$. Then, from (10), we have

$$T = \frac{xF}{1-B} \qquad \text{and} \qquad B = \frac{xF}{(1-B)^d}. \tag{12}$$

From (11) and (12), it follows that

$$F = 1 + \frac{wxF}{1-B} = 1 + wB\,(1-B)^{d-1} \tag{13}$$

and

$$B = x\Big(1 + wB\,(1-B)^{d-1}\Big)\,(1-B)^{-d}\,. \tag{14}$$

Now, from (13), we get

$$\begin{aligned}\left[x^n w^k\right] F &= \left[x^n w^{k-1}\right] B\,(1-B)^{d-1} \\ &= \left[x^n w^{k-1}\right] \sum_{i\geq 0} \tbinom{d-1}{i} (-1)^i B^{i+1}.\end{aligned}$$

By the Lagrange inversion formula and (14), we have

$$\begin{aligned}
[x^n w^k]F &= \sum_{i\geq 0}[b^{n-i-1}w^{k-1}]\tfrac{i+1}{n}\tbinom{d-1}{i}(-1)^i(1+wb\,(1-b)^{d-1})^n\,(1-b)^{-dn}\\
&= \sum_{i\geq 0}[b^{n-i-1}w^{k-1}]\tfrac{i+1}{n}\tbinom{d-1}{i}(-1)^i\sum_{j\geq 0}\tbinom{n}{j}w^j b^j\,(1-b)^{j(d-1)}\,(1-b)^{-dn}\\
&= \tfrac{1}{n}\tbinom{n}{k-1}\sum_{i\geq 0}[b^{n-i-k}](i+1)\tbinom{d-1}{i}(-1)^i\,(1-b)^{(k-1)(d-1)-dn}.
\end{aligned}$$

By the binomial theorem, we obtain

$$\begin{aligned}
[x^n w^k]F &= \tfrac{1}{n}\tbinom{n}{k-1}\sum_{i\geq 0}[b^{n-i-k}](i+1)\tbinom{d-1}{i}(-1)^i\sum_{j\geq 0}\tbinom{(k-1)(d-1)-dn}{j}(-b)^j\\
&= \tfrac{1}{n}\tbinom{n}{k-1}\sum_{i\geq 0}[b^{n-i-k}](i+1)\tbinom{d-1}{i}(-1)^i\sum_{j\geq 0}\tbinom{dn-(k-1)(d-1)+j-1}{j}b^j\\
&= \tfrac{1}{n}\tbinom{n}{k-1}\sum_{i\geq 0}(i+1)\tbinom{d-1}{i}(-1)^i\tbinom{dn-(k-1)(d-1)+n-i-k-1}{n-i-k}.
\end{aligned}$$

By using the identity $\binom{-a}{i}(-1)^i=\binom{a+i-1}{i}$, we have

$$\begin{aligned}
[x^n w^k]F &= \tfrac{1}{n}\tbinom{n}{k-1}\sum_{i\geq 0}\left[\tbinom{i-d}{i}-(d-1)\tbinom{i-d}{i-1}\right]\tbinom{(d+1)n-kd+d-i-2}{n-i-k}\\
&= \tfrac{1}{n}\tbinom{n}{k-1}\left[\tbinom{(d+1)n-kd-1}{n-k}-(d-1)\tbinom{(d+1)n-kd-1}{n-k-1}\right]\\
&= \tfrac{1}{n}\tbinom{n}{k-1}\cdot\frac{n}{d(n-k)+k}\tbinom{d(n-k)+n-1}{n-k}\\
&= \frac{1}{d(n-k)+k}\tbinom{n}{k-1}\tbinom{d(n-k)+n-1}{n-k}.
\end{aligned}$$

□

By summing over all k in (9), we obtain the following result.

Corollary 4.2. *The number of d-dimensional plane forests on n vertices is given by*

$$\sum_{k=1}^{n}\frac{1}{d(n-k)+k}\binom{n}{k-1}\binom{d(n-k)+n-1}{n-k}. \tag{15}$$

We rediscover the formula for the number of d-dimensional plane trees on n vertices, already obtained in (5), by setting $k=1$ in (9). Further, the following results are obtained by setting $d=1$ and $d=2$ in (9).

Corollary 4.3. *There are*

$$\frac{1}{n}\binom{n}{k-1}\binom{2n-k-1}{n-k} \quad \text{and} \quad \frac{1}{2n-k}\binom{n}{k-1}\binom{3n-2k-1}{n-k}$$

plane forests and noncrossing forests, respectively, on n vertices with k components.

5 Descents

In the sequel, we obtain a formula for the number of d-dimensional plane trees with a given number of descents.

Theorem 5.1. *There are*

$$\frac{1}{n-1}\binom{n+k-2}{k}\binom{d(n-1)-k}{n-k-2} \tag{16}$$

d-dimensional plane trees on n vertices with k descents.

Proof. Let $T = T(x, z)$ be the generating function for d-dimensional plane trees where x marks an edge and z marks a descent. Moreover, let $\tilde{T} = \tilde{T}(x, z)$ be the analogous generating function for d-dimensional plane trees where the root is not vertex number 1, but rather the vertex with the highest number (so that all edges going out from the root are descents). Then, we have

$$T(x, z) = \frac{1}{1 - xT(x, z)^{d-1}\tilde{T}(x, z)}, \tag{17}$$

since in each butterfly, the root has a lower number than the vertices in $d-1$ wings, but a higher number than the vertices in the d^{th} wing. Likewise,

$$\tilde{T}(x, z) = \frac{1}{1 - xzT(x, z)^{d-1}\tilde{T}(x, z)}, \tag{18}$$

since all root edges are descents in this case. From (17), we have

$$\tilde{T} = \frac{T - 1}{xT^d}.$$

Now, replacing $\tilde{T}$ in (18) by $(T - 1)/xT^d$ and collecting the like terms, we get

$$xT^{d+1} + (z - 1)T^2 + (1 - 2z)T + z = 0. \tag{19}$$

Now, let $T = 1 + S$. Equation (19) becomes

$$S = \frac{x(1+S)^{d+1}}{(1+S) - zS},$$

which is in a form we can apply the Lagrange inversion formula. So,

$$\begin{aligned}
[x^{n-1}z^k]T &= [x^{n-1}z^k]S \\
&= \tfrac{1}{n-1}[s^{n-2}z^k](1+s)^{(d+1)(n-1)}((1+s)-zs)^{-(n-1)} \\
&= \tfrac{1}{n-1}[s^{n-2}z^k]\sum_{j\geq 0}\tbinom{-(n-1)}{j}(-zs)^j(1+s)^{(d+1)(n-1)-(n-1)-j}.
\end{aligned}$$

So, we have

$$\begin{aligned}
[x^{n-1}z^k]T &= \frac{1}{n-1}[s^{n-2}z^k]\sum_{j\geq 0}\binom{n+j-2}{j}z^j s^j(1+s)^{d(n-1)-j} \\
&= \frac{1}{n-1}[s^{n-k-2}]\binom{n+k-2}{k}(1+s)^{d(n-1)-k} \\
&= \frac{1}{n-1}[s^{n-k-2}]\binom{n+k-2}{k}\sum_{j\geq 0}\binom{d(n-1)-k}{j}s^j \\
&= \frac{1}{n-1}\binom{n+k-2}{k}\binom{d(n-1)-k}{n-k-2}.
\end{aligned}$$

□

We can obtain the formula for the total number of d-dimensional plane trees on n vertices by summing over all k in (16). Setting $d = 2$ in (16), we get the following result that was obtained by David Hough [9, Theorem 2.2].

Corollary 5.2. *There are*

$$\frac{1}{n-1}\binom{n+k-2}{n-2}\binom{2n-k-2}{n}$$

noncrossing trees on n vertices with k descents.

Corollary 5.3. *The number of increasing d-dimensional plane trees on n vertices is*

$$\frac{1}{n-1}\binom{dn-d}{n-2}.$$

Proof. Set $k = 0$ in (16). □

6 Outdegree sequences

We prove the following result.

Theorem 6.1. *The number of d-dimensional plane trees on n vertices with outdegree sequence $(d_0, d_1, \ldots, d_r)$, where d_r is the maximum outdegree in the trees is*

$$\sum_{k=1}^{r} \frac{k}{n-1} \binom{n-1}{d_0, \ldots, d_k-1, \ldots, d_k} \binom{k+d-1}{k}^{d_k-1} \prod_{\substack{i=0 \\ i\neq k}}^{r} \binom{i+d-1}{i}^{d_i}, \quad (20)$$

where an empty product is equal to 1.

Proof. Let v be a vertex of degree k in a tree T in which the maximum outdegree of a vertex in T is r. There is thus a sequence of k butterflies attached at v. These butterflies are again distributed among d wings. This is equivalent to choosing k objects from a set with d elements such that repetitions are allowed. The number of such ways is

$$\binom{d+k-1}{k}.$$

So, the generating function $B = B(x)$ for the butterflies is thus

$$B = x \sum_{i\geq 0} \binom{d+i-1}{i} B^i.$$

Now, let w_i mark a vertex of outdegree i either in the trees or in butterflies. It follows that

$$B := B(x, w_0, w_1, \ldots) = x \sum_{i\geq 0} \binom{d+i-1}{i} w_i B^i. \quad (21)$$

The equation for the trees is thus

$$T = x \sum_{i\geq 0} w_i B^i.$$

Now, we use (21) and the Lagrange inversion formula to obtain

$$\begin{aligned}
&[w_0^{d_0} w_1^{d_1} \cdots w_r^{d_r} x^n] x w_k B^k \\
&\quad = [w_0^{d_0} w_1^{d_1} \cdots w_k^{d_k-1} \cdots w_r^{d_r} x^{n-1}] B^k \\
&\quad = \frac{k}{n-1} \binom{n-1}{d_0, \ldots, d_k-1, \ldots, d_k} \binom{d}{1}^{d_1} \\
&\qquad\qquad \times \binom{d+1}{2}^{d_2} \cdots \binom{d+k-1}{k}^{d_k-1} \cdots \binom{d+r-1}{r}^{d_r}.
\end{aligned}$$

We obtain the required formula by summing over all k. □

We obtain the equivalent formula for the number of plane trees or noncrossing trees with a given outdegree sequence by letting $d = 1$ or $d = 2$, respectively, in (20).

7 Conclusion

In this paper, we have unified various results of plane and noncrossing trees by considering plane trees with d wings of butterflies. We have used root degree, vertices of a certain type, leaves, endpoints, forests, descents, and outdegree sequences as the statistics of enumeration. There are various bijections of plane and noncrossing trees that have been constructed. These bijections can also be generalized by considering d-dimensional plane trees.

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