BULLETIN OF THE ICA
Volume **108** (2026), 104–150

Parking functions with a fixed set of lucky cars

Pamela E. Harris and Lucy Martinez

Abstract. In a parking function, a lucky car is a car that parks in its preferred parking spot, and the parking outcome is the permutation encoding the order in which the cars park on the street. We give a characterization for the set of parking outcomes arising from parking functions with a fixed set of lucky cars. This characterization involves the descent bottom set of a permutation, and we use the characterization to give a formula for the number of parking functions with a fixed set of lucky cars. Our work includes the cases where the number of cars is equal to the number of parking spots and where there are more spots than cars. We also give product formulas for the number of weakly increasing parking functions having a fixed set of lucky cars. When the number of cars equals the number of spots, this is a product of Catalan numbers.

1 Introduction

Throughout we let $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $[n] := \{1, 2, \ldots, n\}$. The set of parking functions of length n consists of all n-tuples $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$ whose rearrangement into weakly increasing order $\alpha^{\uparrow} = (b_1, b_2, \ldots, b_n)$ satisfies $b_i \leq i$ for all $i \in [n]$. The use of the name "parking function" arises as these n-tuples encode the parking preferences of n cars attempting to park on a one-way street consisting of n parking spots. The cars enter the street in sequential order, from car 1 to car n, with car i parking in the first spot it encounters at or past its preference a_i. If $\alpha = (a_1, a_2, \ldots, a_n)$ denotes the preference list for all cars and all cars are able to park in the n spots on the street, then α is a *parking function of length* n, and the order in which the cars park on the street is called the *outcome* of α and

Key words and phrases: parking function, permutation, (m, n)-parking function, admissible sets, lucky statistic.
Mathematics Subject Classifications: MSC2020 05A05, 05A10, 05A15
Corresponding author: Lucy Martinez <lucy.martinez@rutgers.edu>

Received: 11 October 2024
Accepted: 19 August 2025

is denoted $\mathcal{O}(\alpha)$. More precisely, if $\mathfrak{S}_n$ denotes the set of permutations of $[n]$ written in one-line notation, then the outcome of α is

$$\mathcal{O}(\alpha) = \pi_1\pi_2\cdots\pi_n,$$

where π_i denotes that car π_i parked in spot i on the street. We let PF_n denote the set of parking functions of length n and Konheim and Weiss established $|\mathrm{PF}_n| = (n+1)^{n-1}$, see [21].

In a parking function, a car is said to be *lucky* if it parks in its preferred parking spot. For example, the parking function $(1,1,1,1)$ has outcome $\mathcal{O}\big((1,1,1,1)\big) = 1234$, and only the first car is lucky. If the entries in a parking function are all distinct, i.e., $\alpha = (\pi_1, \pi_2, \ldots, \pi_n)$ where $\pi = \pi_1\pi_2\cdots\pi_n$ is a permutation of $[n]$, then $\mathcal{O}(\alpha) = \pi^{-1}$, and all cars are lucky. Gessel and Seo [17] gave a product formula for the generating function for the number of parking functions based on the number of lucky cars, which is given by

$$L_n(q) = \sum_{\alpha\in \mathrm{PF}n} q^{\mathsf{lucky}(\alpha)} = q\prod_{i=1}^{n-1}\big(i + (n-i+1)q\big). \tag{1}$$

Gessel and Seo established Equation (1) via ordered set partitions and generating function techniques. Shin [24] gave a bijective proof of Gessel and Seo's result by constructing a nonrecursive bijection between forests and parking functions.

Other work related to the lucky statistic includes that of Harris, Kretschmann, and Martínez Mori [20] who showed that the number of parking preference lists, elements of $[n]^n$, with exactly $n-1$ lucky cars is the same as the total number of comparisons performed by the `Quicksort` algorithm given all possible orderings of an array of size n. Also, Aguillon et al. established that the set of parking functions with exactly one unlucky car parking one spot away from its preference is in bijection with the set of ideal states in the Towers of Hanoi game [2, Theorem 1].

The main focus of this paper is to study lucky sets as "admissible sets" in parking functions. In the study of permutations, given a permutation characteristic or statistic and a set of indices or values $S \subset [n]$, we say that S is an admissible set if there exist permutations with exactly that characteristic/statistic at those indices/values. For example, in a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, we say i is a peak if $\pi_{i-1} < \pi_i > \pi_{i+1}$. Then, the admissible peak sets of a permutation consist of the subsets of $[n]$ that do not contain 1 nor n, nor any consecutive values. This implies that there are

Fibonacci many peak sets. Once the admissible sets are understood, it is then natural to fix an admissible set and then characterize and enumerate the set of permutations with the given property exactly at the set S. For admissible peak sets, Billey, Burdzy, and Sagan [4] showed that the number of $w \in \mathfrak{S}_n$ with peak set S is a power of two times a polynomial in n (called the *peak polynomial*), and they give techniques for explicit computation of this polynomial in special cases. They also conjectured the coefficients of a peak polynomial (when expanded in the basis of binomial coefficients) are always positive, which was proven in the affirmative by Diaz-Lopez, Harris, Insko, and Omar [12]. Other work includes that of the concept of pinnacle sets, which are the values at peaks rather than their position. Davis, Nelson, Petersen, and Tenner [11] gave a complete characterization of admissible pinnacle sets of permutations and a recursive formula for the number of permutations in $\mathfrak{S}_n$ with a given pinnacle set. González et al. [18] studied the analogous question for signed permutations and gave a closed formula for the number of type B and D admissible pinnacle sets. A similar story is also known for descents in permutations, and we point the interested reader to the work of Diaz-Lopez et al. [13].

Although work on admissible sets in permutations have been studied extensively, such ideas have rarely been explored in the context of parking functions. One instance is the work by Cruz et al. [10] in which they gave a recursive formula for the number of parking functions with a given descent set $I \subseteq [n]$. In the related setting of Stirling permutations, Colmenarejo et al. [8] studied Stirling permutations as parking functions and considered the questions of what subsets of cars are lucky and how many Stirling permutations have certain sets of lucky cars. They established that there are $(n-1)!$ Stirling permutations with only the first car being lucky and that there are Catalan many Stirling permutations with a maximal number of lucky cars. They also studied cases in which a small number of cars are lucky; however, general results for counting Stirling permutations with a fixed set of lucky cars remains an open problem. These examples illustrate that the lucky statistic of parking functions gives rise to interesting connections with other well-studied families of combinatorial objects.

Our work studies the notion of fixed lucky sets (a fixed set of lucky cars) in parking functions, which plays a role parallel to pinnacle sets in permutations. In this way, we bring a perspective rooted in permutation statistics to the study of parking functions. To make our approach precise, we let $\mathsf{Lucky}(\alpha)$ denote the set of lucky cars of $\alpha \in \mathrm{PF}_n$. For example, $\mathsf{Lucky}\big((1,1,1,1)\big) = \{1\}$ while $\mathsf{Lucky}(\alpha) = [n]$ whenever

$$\alpha = (\pi_1, \pi_2, \ldots, \pi_n) \quad \text{and} \quad \pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n.$$

As expected, the first car in any parking function is always lucky, so $1 \in \mathsf{Lucky}(\alpha)$ for all $\alpha \in \mathrm{PF}_n$. For any subset $I \subseteq [n]$, we define

$$\mathsf{LuckyPF}_n(I) := \{\alpha \in \mathrm{PF}_n : \mathsf{Lucky}(\alpha) = I\},$$

which is the set of parking functions of length n with lucky cars being the cars in the set I and unlucky cars being the cars in the set $[n] \setminus I$. For $I \subseteq [n]$, if $\mathsf{LuckyPF}_n(I) \neq \varnothing$, then I is said to be a *lucky set* of PF_n. For example, as the first car is always a lucky car regardless of the preference of the other cars, a set I not containing 1 can never be a lucky set of PF_n for any $n \geq 1$. We stress that lucky sets contain the lucky cars rather than the position at which the lucky cars park.

Among our contributions, in Theorem 2.1 we characterize the lucky sets of PF_n by proving that a subset $I \subseteq [n]$ is a lucky set of PF_n if and only if $1 \in I$. An immediate consequence is that there are 2^{n-1} distinct lucky sets of PF_n (Corollary 2.2). Curiously, not every permutation can be an outcome of a parking function with a fixed set of lucky cars. For example, the permutation $\pi = 312 \in \mathfrak{S}_3$ cannot be the outcome of a parking function $\alpha \in \mathrm{PF}_3$ with lucky set $\{1\}$. This is because car $3 = \pi_1$ parks in spot 1, however when car 3 enters to park, spot 1 is unoccupied. This means that parking in spot 1 would make car 3 a lucky car, but we assumed that the set of lucky cars was $\{1\}$, reaching a contradiction. Based on this observation, a natural question to ask is the following: *For a fixed subset $I \subseteq [n]$ (containing 1), what permutations in $\mathfrak{S}_n$ arise as the outcome of parking functions with lucky set I?* We answer this question by providing a full characterization of when a given subset of cars can be the set of lucky cars of a given outcome permutation.

Theorem 2.7. *Fix a lucky set $I \subseteq [n]$ of PF_n. Then*

$$\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$$

is the outcome of an $\alpha \in \mathsf{LuckyPF}_n(I)$ if and only if

(1) $\pi_1 \in I$,

(2) *if $\pi_i \in I$ and $\pi_{i+1} \notin I$, then $\pi_i < \pi_{i+1}$, and*

(3) *if $\pi_{i-1} > \pi_i$, then $\pi_i \in I$.*

The proof of Theorem 2.7 involves descent bottom sets of permutations (Definition 2.5), which are the values π_{i+1} whenever $\pi_i > \pi_{i+1}$.

Any descent bottom in an outcome of a parking function would imply that car π_{i+1} is lucky since it parked in spot $i+1$ having found spot i vacant,

as car π_i had not yet entered the street to park. This perspective draws inspiration from prior work on peak sets of permutations and Stirling permutations, and extends these ideas into the parking function setting. We let $\mathcal{O}_n(I)$ denote the set of outcome permutations for parking functions with lucky set I. If $I = \{1\}$, then $\pi = 123\cdots n$ is the only possible outcome of a parking function having only car 1 being lucky. Hence $\left|\mathcal{O}_n(I)\right| = 1$. However, even though Theorem 2.7 gives a characterization for the set $\mathcal{O}_n(I)$, determining a formula for the cardinality of the set $\mathcal{O}_n(I)$ has proven difficult for general lucky set I. We give a result for when the first k cars are lucky.

Theorem 2.10. *Let $I = \{1, 2, 3, \ldots, k\} \subseteq [n]$ be a lucky set of PF_n. Then $\left|\mathcal{O}_n(I)\right|$ is*

$$\sum_{\substack{J=\{j_1=1,\, j_2,\, j_3,\, \ldots,\, j_k\} \\ J\subseteq[n]}} k!\binom{n-k}{j_2-j_1-1,\, j_3-j_2-1,\, \ldots,\, j_k-j_{k-1}-1,\, n-j_k}.$$

The proof of Theorem 2.10 relies on specifying the spots that are occupied by the lucky cars, which must always include the first spot, and noting that those cars can park in those spots in any order.

In our main result, we utilize the set of outcomes $\mathcal{O}_n(I)$ to give a formula for the number of parking functions of length n with lucky set I.

Theorem 2.17. *Fix a lucky set $I \subseteq [n]$ of PF_n. Let $\mathcal{O}_n(I)$ denote the set of outcome permutations for parking functions with lucky set I. Then the number of parking functions with lucky set I is*

$$\left|\mathsf{LuckyPF}_n(I)\right| = \sum_{\pi=\pi_1\pi_2\cdots\pi_n\in\mathcal{O}_n(I)} \left(\prod_{\pi_j\notin I} \ell(\pi_j)\right), \tag{2}$$

where, for each $i \in [n]$, $\ell(\pi_i)$ denotes the length of the longest subsequence $\pi_j\pi_{j+1}\cdots\pi_{i-1}$ with $\pi_t < \pi_i$ for all $j \le t \le i-1$.

The proof of Theorem 2.17 relies on the characterization of the permutations that are outcomes of parking functions with lucky set I (as described in Theorem 2.7) and an argument often referred to as "counting through permutations." This technique allows one to count the number of parking functions with a given parking outcome. For other work that uses this technique for counting parking functions, we refer the reader to [1, 9, 16, 25].

We also consider the subset $\mathrm{PF}_n^{\uparrow}$ of weakly increasing parking functions. The only possible outcome of a weakly increasing parking function is the identity permutation $\pi = 123\cdots n$. Hence, restricting to the set $\mathrm{PF}_n^{\uparrow}$ reduces Equation (2) to just the product, which we show is a product of Catalan numbers, see [22]. For $n \geq 1$, the n^{th} Catalan number is $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$. We can now state the result.

Theorem 2.20. *Fix a lucky set $I = \{i_1, i_2, i_3, \ldots, i_k\} \subseteq [n]$ of $\mathrm{PF}_n^{\uparrow}$ and assume $1 = i_1 < i_2 < \cdots < i_k \leq n$. If $\mathsf{LuckyPF}_n^{\uparrow}(I)$ denotes the set of weakly increasing parking functions in $\mathrm{PF}_n^{\uparrow}$ with lucky set I, then*

$$\left|\mathsf{LuckyPF}_n^{\uparrow}(I)\right| = \prod_{j=1}^{k} \mathrm{Cat}_{x_j}, \tag{3}$$

where $x_j = i_{j+1} - i_j - 1$ for each $j \in [k-1]$ and $x_k = n - i_k$.

In Section 3, we generalize all of the aforementioned results to the set of (m,n)-parking functions, which are parking functions with m cars and n parking spots with $m \leq n$. We characterize the lucky sets (Theorem 3.1) and characterize the outcomes of (m,n)-parking functions with a fixed set of lucky cars.

Theorem 3.8. *Fix a lucky set $I \subseteq [m]$ of $\mathrm{PF}_{m,n}$. Then $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_{m,n}$ is the outcome of $\alpha \in \mathsf{LuckyPF}_n(I)$ if and only if $\mathtt{X}\pi = \mathtt{X}\pi_1\pi_2\cdots\pi_n$ satisfies the following:*

1. *if $\pi_{i-1} = \mathtt{X}$ and $\pi_i \in [m]$, then $\pi_i \in I$,*
2. *if $\pi_i \in I$ and $\pi_{i+1} \notin I \cup \{\mathtt{X}\}$, then $\pi_i < \pi_{i+1}$, and*
3. *if $\pi_{i-1}, \pi_i \in [m]$ and $\pi_{i-1} > \pi_i$, then $\pi_i \in I$.*

We also give a formula for the number of outcomes for m cars parking in n spots with the first k cars being lucky (Theorem 3.10). For $n \geq m$, we give a formula for the number of (m,n)-parking functions with a fixed set of lucky cars (Theorem 3.14). We also give a product formula for the number of weakly increasing (m,n)-parking functions having lucky set I (Theorem 3.32). However, this case is much more complex than the case when $n = m$ and requires some additional technical machinery and a series of combinatorial results to arrive at the main enumerative and characterizing results. Among these technical results, we establish that the number of permutations that are outcomes of weakly increasing (m,n)-parking functions with lucky set I is the same as the number of weak

compositions of $n-m$ into $|I|+1$ parts (Lemma 3.17). We then use weak compositions to construct subwords, which are used to count the number of weakly increasing (m,n)-parking functions with a fixed lucky set I and outcome π (Lemma 3.31). We conclude the article with Section 4, where we state a few open problems for further study.

2 Same number of cars as spots

In this section, we begin by describing all of the possible lucky sets. We then characterize the types of permutations that are outcomes of parking functions with a fixed set of lucky cars, and we also give a formula for the number of parking functions with a fixed set of lucky cars.

2.1 Lucky sets

We begin by establishing necessary and sufficient conditions on the set $I \subseteq [n]$ so that it is a lucky set of PF_n, i.e., it is a set for which there is a parking function with I being the cars that are lucky and $[n] \setminus I$ being the set of unlucky cars.

Theorem 2.1. *A subset $I \subseteq [n]$ is a lucky set of PF_n if and only if $1 \in I$.*

Proof. ($\Rightarrow$) If I is a lucky set of PF_n, then there exists a parking function $\alpha \in \mathrm{PF}_n$ such that $\mathsf{Lucky}(\alpha) = I$. As the first car is always a lucky car for any $\alpha \in \mathrm{PF}_n$, this implies that $1 \in I$.

($\Leftarrow$) Assume $I = \{i_1 = 1, i_2, i_3, \ldots, i_k\}$, with $k \in [n]$. Without loss of generality assume that $i_1 = 1 < i_2 < i_3 < \cdots < i_k$. Let $\alpha = (a_1, a_2, \ldots, a_n)$ where

$$a_x = \begin{cases} x & \text{if } x \in I, \\ i_\ell & \text{if } i_\ell < x < i_{\ell+1}. \end{cases}$$

By construction α is weakly increasing and satisfies that $a_i \leq i$ for all $i \in [n]$. Thus $\alpha \in \mathrm{PF}_n$. Every weakly increasing parking function parks the cars in sequential order, i.e., car i parks in spot i for all $i \in [n]$. For all $x \in I$, car x has preference x and parks in spot x. Thus every car in I is a lucky car of α. On the other hand, if $x \notin I$, then car x has preference strictly smaller than x and parks in spot x. Thus, these cars would be unlucky. Therefore, $\mathsf{Lucky}(\alpha) = I$, and I is a lucky set of PF_n, as desired. □

Theorem 2.1 immediately implies the following corollaries.

Corollary 2.2. *For $n \geq 1$, there are 2^{n-1} distinct lucky sets of PF_n.*

Proof. Since the first car is always lucky, then for each $1 < i \leq n$ it is only a matter of whether car i is lucky or not. This yields 2^{n-1} lucky sets of PF_n, as claimed. □

Corollary 2.3. *If $I \subseteq [n]$ is a lucky set of PF_n, then it is a lucky set of PF_m for all $m \geq n$.*

Proof. If $\alpha = (a_1, a_2, \ldots, a_n)$ is such that $\mathsf{Lucky}(\alpha) = I$, then appending $m - n$ copies of the value n to α satisfies that

$$\mathsf{Lucky}\big((a_1, a_2, \ldots, a_n, \underbrace{n, n, \ldots, n}_{m-n \text{ times}})\big) = I. \qquad \square$$

2.2 Outcomes of parking functions with a fixed lucky set

Not every permutation is the outcome of a parking function with a fixed set of lucky cars.

Example 2.4. Let $\alpha \in \mathrm{PF}_4$ and suppose that $\mathsf{Lucky}(\alpha) = \{1, 4\}$. Notice that $\pi = 1423 \in \mathfrak{S}_4$ is not a possible outcome since car 2 would park in spot 3—and would have done so after finding spot 2 unoccupied on the street—as it attempts to park. Thus, the only way car 2 would park in spot 3 is if it preferred that spot, in which case car 2 would be lucky, implying that $2 \in \mathsf{Lucky}(\alpha)$, a contradiction.

The previous example illustrates that if a permutation is to be the outcome of a parking function with a fixed set of lucky cars, then the descents in that permutation force certain cars to be lucky. Before proving this, we adapt the usual definition of a descent and descent bottom for our purposes.

Definition 2.5. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, an index $1 < i \leq n$ is a *descent* of π if $\pi_{i-1} > \pi_i$, and the value π_i is called a *descent bottom* of π. For convenience, we say $i = 1$ is always a descent and π_1 is always a descent bottom. The set

$$\mathrm{Des}(\pi) := \{i \in [n] : i \text{ is a descent of } \pi\},$$

is called the *descent set* of π, and the set

$$\mathrm{DBottom}(\pi) := \{\pi_i \in [n] : \pi_i \text{ is a descent bottom of } \pi\},$$

is called the *descent bottom set* of π.

By using generating functions and well-known bijections between parking functions, Prüfer codes, and labeled Dyck paths, Schumacher [23] gave explicit formulas for the total number of descents among parking functions of length n with exactly i ties.

In our work, we use the descent bottom set of a permutation in connection with describing parking functions with a fixed set of lucky cars. We begin by establishing that if $\pi \in \mathfrak{S}_n$ is the outcome of a parking function α with I as a lucky set, then all descent bottoms of π must be lucky cars of α.

Lemma 2.6. *Fix $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. For any $\alpha \in \mathrm{PF}_n$ with outcome $\mathcal{O}(\alpha) = \pi$, if $i \in \mathrm{Des}(\pi)$, then $\pi_i \in \mathsf{Lucky}(\alpha)$. In other words,*

$$\mathrm{DBottom}(\pi) \subseteq \mathsf{Lucky}(\alpha).$$

Proof. Let α be a parking function of length n with outcome

$$\mathcal{O}(\alpha) = \pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n.$$

For each $i \in [n]$, since $\mathcal{O}(\alpha) = \pi = \pi_1\pi_2\cdots\pi_n$, we know that car π_i parked in spot i. If $i = 1$, then car π_1 was the first car with preference for spot 1 and parked there; hence, it is a lucky car. Implying that $\pi_1 \in \mathsf{Lucky}(\alpha)$. By definition, for any $1 < i \leq n$, if $i \in \mathrm{Des}(\pi)$, then $\pi_{i-1} > \pi_i$. This implies that car π_i parks on the street before car π_{i-1} enters the street to park. Thus, when car π_i entered the street to park, spot $i-1$ is unoccupied. Assume for contradiction that car π_i has a preference $1 \leq a_i \leq i-1$. Then car π_i would park in the first available spot past a_i. Among the spots numbered $a_i, a_i+1, \ldots, i-1$ spot $i-1$ remains available. So car π_i would park in spot x satisfying $a_i \leq x \leq i-1 < i$, contradicting the assumption that car π_i parks in spot i. Thus, car π_i parks in spot i and must have done so by preferring spot i. This makes car π_i a lucky car, and so whenever $i \in \mathrm{Des}(\pi)$ we have that $\pi_i \in \mathsf{Lucky}(\alpha)$, as claimed. □

Lemma 2.6 establishes that descent bottoms are lucky cars. However, not all lucky cars have to be descent bottoms. For example, if $I = [n]$, then all cars are lucky, and a possible outcome is $\pi = 123\cdots n$, which has $\pi_1 = 1$ as the only descent bottom. We now give a complete characterization for permutations that are outcomes of a parking function with a fixed set of lucky cars.

Theorem 2.7. *Fix a lucky set $I \subseteq [n]$ of PF_n. Then*

$$\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$$

is the outcome of an $\alpha \in \mathsf{LuckyPF}_n(I)$ if and only if

(1) $\pi_1 \in I$,

(2) *if $\pi_i \in I$ and $\pi_{i+1} \notin I$, then $\pi_i < \pi_{i+1}$, and*

(3) *if $\pi_{i-1} > \pi_i$, then $\pi_i \in I$.*

Proof. ($\Rightarrow$) For the forward direction we verify that when π is an outcome of a parking function with lucky set I, it satisfies each of the needed conditions.

For (1): In any parking function the first car to prefer spot 1 will park there, hence, that car is a lucky car. So if $\pi = \pi_1\pi_2\cdots\pi_n = \mathcal{O}(\alpha)$, then $\pi_1 \in I$.

For (2): Suppose $\pi_i \in I$ and $\pi_{i+1} \notin I$. If $\pi_i < \pi_{i+1}$, then we are done. For the sake of contradiction, suppose that $\pi_i > \pi_{i+1}$. This means that when car π_{i+1} parks in spot $i+1$, car π_i has not yet entered the street to park. Hence, spot i is empty when car π_{i+1} enters the street to park. However, car π_{i+1} is not lucky, so it must have preferred a spot numbered x, where $x < i+1$. As spot i is empty, car π_{i+1} would have parked in a spot numbered y with $y \leq i$. Hence, car π_{i+1} would not have parked in spot $i+1$, a contradiction.

For (3): Suppose that $\pi_{i-1} > \pi_i$. Then, by Lemma 2.6, $\pi_i \in I$.

($\Leftarrow$) Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ satisfy conditions (1), (2), and (3) above. We use the following notation:

- $\pi_x = y$ denotes that car y parked in spot x on the street,
- $\pi_u^{-1} = v$ denotes that car u parked in spot v on the street.

By the same argument as in [19, Theorem 2.2], if

$$\pi^{-1} = \pi_1^{-1}\pi_2^{-1}\cdots\pi_n^{-1},$$

then

$$\mathcal{O}\big((\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_n^{-1})\big) = \pi.$$

We now construct $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n$ such that $\mathcal{O}(\alpha) = \pi$ and $\mathsf{Lucky}(\alpha) = I$. For each $\pi_i = 1, 2, \ldots, n$ (in this order) set the value of a_{π_i} as follows:

1. If $\pi_i \in I$, set $a_{\pi_i} = \pi_j^{-1}$ where $j = \pi_i$.
2. If $\pi_i \notin I$, set a_{π_i} as follows:
   (a) If $\pi_{i-1} \in I$ and $\pi_{i-1} < \pi_i$, then set $a_{\pi_i} = \pi_k^{-1}$ where $k = \pi_{i-1}$.
   (b) If $\pi_{i-1} \in I$ and $\pi_{i-1} > \pi_i$, then by Lemma 2.6 we would have that $\pi_i \in I$, a contradiction. Thus, this case never happens.
   (c) If $\pi_{i-1} \notin I$, then set $a_{\pi_i} = \pi_k^{-1}$ where $k = \pi_{i-1}$.

We must now establish the following claims:

Claim 1. As constructed, α is a parking function.

Prior to proving this claim, we recall the characterization of parking functions from the introduction: A tuple β is a parking function if and only if its rearrangement into weakly increasing order, denoted $\beta^{\uparrow} = (b'_1, b'_2, \ldots, b'_n)$, satisfies $b'_i \leq i$ for all $i \in [n]$. We now establish this inequality condition for the constructed tuple α. By construction, for each $i \in [n]$, the entry a_i in α is assigned at most the parking spot where car i ends up parking. Namely,

$$a_i \leq \pi_i^{-1} \tag{4}$$

for all $i \in [n]$. Here π^{-1} is a permutation of $[n]$ so its rearrangement into weakly increasing order is the identity permutation $12 \cdots n$. Then rearranging the entries of α into weakly increasing order, and denoting this by $\alpha^{\uparrow} = (a'_1, a'_2, \ldots, a'_n)$, together with Inequality (4) ensures that $a'_i \leq i$ for all $i \in [n]$. Thus α is a parking function.

Claim 2. The set of lucky cars of α is precisely I, i.e., $\mathsf{Lucky}(\alpha) = I$.

The only preferences used in α are the values π_i^{-1} for $i \in I$. Moreover, all of the lucky cars in I get the first instance of those preference, as any unlucky car is given the preference of the nearest lucky car parked to its left, which makes those cars unlucky. This implies that the set of lucky cars in α is precisely I.

Claim 3. The outcome of α is π, i.e., $\mathcal{O}(\alpha) = \pi$.

By the same argument as in [19, Theorem 2.2],

$$\mathcal{O}\big((\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_n^{-1})\big) = \pi.$$

As α is constructed from π^{-1} an entry at a time—giving car $j = \pi_i$ preference π_j^{-1} whenever it is lucky and giving the unlucky cars the

preference of the previous lucky car—the outcome of α is exactly the same as the outcome of π^{-1} when we consider π^{-1} as a parking function with entries $(\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_n^{-1})$. Thus $\mathcal{O}(\alpha) = \pi$ as desired.

This completes the proof of Theorem 2.7. □

2.3 Counting outcomes where the first k cars are lucky

In Theorem 2.7 we have a complete characterization for the permutations that arise as outcomes of parking functions with a fixed set of lucky cars. We now set some notation to describe those permutations.

Definition 2.8. Fix a lucky set I of PF_n. The set of *outcomes* for I is the subset of $\mathfrak{S}_n$ defined as

$$\mathcal{O}_n(I) := \big\{\pi \in \mathfrak{S}_n : \mathcal{O}(\alpha) = \pi \text{ for some } \alpha \in \mathsf{LuckyPF}_n(I)\big\}.$$

Example 2.9. Let $n = 5$ and $I = \{1, 4\}$. By Theorem 2.7,

$$\pi = \pi_1\pi_2\pi_3\pi_4\pi_5 \in \mathcal{O}_4(\{1, 4\})$$

if and only if

(1) $\pi_1 = 1$ or $\pi_1 = 4$,

(2) if $\pi_i \in I$ and $\pi_{i+1} \notin I$, then $\pi_i < \pi_{i+1}$, and

(3) if $\pi_i - 1 > \pi_i$, then $\pi_i \in I$.

Condition (3) implies that $\mathrm{DBottom}(\pi) \subseteq I$, which we proved in Lemma 2.6. Let us construct these permutations one condition at a time. There are $2 \cdot 4! = 48$ permutations in $n = 5$ satisfying Condition (1), as $\pi = 1\sigma$ with $\sigma \in \mathfrak{S}_{\{2,3,4,5\}}$ or $\pi = 4\sigma$ with $\sigma \in \mathfrak{S}_{\{1,2,3,5\}}$. By Condition (2):

- If $\pi = 1\sigma$ with $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4 \in \mathfrak{S}_{\{2,3,4,5\}}$, then $2 \le \sigma_1 \le 5$. To ensure π also satisfies Condition (3), σ can either have no descents or a descent bottom at 4. Hence π can only be 12345 or 12354.
- If $\pi = 4\sigma$ with $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4 \in \mathfrak{S}_{\{1,2,4,5\}}$, then Condition (3) implies that σ must have only a descent bottom at 1. Hence π can only be 41235 or 45123.

Thus $\mathcal{O}_5(\{1, 4\}) = \{12345, 12354, 41235, 45123\}$.

We now give a formula for the number of parking outcomes when the first k cars are lucky.

Theorem 2.10. *Let $I = \{1, 2, 3, \ldots, k\} \subseteq [n]$ be a lucky set of PF_n. Then $\left|\mathcal{O}_n(I)\right|$ is*

$$\sum_{\substack{J=\{j_1=1,\, j_2,\, j_3,\, \ldots,\, j_k\} \\ J\subseteq[n]}} k!\binom{n-k}{j_2-j_1-1,\, j_3-j_2-1,\, \ldots,\, j_k-j_{k-1}-1,\, n-j_k}.$$

Proof. Let $I = \{1, 2, 3, \ldots, k\}$ be the set of lucky cars. Fix a set of indices of parking spots $J = \{j_1 = 1, j_2, \ldots, j_k\} \subseteq [n]$, satisfying $j_1 < j_2 < \ldots < j_k \leq n$, in which the k lucky cars park. The lucky cars can park in any order among the spots in J, so there are $k!$ many ways for those cars to appear in an outcome permutation with lucky set I. Since the lucky cars park on the street first, we need to park the $n - k$ unlucky cars, which are numbered $k+1, k+2, \ldots, n$. The unlucky cars must park in the gaps between any two consecutively parked lucky cars. The gaps between any two consecutively parked cars have size $j_{i+1} - j_i - 1$ whenever $1 \leq i \leq k-1$, and (at the end of the street) the gap from the last parked lucky car in position j_k and the end of the street has length $n - j_k$. We can select the unlucky cars that park in each respective gap in $\binom{n-k}{j_2-j_1-1,\, j_3-j_2-1,\, \ldots,\, j_k-j_{k-1}-1,\, n-j_k}$ ways. For each such selection of cars parking in those gaps, the unlucky cars must park in increasing order, thereby ensuring that those cars create no descents in the outcome as that would imply that an unlucky car is lucky, which would give a contradiction.

Taking the sum over all possible subsets J containing 1 and of size k, we find that the number of outcomes of parking functions with lucky cars in the set $I = \{1, 2, 3, \ldots, k\}$ is given by $\left|\mathcal{O}_n(I)\right|$, which is equal to

$$\sum_{\substack{J=\{j_1=1,\, j_2,\, j_3,\, \ldots,\, j_k\} \\ J\subseteq[n]}} k!\binom{n-k}{j_2-j_1-1,\, j_3-j_2-1,\, \ldots,\, j_k-j_{k-1}-1,\, n-j_k}$$

as claimed. □

We conclude by noting that, although we have a complete characterization for the elements of $\mathcal{O}_n(I)$ for any lucky set I, giving a closed formula for the cardinality of the set in general remains an open problem.

2.4 Counting parking functions with a fixed set of lucky cars

Throughout this section I is a lucky set of PF_n. Hence, by Theorem 2.1, $1 \in I$. We begin with the following definitions.

Definition 2.11. For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ let $\ell(\pi_i)$ be the length of the longest subsequence $\pi_j\pi_{j+1}\cdots\pi_{i-1}$ with $\pi_t < \pi_i$ for all $j \le t \le i-1$.

Example 2.12. If $\pi = 195348267 \in \mathfrak{S}_9$, then $\ell(\pi_1 = 1) = 0$, $\ell(\pi_2 = 9) = 1$, $\ell(\pi_3 = 5) = 0$, $\ell(\pi_4 = 3) = 0$, $\ell(\pi_5 = 4) = 1$, $\ell(\pi_6 = 8) = 3$, $\ell(\pi_7 = 2) = 0$, $\ell(\pi_8 = 6) = 1$, $\ell(\pi_9 = 7) = 2$.

For each $i \in [n]$, the definition of $\ell(\pi_i)$ counts the number of cars arriving before car π_i that parked contiguously and immediately to the left of car π_i. We show (in Lemma 2.15) that, in the case that car π_i is an unlucky car, the number $\ell(\pi_i)$ is exactly the number of potential spots car π_i could prefer and which car π_i would find occupied by cars that arrived and parked before it. We make this precise next.

Definition 2.13. Fix a lucky set I of PF_n and a permutation

$$\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{O}_n(I).$$

For each $i \in [n]$, let $\mathrm{Pref}_I(\pi_i)$ denote the set of preferences of car π_i in a parking function α of length n satisfying $\mathsf{Lucky}(\alpha) = I$ and $\mathcal{O}(\alpha) = \pi$.

Example 2.14 (Continuing Example 2.12)**.** If $\pi = 195348267 \in \mathfrak{S}_9$ and car $\pi_8 = 6$ is unlucky, then it could only prefer the spot occupied by car $\pi_7 = 2$, which parked immediately to its left. This agrees with $\ell(\pi_8 = 6) = 1$. If car $\pi_6 = 8$ is unlucky, then it could prefer the spots occupied by cars $\pi_3 = 5$, $\pi_4 = 3$, and $\pi_5 = 4$, as those cars are parked immediately to the left of car $\pi_6 = 8$ and arrived prior to car $\pi_6 = 8$. So car $\pi_6 = 8$ has 3 preferences, which would make it an unlucky car. This agrees with $\ell(\pi_6 = 8) = 3$.

The count for the preferences of unlucky cars illustrated in Example 2.14 holds in general. We prove this next.

Lemma 2.15. *Fix a lucky set I of PF_n and fix $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{O}_n(I)$. Then*

$$\left|\mathrm{Pref}_I(\pi_i)\right| = \begin{cases} 1 & \text{if } \pi_i \in I, \\ \ell(\pi_i) & \text{if } \pi_i \notin I, \end{cases}$$

and the number of possible parking functions α with outcome π and lucky set I is equal to

$$\prod_{i=1}^{n}\left|\mathrm{Pref}_I(\pi_i)\right| = \prod_{\pi_j \notin I}\ell(\pi_j).$$

Proof. Let $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n$ satisfying $\mathsf{Lucky}(\alpha) = I$ and $\mathcal{O}(\alpha) = \pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. For each $i \in [n]$, car π_i is either lucky or unlucky. If car π_i is lucky, then car π_i parking in spot i must have had spot i as its preference. This is a unique preference, which agrees with the value of $\ell(\pi_i) = 1$ for all $\pi_i \in I$.

Now consider the case where $i \in [n]$ for which car π_i is an unlucky car that parks in spot i. Hence, $\pi_i \notin I$. In this case, the only possible preferences that car π_i can have, so as to park in spot i and be unlucky, is the set of spots occupied immediately to the left of spot i provided those spots were occupied by cars that arrived and parked before car π_i. That is, if the sequence of cars $\pi_j\pi_{j+1}\cdots\pi_{i-1}$ is the longest sequence satisfying $\pi_t < \pi_i$ for all $j \leq t \leq i-1$, then the unlucky car π_i can prefer any of the spots numbered $j, j+1, \ldots, i-1$. This is precisely the value of $\ell(\pi_i)$ when $\pi_i \notin I$.

As the car preferences are independent of each other, we have that the product of $\left|\mathrm{Pref}_I(\pi_i)\right|$ over all $i \in [n]$ counts all of the possible preferences for cars 1 to n so that they have outcome π and lucky set I. Thus, the number of possible parking functions α with outcome π and lucky set I is equal to

$$\prod_{i=1}^{n}\left|\mathrm{Pref}_I(\pi_i)\right| = \left(\prod_{\pi_i \in I}\ell(\pi_i)\right)\cdot\left(\prod_{\pi_j \notin I}\ell(\pi_j)\right) = \prod_{\pi_j \notin I}\ell(\pi_j),$$

where the last equality holds as $\ell(\pi_i) = 1$ for all $\pi_i \in I$. □

Example 2.16 (Continuing Example 2.12)**.** Observe that

$$\pi = 195348267 \in \mathfrak{S}_9$$

has $\mathrm{DBottom}(\pi) = \{1 = \pi_1, 2 = \pi_7, 3 = \pi_4, 5 = \pi_3\}$, and so π can be the outcome of a parking function with lucky set I provided $\{1 = \pi_1, 2 = \pi_7, 3 = \pi_4, 5 = \pi_3\} \subseteq I$. Suppose that $I = \{1, 2, 3, 5, 7\}$. Then by Lemma 2.15 the number of parking functions α satisfying $\mathcal{O}(\alpha) = \pi$ and $\mathsf{Lucky}(\alpha) = I$ is given by

$$\begin{aligned}\prod_{i=1}^{9}\left|\mathrm{Pref}_I(\pi_i)\right| &= \prod_{\pi_j \notin I}\ell(\pi_j)\\ &= \ell(\pi_5 = 4)\cdot\ell(\pi_8 = 6)\cdot\ell(\pi_6 = 8)\cdot\ell(\pi_2 = 9)\\ &= 1\cdot 1\cdot 3\cdot 1\\ &= 3.\end{aligned}$$

In fact, the only parking functions $\alpha \in \mathrm{PF}_9$ with lucky set I and outcome π are

$$(1,7,4,4,3,7,9,\boxed{3},1),$$
$$(1,7,4,4,3,7,9,\boxed{4},1),$$
$$(1,7,4,4,3,7,9,\boxed{5},1),$$

where we box the difference between these parking functions, which denotes the possible preferences of car $\pi_6 = 8$ being spots 3, 4, and 5, as those spots are occupied by earlier cars $\pi_3 = 5$, $\pi_4 = 3$, and $\pi_5 = 4$, respectively. Hence, car 8 is forced to park in spot 6.

If instead we consider the lucky set $J = \{1,2,3,5,8\}$, then by Lemma 2.15 the number of parking functions α satisfying $\mathcal{O}(\alpha) = \pi$ and $\mathsf{Lucky}(\alpha) = J$ is

$$\begin{aligned}\prod_{i=1}^{9}\left|\mathrm{Pref}_J(\pi_i)\right| &= \prod_{\pi_j \notin J} \ell(\pi_j) \\ &= \ell(\pi_5 = 4)\cdot \ell(\pi_8 = 6)\cdot \ell(\pi_9 = 7)\cdot \ell(\pi_2 = 9) \\ &= 1\cdot 1\cdot 2\cdot 1 \\ &= 2.\end{aligned}$$

Hence, the parking functions $\alpha \in \mathrm{PF}_9$ with lucky set

$$J = \{1,2,3,5,8\}$$

and outcome π are

$$(1,7,4,4,3,7,\boxed{7},6,1),$$
$$(1,7,4,4,3,7,\boxed{8},6,1),$$

where we box the difference between these parking functions, which denotes the possible preferences of car 7 being spots 7 and 8, as those spots are occupied by cars 2 and 6, respectively. Hence, car 7 is forced to park in spot 9.

We now give a formula for the number of parking functions with a fixed set of lucky cars. Our proof technique is similar to that of Colmenarejo et al., as presented in [9, Corollary 3.1].

Theorem 2.17. *Fix a lucky set I of PF_n. Let $\mathcal{O}_n(I)$ denote the set of outcome permutations for parking functions with lucky set I. Then the number of parking functions with lucky set I is*

$$\left|\mathsf{LuckyPF}_n(I)\right| = \sum_{\pi=\pi_1\pi_2\cdots\pi_n \in \mathcal{O}_n(I)} \left(\prod_{\pi_j \notin I} \ell(\pi_j)\right). \tag{5}$$

Proof. Fix a lucky set $I \subseteq [n]$ of PF_n. By Lemma 2.15, for any $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{O}_n(I)$ the number of parking functions $\alpha \in \mathsf{LuckyPF}_n(I)$ with outcome π is given by $\prod_{\pi_j \notin I} \ell(\pi_j)$. To account for all parking functions with lucky set I, we must sum over all permutations π arising as the outcomes of such parking functions. Those are the permutations characterized in Theorem 2.7, and that set is denoted by $\mathcal{O}_n(I)$. This completes the proof. □

Example 2.18. In Example 2.9 we determine that the set of outcomes of parking functions of length $n = 5$ with lucky set $I = \{1, 4\}$ is $\mathcal{O}_n(I) = \{12345, 12354, 41235, 45123\}$. Next, for each $\pi \in \mathcal{O}_5(I)$ we use Lemma 2.15 to compute the number of parking functions with lucky set I and outcome π:

- If $\pi = 12345$, then there are $\prod_{\pi_j \in \{2,3,5\}} \ell(\pi_j) = 1 \cdot 2 \cdot 4 = 8$ parking functions.
- If $\pi = 12354$, then there are $\prod_{\pi_j \in \{2,3,5\}} \ell(\pi_j) = 1 \cdot 2 \cdot 3 = 6$ parking functions.
- If $\pi = 41235$, then there are $\prod_{\pi_j \in \{2,3,5\}} \ell(\pi_j) = 1 \cdot 2 \cdot 4 = 8$ parking functions.
- If $\pi = 45123$, then there are $\prod_{\pi_j \in \{2,3,5\}} \ell(\pi_j) = 1 \cdot 2 \cdot 1 = 2$ parking functions.

Thus, by Theorem 2.17, the number of parking functions of length 5 with lucky set $I = \{1, 4\}$ is given by

$$\begin{aligned}\left|\mathsf{LuckyPF}_5(\{1,4\})\right| &= \sum_{\pi \in \mathcal{O}_5(\{1,4\})} \left(\prod_{\pi_j \in \{2,3,5\}} \ell(\pi_j) \right) \\ &= 8 + 6 + 8 + 2 = 24.\end{aligned}$$

2.5 Weakly increasing parking functions with a fixed set of lucky cars

In this section, we let $\mathrm{PF}_n^{\uparrow}$ denote the set of parking functions

$$\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n$$

satisfying $a_i \leq a_{i+1}$ for all $i \in [n-1]$. The number of elements in $\mathrm{PF}_n^{\uparrow}$ is given by the n^{th} Catalan number

$$\mathrm{Cat}_n = \tfrac{1}{n+1}\tbinom{2n}{n},$$

which is OEIS sequence A000108, see [22].

We begin by noting that the proof of Theorem 2.1 directly establishes that any subset of $[n]$ including 1 is a lucky set for $\mathrm{PF}_n^{\uparrow}$. We state this formally next.

Lemma 2.19. *A subset $I \subseteq [n]$ is a lucky subset of $\mathrm{PF}_n^{\uparrow}$ if and only if $1 \in I$.*

For a fixed lucky set I, let $\mathsf{LuckyPF}_n^{\uparrow}(I)$ denote the set of weakly increasing parking functions $\alpha \in \mathrm{PF}_n^{\uparrow}$ with $\mathsf{Lucky}(\alpha) = I$. We now give a formula for the number of weakly increasing parking functions with a fixed set of lucky cars.

Theorem 2.20. *Fix a lucky set $I = \{i_1, i_2, i_3, \ldots, i_k\} \subseteq [n]$ of $\mathrm{PF}_n^{\uparrow}$ and assume $1 = i_1 < i_2 < \cdots < i_k \leq n$. Then*

$$\left|\mathsf{LuckyPF}_n^{\uparrow}(I)\right| = \prod_{j=1}^{k} \mathrm{Cat}_{x_j}, \tag{6}$$

where $x_j = i_{j+1} - i_j - 1$ for each $j \in [k-1]$ and $x_k = n - i_k$.

Proof. If $\alpha \in \mathrm{PF}_n^{\uparrow}$, then the cars park in order and, hence, $\mathcal{O}(\alpha) = \pi = 123\cdots n \in \mathfrak{S}_n$. Namely, for all $i \in [n]$, car i parks in spot i. Thus, regardless of lucky set I, the outcome of a weakly increasing parking function α has no descents nor descent bottoms except at π_1. Thus, if $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n^{\uparrow}$ has lucky set I, then for each $i_j \in I$, with $1 \leq j \leq k$, car i_j has preference $a_{i_j} = i_j$. While for any $j \in [k-1]$, the cars $i_j + 1$, $i_j + 2$, $\ldots$, $i_{j+1} - 1$ must have weakly increasing preferences $a_{i_j+1} \leq a_{i_j+2} \leq \cdots \leq a_{i_{j+1}-1}$, respectively, with $i_j \leq a_x \leq x - 1$ for all $x \in [i_j + 1, i_{j+1} - 1]$. The number of lists satisfying this condition is precisely the number of weakly increasing parking functions of length $i_{j+1} - 1 - (i_j + 1) + 1 = i_{j+1} - i_j - 1$; the number of which is given by the Catalan number $\mathrm{Cat}_{i_{j+1}-i_j-1}$. In a similar way, the preferences for cars $i_k + 1$, $i_k + 2$, $\ldots$, n must also be weakly increasing and satisfy $i_k \leq a_x \leq x - 1$ for all $x \in [i_k + 1, n]$; the number of which is given by the Catalan number Cat_{n-i_k}. The result follows from taking the product of the resulting Catalan numbers, which gives the formula in Equation (6). □

Example 2.21 (Continuing Example 2.18)**.** Let $n = 5$ and $I = \{1, 4\}$. Let $i_1 = 1$ and $i_2 = 4$. Then, using the definition of x_j as given in Theorem 2.20, observe that $x_1 = 4 - 1 - 1 = 2$ and $x_2 = 5 - 4 = 1$. Then by Theorem 2.20, the number of weakly increasing parking functions of length 5 with lucky set $\{1, 4\}$ is $\mathrm{Cat}_2 \cdot \mathrm{Cat}_1 = 2 \cdot 1 = 2$. One can readily confirm that the only

weakly increasing parking functions of length 5 with lucky set $I = \{1, 4\}$ are $(1, 1, 1, 4, 4)$ and $(1, 1, 2, 4, 4)$.

3 More spots than cars

One common generalization of parking functions is the case where there are more parking spots than cars. We let $\mathrm{PF}_{m,n}$ denote the set of (m, n)-parking functions with m cars and n spots, and we assume that $1 \leq m \leq n$. Then, the set $\mathrm{PF}_{m,n}$ consists of all m-tuples $\beta = (b_1, b_2, \ldots, b_m) \in [n]^m$ whose weakly increasing rearrangement denoted $\beta^{\uparrow} = (b'_1, b'_2, \ldots, b'_m)$ satisfies

$$b'_i \leq n - m + i \text{ for all } i \in [m]. \tag{7}$$

Konheim and Weiss established that

$$|\mathrm{PF}_{m,n}| = (n + 1 - m)(n + 1)^{m-1},$$

see [21, Lemma 2].

We now consider the analogous question of how many parking functions in $\mathrm{PF}_{m,n}$ have a fixed set of lucky cars. In this case, the analysis only differs in that lucky cars have more options among the spots in which they might park. Also among the possible outcomes, the only change is made by first selecting an m-element subset of $[n]$ to index the location of where the m cars park on the street, as the remaining spots would be empty. So the proof of some results are analogous to the case when $m = n$, in which case we omit their details, but we give their formal statements below. Whenever the argument differs substantially, we provide a detailed proof.

This section is organized analogously to the case where $m = n$. That is, we begin by describing all of the possible lucky sets. We then characterize the types of permutations that are outcomes of parking functions with a fixed set of lucky cars, and then we give a formula for the number of parking functions with a fixed set of lucky cars.

3.1 Lucky sets for (m, n)-parking functions

As before, given $\alpha \in \mathrm{PF}_{m,n}$, we let $\mathsf{Lucky}(\alpha)$ denote the set of lucky cars of α. Given a subset $I \subset [m]$, we let

$$\mathsf{LuckyPF}_{m,n}(I) = \big\{\alpha \in \mathrm{PF}_{m,n} : \mathsf{Lucky}(\alpha) = I\big\}.$$

Whenever $I \subseteq [m]$ is such that $\mathsf{LuckyPF}_{m,n}(I) \neq \varnothing$, we say that I is a lucky set of $\mathrm{PF}_{m,n}$. We begin by characterizing and enumerating the lucky sets of $\mathrm{PF}_{m,n}$.

Theorem 3.1. *A subset $I \subseteq [m]$ is a lucky set of $\mathrm{PF}_{m,n}$ if and only if $1 \in I$.*

Corollary 3.2. *For $m \geq 1$, there are 2^{m-1} distinct lucky sets of $\mathrm{PF}_{m,n}$.*

Corollary 3.3. *If $I \subseteq [m]$ is a lucky set of $\mathrm{PF}_{m,n}$, then it is a lucky set of $\mathrm{PF}_{k,\ell}$ for all $k \geq m$ and $\ell \geq n$ with $\ell \geq k$.*

3.2 Outcomes of (m, n)-parking functions with a fixed lucky set

Next we define the outcome of a parking function when there are more parking spots than cars.

Definition 3.4. Let $\mathfrak{S}_{m,n}$ denote the set of permutations of the multiset $\{\mathtt{X}, \ldots, \mathtt{X}\} \cup [m]$ with $\mathtt{X}$ having multiplicity $n-m$. Given a parking function $\alpha = (a_1, a_2, \ldots, a_m) \in \mathrm{PF}_{m,n}$, define the *outcome* of α by

$$\mathcal{O}(\alpha) = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_{m,n}$$

where $\pi_i = j \in [m]$ denotes that car j parked in spot i, and $\pi_i = \mathtt{X}$ indicates that spot i remained vacant.

For the sake of simplicity in our arguments, we assume that $\mathtt{X} > i$ for any $i \in \mathbb{N}$. Moreover, to every element $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_{m,n}$ we prepend $\pi_0 = \mathtt{X}$. We now give our definition for descents and descent bottoms for permutations in $\mathfrak{S}_{m,n}$.

Definition 3.5. Let $\pi_0 = \mathtt{X}$ and $\pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_{m,n}$. Let $\pi = \pi_0 \pi_1 \pi_2 \cdots \pi_n = \mathtt{X}\pi_1 \pi_2 \cdots \pi_n$. *Descent bottoms* of π are values $\pi_i \in [m]$ for which $\pi_{i-1} = \mathtt{X}$ or $\pi_{i-1} \in [m]$ and $\pi_{i-1} > \pi_i$. In these cases, i is called a *descent* of π. The set

$$\mathrm{Des}_{m,n}(\pi) := \{i \in [n] : i \text{ is a descent of } \pi \in \mathfrak{S}_{m,n}\}$$

is called the *descent set* of π, and

$$\mathrm{DBottom}_{m,n}(\pi) := \{\pi_i \in [n] : \pi_i \text{ is a descent bottom of } \pi \in \mathfrak{S}_{m,n}\}$$

is called the *descent bottom set* of π.

We illustrate the above definitions with the following example.

Example 3.6. Let $\alpha \in \mathrm{PF}_{5,7}$ and

$$\pi = \pi_0\pi_1\pi_2\pi_3\pi_4\pi_5\pi_6\pi_7 = \mathtt{X453X1X2}.$$

The descent bottoms of π are $\pi_1 = 4$, $\pi_3 = 3$, $\pi_5 = 1$, and $\pi_7 = 2$. All of these cars must be in the lucky set of α, as those cars would have found the spot immediately to their left vacant upon their attempt to park, either because that spot remained empty after all cars parked or because the car that ultimately parked there had not yet entered the street to park. Hence, if $\mathcal{O}(\alpha)$ is to be equal to π, then $\{1, 2, 3, 4\} \subseteq \mathsf{Lucky}(\alpha)$.

By using Definition 3.5, we can extend Lemma 2.6 to the case where there are more spots than cars.

Lemma 3.7. *Fix $\pi \in \mathfrak{S}_{m,n}$. For any $\alpha \in \mathrm{PF}_{m,n}$ with outcome $\mathcal{O}(\alpha) = \pi$, with $\pi \in \mathfrak{S}_{m,n}$, if $i \in \mathrm{Des}(\mathtt{X}\pi)$, then $\pi_i \in \mathsf{Lucky}(\alpha)$. In other words, $\mathrm{DBottom}_{m,n}(\pi) \subseteq \mathsf{Lucky}(\alpha)$.*

Proof. The proof of Lemma 3.7 is analogous to the proof of Lemma 2.6, as it is immaterial to car π_i whether $\pi_{i-1} = \mathtt{X}$ or $\pi_{i-1} \in [m]$ since in either case car π_i would find spot $i - 1$ empty and, hence, would park in spot i and be a lucky car. □

As before, Lemma 3.7 establishes that, if α has outcome π, then all descent bottoms of π are lucky cars of α. We also provide an analogous result to Theorem 2.7 to characterize which permutations in $\mathfrak{S}_{m,n}$ are the outcomes of a parking function in $\mathrm{PF}_{m,n}$ with a fixed set of lucky cars.

Theorem 3.8. *Fix a lucky set $I \subseteq [m]$ of $\mathrm{PF}_{m,n}$. Then $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_{m,n}$ is the outcome of $\alpha \in \mathsf{LuckyPF}_n(I)$ if and only if $\mathtt{X}\pi = \mathtt{X}\pi_1\pi_2\cdots\pi_n$ satisfies the following:*

1. *if $\pi_{i-1} = \mathtt{X}$ and $\pi_i \in [m]$, then $\pi_i \in I$,*
2. *if $\pi_i \in I$ and $\pi_{i+1} \notin I \cup \{\mathtt{X}\}$, then $\pi_i < \pi_{i+1}$, and*
3. *if $\pi_{i-1}, \pi_i \in [m]$ and $\pi_{i-1} > \pi_i$, then $\pi_i \in I$.*

Proof. Before we proceed with the technical details, we make the following remarks regarding the items in the theorem statement:

1. The statement "If $\pi_{i-1} = \mathtt{X}$ and $\pi_i \in [m]$, then $\pi_i \in I$" means that a car π_i parking after an empty spot must be lucky.

2. The statement "If $\pi_i \in I$ and $\pi_{i+1} \notin I \cup \{\mathtt{X}\}$, then $\pi_i < \pi_{i+1}$" means that anytime we have two consecutively parked cars, if the first car is lucky and second is not, then there is an ascent in the outcome.
3. The statement "If $\pi_{i-1}, \pi_i \in [m]$ and $\pi_{i-1} > \pi_i$, then $\pi_i \in I$" means that, if there are two consecutively parked cars and there is a descent in the outcome, then second car is lucky.

We use these facts and proceed with the technical details next.

($\Rightarrow$) Suppose that $\pi = \pi_1\pi_2\cdots\pi_n = \mathcal{O}(\alpha)$ and that the lucky cars of α are those indexed by the set I. Any car parking after an empty spot is lucky, so it must be in I, which is precisely Condition (1). Also whenever there are two consecutively parked cars π_i and π_{i+1}, if π_i is lucky and the second is not lucky, then car π_{i+1} had to be in the queue later, and hence $\pi_i < \pi_{i+1}$. This is precisely the statement of Condition (2). Moreover, whenever two consecutively parked cars π_{i-1} and π_i satisfy $\pi_{i-1} > \pi_i$, then car π_i arrived and parked prior to car π_{i-1} entering and parking, so it must be a lucky car. This is precisely Condition (3). So π satisfies the given conditions.

($\Leftarrow$) Assume $\pi = \pi_1\pi_2\cdots\pi_m \in \mathfrak{S}_{m,n}$ and $\mathtt{X}\pi$ satisfies Conditions (1), (2), and (3). We use the following notation:

- For all $i \in [n]$, $\pi_i = j$ means car j parked in spot i, and if $\pi_i = \mathtt{X}$, then no car parks in spot i.
- Let $J = \{i \in [n] : \pi_i \in [m]\}$. Further, order the elements of J from smallest to largest so that $J = \{j_1 < j_2 < \cdots < j_m\}$. For each $i \in [m]$, let $\tau_i = \pi_{j_i}$ and define

$$\tau = \tau_1\tau_2\cdots\tau_m. \tag{8}$$

- For any $u \in J$, define $\pi_v^{-1} = u$, which denotes that car v parks in spot u on the street.

Next, we construct a parking function $\alpha = (a_1, a_2, \ldots, a_m) \in \mathrm{PF}_{m,n}$, and we prove that α satisfies $\mathcal{O}(\alpha) = \pi$ and $\mathsf{Lucky}(\alpha) = I$.

To construct α let τ be as defined in Equation (8). For each $\tau_i = 1, 2, \ldots, m$ (in this order) set the value of a_j where $j = \tau_i$ as follows:

Step (1): If $\tau_i = j \in I$, set $a_j = \pi_j^{-1}$.

Step (2): If $\tau_i = j \notin I$, set a_j as follows:

(a) If $\tau_{i-1} \in I$ and $\tau_{i-1} < \tau_i$, then set $a_j = \pi_k^{-1}$ where $k = \tau_{i-1}$.

(b) If $\tau_{i-1} \in I$ and $\tau_{i-1} > \tau_i$, then in fact τ_i would be in I, a contradiction. Thus, this case never happens.

(c) If $\tau_{i-1} \notin I$, then set $a_j = \pi_k^{-1}$ where $k = \tau_{i-1}$.

We must now establish the following claims:

Claim 1. As constructed, α is a parking function in $\mathrm{PF}_{m,n}$.

We now show that the weakly increasing rearrangement of α satisfies the inequality in Equation (7). By construction, for each $i \in [m]$, the entry a_i in α is assigned at most the parking spot where car i ends up parking. Namely,

$$a_i \leq \pi_i^{-1} \tag{9}$$

for all $i \in [m]$. Thus, the entries in $\alpha = (a_1, a_2, \ldots, a_m)$ are entry-wise less than or equal to the entries in $(\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_m^{-1})$. By definition of J, the set $\{\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_m^{-1}\} = J$, where J consists of the indices at which the cars parked given the outcome π. Now, reorder the entries $(\pi_1^{-1}, \pi_2^{-1}, \ldots \pi_m^{-1})$ in weakly increasing order and denote this by $(\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_m^{-1})^{\uparrow}$. Again by definition of J, we have that

$$(\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_m^{-1})^{\uparrow} = j_1 j_2 j_3 \cdots j_m.$$

We now consider $(\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_m^{-1})^{\uparrow}$ as a parking function. We intend to prove that the i^{th} entry in the tuple $(\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_m^{-1})^{\uparrow}$ is less than or equal to $n - m + i$, for all $i \in [m]$. This is equivalent to the claim that $j_i \leq n - m + i$, for each $i \in [m]$. Assume for contradiction that $j_i \geq n - m + i + 1$. Then car i parks at a spot numbered $m - n + i + 1$ or larger. After car i parks, there are at most $n - (n - m + i + 2) + 1 = m - i - 1$ spots remaining at the end of the street, yet there are $m - i$ cars left to park. Hence, some car would be unable to park. This is a contradiction because we started with J being the set of indices at which cars parked with outcome π.

Hence $j_i \leq n - m + i$ for all $i \in [m]$. Then, rearranging the entries of α into weakly increasing order, and denoting this by $\alpha^{\uparrow} = (a'_1, a'_2, \ldots, a'_m)$, together with Inequality (9) ensures that $a'_i \leq m - n + i$ for all $i \in [m]$. Thus, α is a parking function in $\mathrm{PF}_{m,n}$.

Claim 2. The set of lucky cars of α is precisely I, i.e., $\mathsf{Lucky}(\alpha) = I$.

The only preferences used to create α are the values $\pi_{\tau_i}^{-1}$ when $\tau_i \in I$. Moreover, all of the lucky cars in I get the first instance of those

preference as any unlucky car is given the preference of the nearest lucky car parked to its left, which makes those cars unlucky. This implies that the set of lucky cars in α is precisely I.

Claim 3. The outcome of α is π, i.e., $\mathcal{O}(\alpha) = \pi$.

Consider the first car parked on the street, by Equation (8) this is denoted $\tau_1 = \pi_{j_1}$. By definition of J, car $\tau_1 = \pi_{j_1}$ occupies spot j_1 on the street. Since the first parked car on the street is always lucky, the preference for car $\tau_1 = \pi_{j_1}$ was set to be $a_{\tau_1} = \pi_{\tau_1}^{-1}$ in Step (1). As car $\tau_1 = \pi_{j_1}$ is lucky, then it must park in its preference $\pi_{\tau_1}^{-1}$. Whenever we write $\pi_x^{-1} = y$, it means that car x parks in spot y. Thus $\pi_{\tau_1}^{-1} = j_1$ means that car $\tau_1 = \pi_{j_1}$ parks in spot j_1. This implies that $\pi_{j_1} = \tau_1$, as this denotes that car τ_1 parked in spot j_1. Note that this implies that $\pi_0 = \pi_1 = \pi_2 = \cdots = \pi_{\tau_1 - 1} = \mathtt{X}$.

Assume for induction that, for all $1 < \ell < m$, the car τ_ℓ (the ℓ^{th} car parked on the street) with preference a_{τ_ℓ} satisfies $\pi_{j_\ell} = \tau_\ell$.

Now consider the m^{th} car. This is car τ_m. The preference for car τ_m depends on whether the car was lucky or unlucky. We consider each case next.

- If car τ_m is lucky, then in Step (1) we set the preference $a_{\tau_m} = \pi_{\tau_m}^{-1}$. Since car τ_m is the m^{th} car, it has to park in the m^{th} occupied spot on the street, which by definition of J is numbered j_m. So $\pi_{\tau_m}^{-1} = j_m$, since whenever we write $\pi_x^{-1} = y$ means that car x parked in spot y. Since car τ_m parked in spot j_m, we have that $\pi_{j_m} = \tau_m$ as desired.
- If car τ_m is unlucky, then in Step (2) its preference was set to be $a_{\tau_m} = \pi_{\tau_{m-1}}^{-1}$. By induction hypothesis car τ_{m-1} satisfies $\pi_{j_{m-1}} = \tau_{m-1}$. Meaning that car τ_{m-1} parked in the spot numbered j_{m-1}. Car τ_m wants to park in spot $\pi_{\tau_{m-1}}^{-1}$. However, $\pi_{j_{m-1}} = \tau_{m-1}$ implies that $\pi_{\tau_{m-1}}^{-1} = j_{m-1}$. This tells us that car τ_m will find its preferred spot occupied by car τ_{m-1}. Thus, car τ_m will proceed to the next available spot, which by definition of J must be j_m. Therefore, $\pi_{j_m} = \tau_m$.

Thus, all of the m cars with preferences in α have outcome agreeing with the non-$\mathtt{X}$ entries in π. Whenever $j \notin J$, then $\pi_j = \mathtt{X}$. Therefore, $\mathcal{O}(\alpha) = \mathtt{X}\pi$ as claimed.

This completes the proof of Theorem 3.8. □

3.3 Counting outcomes of (m, n)-parking functions where the first k cars are lucky

In Theorem 3.8 we give a complete characterization for the permutations that arise as outcomes of parking functions with a fixed set of lucky cars. We now set some notation to describe those permutations.

Definition 3.9. Fix a lucky set I of $\mathrm{PF}_{m,n}$. Let $\mathcal{O}_{m,n}(I)$ denote the set of outcomes arising from the parking functions in $\mathrm{PF}_{m,n}$ with lucky set I.

We now give a formula for the number of outcomes where the first k cars are lucky. This result generalizes Theorem 2.10 to the case where there are more spots than cars.

Theorem 3.10. *Let $I = \{1, 2, 3, \ldots, k\} \subseteq [m]$ be a lucky set of $\mathrm{PF}_{m,n}$ and let $J = \{j_1, j_2, j_3, \ldots, j_k\} \subseteq [n]$ with $1 \leq j_1 \leq n - m + 1$. Let $\mathrm{WComp}(m-k,\, k)$ denote the set of weak compositions of $m-k$ with k parts. Let $\mathrm{WComp}_J(m - k,\, k)$ be the subset of $\mathrm{WComp}(m - k,\, k)$ for which $(t_1, t_2, \ldots, t_k) \in \mathrm{WComp}(m - k,\, k)$ satisfies $(t_1, t_2, \ldots, t_k) \leq (j_2 - j_1 + 1,\ j_3 - j_2 + 1,\ \ldots,\ n - j_k)$ point-wise. Then*

$$\left|\mathcal{O}_{m,n}(I)\right| \quad = \sum_{(t_1,t_2,t_3,\ldots,t_k)\in \mathrm{WComp}_J(m-k,k)} k!\binom{m-k}{t_1,\, t_2,\, t_3,\, \ldots,\, t_k}.$$

Proof. Let $I = \{1, 2, 3, \ldots, k\} \subseteq [m]$ and fix a set of indices of parking spots $J = \{j_1, j_2, \ldots, j_k\} \subseteq [n]$, satisfying $j_1 < j_2 < \cdots < j_k \leq n$, in which the k lucky cars in I park. The lucky cars in I can park in any order among the spots indexed by J, so there are $k!$ ways for those cars to appear in an outcome permutation. As those cars all park on the street first, we need to now park all of the remaining unlucky cars, of which there are $m-k$. These unlucky cars must park in the gaps between any two consecutively parked lucky cars. If $j_1 > 1$, then there is at least one empty spot at the beginning of the street. Any unlucky car must park to the right of the first lucky car on the street. Namely, any unlucky car must park past spot j_1, as otherwise the unlucky car would be lucky. Moreover, we claim that $j_1 \leq n - m + 1$, which means that the first lucky car can park in any of the first $n - m + 1$ spots on the street. Assume for contradiction that $j_1 > n - m + 1$. Then observe that there are $m - k$ (unlucky) cars left to park after the k lucky cars park. Now, the number of spots to the right of j_1 is $n - j_1 + 1 - k$ and $n - j_1 + 1 - k \leq n - (n - m + 2) + 1 - k = m - k - 1$. This means that there are fewer spots to the right of j_1 than cars left to park. This is a contradiction because all unlucky cars must park to the right of any lucky

car, and if $j_1 > n-m+1$, then some unlucky car would fail to park. Thus, $1 \leq j_1 \leq n-m+1$, as claimed.

Next, the gaps between any two consecutively parked cars have size $j_{i+1} - j_i - 1$ whenever $1 \leq i \leq k-1$, and the gap at the end of the street (from the last parked lucky car in position j_k to the end of the street) has length $n - j_k$. We must select what cars park in each respective gap. Whenever we select cars to park in those gaps, the unlucky cars must park immediately to the right of a lucky car and must appear in increasing order. Moreover, the gaps are limited in size and may contain up to $j_{i+1} - j_i + 1$ cars where $1 \leq i \leq k-1$, while the gap at the end of the street may contain up to $n - j_k$ cars. The number of cars we must park in these gaps must equal the number of unlucky cars, which is $m - k$. As some gaps may not have any unlucky cars parked in them, selecting the number of cars to park in the gaps under these constraints can be accounted by $(t_1, t_2, \ldots, t_k)$, a weak composition of $m-k$ with k parts, which satisfies $(t_1, t_2, \ldots, t_k) \leq (j_2 - j_1 + 1, j_3 - j_2 + 1, \ldots, n - j_k)$ point-wise. Then given the composition $(t_1, t_2, \ldots, t_k)$, the number of ways to select what cars park in each respective gap is given by $\binom{m-k}{t_1, t_2, \ldots, t_k}$. Each selection of cars parking in those gaps must have the selected cars park in increasing order so as to ensure they create no descents in the outcome, which would imply some car not in I is lucky.

Taking the sum over all possible restricted weak compositions in

$$\mathrm{WComp}_J(m-k,\, k),$$

we find that the number of outcomes with lucky set $I = \{1, 2, 3, \ldots, k\}$ is given by

$$\left|\mathcal{O}_{m,n}(I)\right| \quad = \sum_{(t_1, t_2, t_3, \ldots, t_k) \in \mathrm{WComp}_J(m-k,k)} k! \binom{m-k}{t_1,\, t_2,\, t_3,\, \ldots,\, t_k},$$

as claimed. □

We again remark that, although we have a complete characterization for the elements of $\mathcal{O}_{m,n}(I)$ for any lucky set I, giving a closed formula for the cardinality of the set in general remains an open problem.

3.4 Counting (m, n)-parking functions with a fixed set of lucky cars

If $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_{m,n}$ is an outcome of an (m, n)-parking function and car π_i is an unlucky car, then there must be a car parked in spot $i - 1$,

namely $\pi_{i-1} \in [m]$. Moreover, car π_{i-1} had to have arrived earlier than car π_i to park. So $\pi_{i-1} < \pi_i$. Otherwise, if π_{i-1} were an empty spot, then parking in spot i would have only occurred if car π_i had spot i as its preference. This would make car π_i a lucky car, contrary to the initial assumption. Thus the count for the possible preferences of an unlucky car π_i depends on the number of cars that parked contiguously to the left of spot i by cars arriving to park before car π_i entered the street to park. We define this number next.

Definition 3.11. Fix $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_{m,n}$. For all $\pi_i \in [m]$, let $\ell(\pi_i)$ be the length of the longest subsequence $\pi_j\pi_{j+1}\cdots\pi_{i-1}$ with $\pi_t < \pi_i$ for all $j \le t \le i-1$.

We stress that $\ell(\pi_i)$ is only defined when $\pi_i \in [m]$ and not when $\pi_i = \mathtt{X}$, as the latter indicates an empty spot and not a car.

Definition 3.12. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{O}_{m,n}(I)$. For each $i \in [n]$, if $\pi_i \in [m]$, then let $\mathrm{Pref}_I(\pi_i)$ denote the set of preferences of car π_i in $\alpha \in \mathrm{PF}_{m,n}$ satisfying $\mathsf{Lucky}(\alpha) = I$ and whose outcome is π. Moreover, we let

$$\mathcal{O}^{-1}_{m,n}(I,\pi) = \{\alpha \in \mathrm{PF}_{m,n} : \mathsf{Lucky}(\alpha) = I \text{ and } \mathcal{O}(\alpha) = \pi\}$$

be the set of (m,n)-parking functions with lucky set I and outcome π.

We now state a result for counting (m,n)-parking functions with a fixed lucky set and a given outcome. Since this result and its proof are analogous to that of Lemma 2.15, we omit the details here.

Lemma 3.13. *Fix a lucky set $I \subseteq [m]$ of $\mathrm{PF}_{m,n}$ and fix*

$$\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{O}_{m,n}(I).$$

Then for each $\pi_i \in [m]$,

$$\left|\mathrm{Pref}_I(\pi_i)\right| = \begin{cases} 1 & \text{if } \pi_i \in I, \\ \ell(\pi_i) & \text{if } \pi_i \notin I, \end{cases}$$

and the number of possible (m,n)-parking functions α with outcome π and lucky set I is equal to

$$\left|\mathcal{O}^{-1}_{m,n}(I,\pi)\right| = \prod_{i=1}^{n}\left|\mathrm{Pref}_I(\pi_i)\right| \;=\; \prod_{\pi_j \notin I \cup \{\mathtt{X}\}} \ell(\pi_j).$$

We now give a formula for the number of (m,n)-parking functions with a fixed set of lucky cars.

Theorem 3.14. *Fix a lucky set $I \subseteq [m]$ of $\mathrm{PF}_{m,n}$. If $\mathsf{LuckyPF}_{m,n}(I)$ denotes the set of (m,n)-parking functions α with $\mathsf{Lucky}(\alpha) = I$, then*

$$\left|\mathsf{LuckyPF}_{m,n}(I)\right| = \sum_{\pi=\pi_1\pi_2\cdots\pi_n \in \mathcal{O}_{m,n}(I)} \left(\prod_{\pi_j \notin I \cup \{\mathtt{x}\}} \ell(\pi_j) \right). \tag{10}$$

Proof. Fix a lucky set $I \subseteq [m]$ of $\mathrm{PF}_{m,n}$. By Lemma 3.13, for any

$$\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{O}_{m,n}(I)$$

the number of (m,n)-parking functions $\alpha \in \mathsf{LuckyPF}_{m,n}(I)$ with outcome π is given by $\prod_{\pi_j \notin I\cup\{\mathtt{x}\}} \ell(\pi_j)$. To account for all parking functions with lucky set I, we must sum over all permutations π arising as the outcomes of such parking functions. Those are the permutations characterized in Theorem 3.8, and that set is denoted by $\mathcal{O}_{m,n}(I)$. This completes the proof. □

3.5 Weakly increasing (m,n)-parking functions with a fixed set of lucky cars

In this section, we let $\mathrm{PF}^{\uparrow}_{m,n}$ denote the set of parking functions

$$\alpha = (a_1, a_2, \ldots, a_m) \in \mathrm{PF}_{m,n},$$

which are weakly increasing, i.e., $a_i \le a_{i+1}$ for all $i \in [m-1]$. Theorem 3.1 establishes that any subset of $[m]$ including 1 is a lucky set for $\mathrm{PF}^{\uparrow}_{m,n}$. We state this formally next.

Lemma 3.15. *A subset $I \subseteq [m]$ is a lucky set of $\mathrm{PF}^{\uparrow}_{m,n}$ if and only if $1 \in I$.*

For a fixed lucky set I, let

$$\mathsf{LuckyPF}^{\uparrow}_{m,n}(I) = \left\{\alpha \in \mathrm{PF}^{\uparrow}_{m,n} : \mathsf{Lucky}(\alpha) = I\right\},$$

which is the set of all weakly increasing (m,n)-parking functions with lucky set I. Let

$$\mathcal{O}^{\uparrow}_{m,n}(I) = \left\{\pi \in \mathfrak{S}_{m,n} : \mathcal{O}(\alpha) = \pi \text{ for some } \alpha \in \mathsf{LuckyPF}^{\uparrow}_{m,n}(I)\right\}$$

denote the set of outcomes arising from the parking functions in $\mathrm{PF}^{\uparrow}_{m,n}$ with lucky set I.

We begin with an example where we construct the possible outcomes of a weakly increasing (m,n)-parking function with a fixed set of lucky cars.

Example 3.16. Let $m=7$, $n=10$, and $I=\{1,4,5\}$. Since $\alpha \in \mathrm{PF}^{\uparrow}_{7,10}$, the non-X entries in $\mathcal{O}(\alpha)$ must appear in increasing order. So we may begin with the identity permutation in $\mathfrak{S}_7$ and introduce three X's ensuring that the X's are only placed to the left of a lucky car or at the end of the street. This placement of X's ensures that no other car is forced to be lucky. Hence, the only outcomes that are possible are

$$\mathcal{O}^{\uparrow}_{m,n}(I)=\left\{\begin{array}{llll} \texttt{XXX1234567}, & \texttt{XX123X4567}, & \texttt{XX1234X567}, & \texttt{XX1234567X}, \\ \texttt{X123XX4567}, & \texttt{X123X4X567}, & \texttt{X123X4567X}, & \texttt{X1234XX567}, \\ \texttt{X1234X567X}, & \texttt{X1234567XX}, & \texttt{123XXX4567}, & \texttt{123XX4X567}, \\ \texttt{123XX4567X}, & \texttt{123X4XX567}, & \texttt{123X4X567X}, & \texttt{123X4567XX}, \\ \texttt{1234XXX567}, & \texttt{1234XX567X}, & \texttt{1234X567XX}, & \texttt{1234567XXX} \end{array}\right\}.$$

To count these outcomes it suffices to know the number of weak compositions of $n-m=3$ into $|I|+1=4$ parts. This is given by $\binom{n-m+|I|}{n-m} = \binom{6}{3} = 20$, which is the number of outcomes listed above.

Example 3.16 illustrates the proof technique we use in establishing the following result.

Lemma 3.17. *Fix a lucky set $I \subseteq [m]$ of $\mathrm{PF}^{\uparrow}_{m,n}$. Then the number of permutations in $\mathfrak{S}_{m,n}$ that are outcomes of parking functions in $\mathrm{PF}^{\uparrow}_{m,n}$ with lucky set I is given by*

$$\left|\mathcal{O}^{\uparrow}_{m,n}(I)\right| = \tbinom{n-m+|I|}{n-m}.$$

Proof. Let $\alpha \in \mathrm{PF}^{\uparrow}_{m,n}$ and let $\mathcal{O}(\alpha)=\pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_{m,n}$. As α is weakly increasing, the entries in $[m]$ appear in strictly increasing order in $\mathcal{O}(\alpha)$. By Theorem 3.8, entries in $\mathcal{O}(\alpha)$ that are X impact the lucky set. So we may insert X into the identity permutation $123\cdots m \in \mathfrak{S}_m$ only immediately to the left of a lucky car or to the right of m. As there are $|I|+1$ places where we can insert $n-m$ entries X in $123\cdots m \in \mathfrak{S}_m$, this is equivalent to counting the number of weak compositions of $n-m$ into $|I|+1$ parts. It is well-known that the number of weak compositions of $n-m$ into $|I|+1$ parts is given by $\binom{n-m+|I|+1-1}{n-m} = \binom{n-m+|I|}{n-m}$, as claimed. □

Lemma 3.17 gives the number of permutations that are outcomes of weakly increasing (m, n)-parking functions with a fixed set of lucky cars. We now want to characterize those (m, n)-parking functions directly, and to do so we introduce the following definitions and results.

Definition 3.18. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_{m,n}$ be an outcome of a weakly increasing parking function in $\mathrm{PF}^{\uparrow}_{m,n}$ with lucky set I. Hence, $\pi_j = \mathtt{X}$ for $n - m$ distinct values of $j \in [n]$. Partition π (from left-to-right) into subwords $w_1, w_2, w_3, \ldots, w_k$ where either $w_i = \mathtt{X}$ or w_i consists of a maximally long increasing run of positive integers from the set $[m]$. Whenever $w_i = \mathtt{X}$, let $c_i = 0$, and for each $i \in [k]$ with $w_i \neq \mathtt{X}$ let c_i denote the length of the word w_i. We call $\mathbf{c}(\pi) = (c_1, c_2, \ldots, c_k)$ the *weak composition associated to* π.

In Definition 3.18, we use the naming convention of "weak composition" since for any $\mathbf{c}(\pi) = (c_1, c_2, \ldots, c_k)$ we have that $\sum_{i=1}^{k} c_i = m$ and $n - m$ of the terms in the sum are zero. Thus $\mathbf{c}(\pi)$ is a weak composition of m. We illustrate the definition with the next example.

Example 3.19. Let $m = 7$, $n = 10$, and $I = \{1, 4, 5\}$. Consider $\pi = \mathtt{X123X4X567} \in \mathfrak{S}_{7,10}$. Then the weak composition associated to π is given by $\mathbf{c}(\pi) = (0, 3, 0, 1, 0, 3)$. If we had been given the weak composition $(0, 3, 0, 1, 0, 3)$, we can recover π as follows:

- Since $c_1 = 0$, we let $w_1 = \mathtt{X}$.
- Since $c_2 = 3$, we use the smallest $c_2 = 3$ integers in $[7] = \{1, 2, \ldots, 7\}$ and place them in increasing order to construct $w_2 = 123$.
- Since $c_3 = 0$, we let $w_3 = \mathtt{X}$.
- Since $c_4 = 1$, we then use the next smallest $c_4 = 1$ integer(s) in $\{4, 5, 6, 7\} = [7] \setminus \left[\sum_{i=1}^{3} c_i\right]$ to construct $w_4 = 4$.
- Since $c_5 = 0$, we let $w_5 = \mathtt{X}$.
- Since $c_6 = 3$, we then use the next smallest $c_6 = 3$ integers in $\{5, 6, 7\} = [7] \setminus \left[\sum_{i=1}^{5} c_i\right]$ to construct $w_6 = 567$.

Then $\pi = w_1 w_2 \cdots w_6 = \mathtt{X123X4X567}$, which is the permutation we started with.

From Definition 3.18, given an outcome π we can uniquely determine the weak composition associated to π. As Example 3.19 illustrates, the converse is also true, and we prove this next.

Lemma 3.20. *Let $1 \leq m \leq n$. A weak composition*

$$\mathbf{c} = (c_1, c_2, \ldots, c_k)$$

of m with $n - m$ zeros uniquely determines an outcome permutation $\pi \in \mathfrak{S}_{m,n}$ of a weakly increasing (m, n)-parking function.

Proof. Consider the weak composition $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ of m with $n - m$ zeros. We now construct a set of subwords $w_1, w_2, \ldots, w_k$ from the entries in $\mathbf{c}$, and we use those subwords to construct a permutation in $\mathfrak{S}_{m,n}$. This construction is as follows for each $i \in [k]$:

Step (1): If $c_i = 0$, then set $w_i = \mathtt{X}$.

Step (2): If $c_i > 0$, then set

$$w_i = \left(\left(\sum_{j=1}^{i-1} c_j\right) + 1\right)\left(\left(\sum_{j=1}^{i-1} c_j\right) + 2\right)\cdots\left(\left(\sum_{j=1}^{i-1} c_j\right) + c_i\right),$$

which consists of the consecutive integers $\left(\sum_{j=1}^{i-1} c_j\right)+y$, with $1 \leq y \leq c_i$ listed in increasing order. Moreover, if the top index of the sum is larger than the initial index, then the sum is zero.

Then let $\pi = w_1 w_2 \cdots w_k$. By Step (1) there are $n - m$ entries in π that are $\mathtt{X}$, and by Step (2) the numbers in $[m]$ all appear in π exactly once and in increasing order from left to right. This shows that π is an outcome of a weakly increasing (m, n)-parking function with m cars and n spots, as claimed. The fact that π is unique follows from the fact that the construction of the words $w_1, w_2, \ldots, w_k$ are uniquely determined by the entries $c_1, c_2, \ldots, c_k$ of $\mathbf{c}$, respectively. □

Knowing the outcome permutation (or equivalently a weak composition) is not enough information to know the lucky set of an (m, n)-parking function with that outcome (or equivalently weak composition). This is analogous to the fact that descent bottoms of a permutation in $\mathfrak{S}_n$ only gave a subset of the lucky cars of a parking function with a given outcome, a result we established in Lemma 2.6. The main issue is that the lucky set places a restriction on the positive parts appearing in the weak composition associated to an outcome.

For a fixed lucky set I, we wish to determine the set of weak compositions that correspond to outcomes of elements in $\mathsf{LuckyPF}^{\uparrow}_{m,n}(I)$. Before presenting this result we give a few definitions and technical results.

Definition 3.21. Let $\mathbf{c}$ be the associated weak composition of π. Let $\mathbf{p}$ denote the sequence of positive integers in $\mathbf{c}$, appearing in the same order as in $\mathbf{c}$, and let $|\mathbf{p}|$ denote the number of parts of $\mathbf{p}$. We call $\mathbf{p}$ the *composition associated to* π.

Given a composition $\mathbf{p} = (p_1, p_2, \ldots, p_k)$, define its *partial sums list* by

$$\mathbf{s} = (s_0, s_1, s_2, \ldots, s_k),$$

where $s_0 = 0$ and $s_i = \sum_{j=1}^{i} p_j$ for each $1 \leq i \leq k$.

We illustrate Definition 3.21 with the next example.

Example 3.22. Let $\pi = 1234\mathtt{X}56789\mathtt{X}10\mathtt{X}(11)(12)\mathtt{XX}$. The weak composition associated to π is given by $\mathbf{c} = (4, 0, 5, 0, 1, 0, 2, 0, 0)$, and the composition associated to π is $\mathbf{p} = (4, 5, 1, 2)$. The partial sum list is $\mathbf{s} = (0, 4, 9, 10, 12)$.

Not every composition is the composition associated to an outcome of a weakly increasing (m, n)-parking function with a given lucky set. For example, the only composition associated to an outcome of a weakly increasing parking function with lucky set $I = \{1\}$ is $\mathbf{p} = (m)$, because if $\mathbf{p} = (a, b)$ with $a, b > 0$ and $a + b = m$, then the weak composition associated to $\mathbf{p}$ would have at least one zero between the parts a and b in $\mathbf{p}$. However, such a zero would indicate that in π, the outcome permutation, entry $\pi_{a+1} = \mathtt{X}$, implying that car $a + 1$ would be lucky, a contradiction since we assumed that the only lucky car was the first car as $I = \{1\}$.

To determine all of the possible compositions corresponding to an outcome of an element in $\mathsf{LuckyPF}^{\uparrow}_{m,n}(I)$, we must describe how the entries in the lucky set I restrict the types of parts appearing in the associated composition. The main idea is to begin with a tuple $\mathbf{u}$ associated to the lucky set I and use it to construct a set of tuples $\{\mathbf{v}\}$ by replacing consecutive entries in $\mathbf{u}$ with their sum in $\mathbf{v}$, ensuring that the result is a weak composition that comes from an outcome of a weakly increasing (m, n)-parking function with lucky set I.

Before giving the technical definition we illustrate it with an example.

Example 3.23. Let $m = 22$ and $n = 27$. Consider

$$\mathbf{u} = (1, 3, 1, 4, 2, 5, 3, 2, 1),$$

which is a partition of $m = 22$. We place $n - m = 5$ bars in the tuple $\mathbf{u}$. One way to do this is $\big(|1|3, 1, 4, 2|5, 3|2, 1|\big)$. With such a placement of bars we then add the entries between bars to get $\big(|1|10|8|3|\big)$. Then we replace the bars with "," to get the tuple $\mathbf{v} = (1, 10, 8, 3)$. Of course a distinct placement of bars would yield a different result. If instead we had $\big(||1, 3, 1|4, 2|5, 3, 2|1\big)$, then $\mathbf{v} = (5, 6, 10, 1)$. We also could insert all bars at the end or start of $\mathbf{u}$, in which case $\mathbf{v} = (22)$.

We now give the formal definition of the construction illustrated in Example 3.23.

Definition 3.24. Let $1 \leq m \leq n$ and let $\mathbf{u} = (u_1, u_2, \ldots, u_k)$ be a composition of m. Insert $n - m$ bars into $\mathbf{u}$, denoted "|" and delete any bars immediately following the opening parenthesis "(" or immediately preceding the closing parenthesis ")". Compute the sums of any integers between pairs of bars, or a parenthesis and a bar (at the start/end of the tuple) and replace the bars with commas.

Let $\mathcal{S}(\mathbf{u})$ denote the set all of possible tuples constructed in this way, which we refer to as the *good compositions of* $\mathbf{u}$. Whenever we need to specify the number of parts, we let $\mathcal{S}_x(\mathbf{u})$ denote the set of good compositions of $\mathbf{u}$ with exactly $x + 1$ parts.

The set of good compositions of a specific tuple related to the set of lucky cars helps in determining the parts appearing in the compositions of weakly increasing (m, n)-parking functions with a fixed lucky set. We illustrate this next before proceeding to give a more technical result.

Example 3.25. Let $m = 9$, $n = 11$, and $I = \{1, 4, 5\} \subseteq [9]$. Let $\mathbf{u}$ be the tuple constructed from the consecutive differences of the increasing entries in I with a final entry appended that ensures $\mathbf{u}$ is a composition of $m = 9$. Hence, $\mathbf{u} = (4 - 1, 5 - 4, 9 + 1 - 5) = (3, 1, 5)$. Let $k = n - m = 11 - 9 = 2$, then the good compositions of $\mathbf{u}$ are the elements in the set

$$\mathcal{S}\big((3, 1, 5)\big) = \mathcal{S}_0\big((3, 1, 5)\big) \cup \mathcal{S}_1\big((3, 1, 5)\big) \cup \mathcal{S}_2\big((3, 1, 5)\big) \tag{11}$$

where

$$\begin{aligned}
\mathcal{S}_0\big((3, 1, 5)\big) &= \{(9)\} \\
\mathcal{S}_1\big((3, 1, 5)\big) &= \{(4, 5), (3, 6)\}, \\
\mathcal{S}_2\big((3, 1, 5)\big) &= \{(3, 1, 5)\}.
\end{aligned}$$

We now show that each of the compositions of 9 in $\mathcal{S}\big((3,1,5)\big)$ corresponds to a composition associated to an outcome of a weakly increasing $(9,11)$-parking function with lucky set I. By Lemma 3.17, there should be $\binom{11-9+3}{11-9} = \binom{5}{2} = 10$ outcomes. Using the same technique as in Example 3.16 we construct all of the outcome permutations in $\mathfrak{S}_{9,11}$ with lucky set $I = \{1,4,5\}$ and compute their associated composition:

outcome permutation	associated composition
XX123456789	(9)
X123456789X	(9)
123456789XX	(9)
X123X456789	(3,6)
123XX456789	(3,6)
123X456789X	(3,6)
X1234X56789	(4,5)
1234XX56789	(4,5)
1234X56789X	(4,5)
123X4X56789	(3,1,5)

(12)

Confirming that the good compositions of $(3,1,5)$ are precisely the compositions associated to outcomes of weakly increasing $(9,11)$-parking functions with lucky set $I = \{1,4,5\}$.

However, if instead $n = 10$, then, as $n - m = 1$, we would only be able to place a single bar in $(3,1,5)$. So any good composition associated to an outcome would have at most $n - m + 1 = 2$ parts. Hence, $\mathcal{S}\big((3,1,5)\big) = \{(9),(4,5),(3,6)\}$.

Example 3.25 illustrates a general fact: good compositions of $\mathbf{u}$ constructed with at most $n-m$ bars have at most $n-m+1$ parts. Our next result gives a collection of good compositions for a tuple related to the lucky set.

Proposition 3.26. *Let $1 \leq m \leq n$ and fix a lucky set $I = \{1 = i_1 < i_2 < \cdots < i_k\} \subseteq [m]$ of $\mathrm{PF}_{m,n}$. Let $\mathbf{i} = (i_2 - i_1, i_3 - i_2, \ldots, i_{k-1} - i_{k-2}, i_k - i_{k-1}, m+1-i_k)$. If $\mathbf{p}$ is a good composition of $\mathbf{i}$, namely $\mathbf{p} \in \mathcal{S}(\mathbf{i})$, with at most $n-m+1$ parts, then there exists an outcome permutation whose associated composition is $\mathbf{p}$.*

Proof. Let $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ be a good composition of $\mathbf{i}$ with $k \leq n-m+1$. Then by definition, $\sum_{i=1}^{k} p_i = m$. Since $n - m$ is the number of empty

spots on the street, then any outcome of a weakly increasing (m, n)-parking function with lucky set I must include $n - m$ entries equal to $\mathtt{X}$. As $\mathbf{p}$ has k entries, with $k \leq n - m + 1$, then there are $n - m$ zeros available that we can use to construct a weak composition by inserting a 0 in $\mathbf{p}$ between every two positive values (there are $k - 1$ such places to insert a zero), and we append the remaining $n - m - (k - 1)$ zeros to the end of the weak composition, which we denote by

$$\mathbf{c} = (p_1, 0, p_2, 0, \ldots, p_{k-1}, 0, p_k, \underbrace{0, 0, \ldots, 0}_{\substack{n-m-(k-1) \\ \text{terms}}}) = (c_1, c_2, \ldots, c_{n-m+k}).$$

Using the algorithm in the proof of Lemma 3.20, we construct

$$\pi = \underbrace{123 \cdots p_1}_{p_1 \text{ numbers}} \quad \mathtt{X} \quad \underbrace{(p_1 + 1)(p_1 + 2) \cdots \left(\sum_{x=1}^{2} p_x\right)}_{p_2 \text{ numbers}} \quad \mathtt{X} \cdots \mathtt{X}$$

$$\underbrace{\left(\left(\sum_{x=1}^{k-1} p_x\right) + 1\right)\left(\left(\sum_{x=1}^{k-1} p_x\right) + 2\right) \cdots \left(\sum_{x=1}^{k} p_x\right)}_{p_k \text{ numbers}} \quad \underbrace{\mathtt{X} \cdots \mathtt{X}}_{\substack{n-m-(k-1) \\ \text{terms}}}.$$

Since there are $n-m-(k-1)+(k-1) = n-m$ entries in π that are $\mathtt{X}$ and since all of the numbers in $[m]$ appear in π exactly once and in increasing order from left to right, we know that π is an outcome of a weakly increasing (m, n)-parking function, as claimed. Lastly, by Definition 3.18, the weak composition associated to π is exactly $\mathbf{c}$, and the composition associated to π is exactly $\mathbf{p}$, which concludes the proof. □

We now describe how to generate all outcome permutations (equivalently weak compositions) from the set of good compositions.

Proposition 3.27. *Let $1 \leq m \leq n$ and fix a lucky set $I = \{1 = i_1 < i_2 < \cdots < i_k\} \subseteq [m]$ of $\mathrm{PF}_{m,n}$. Let $\mathbf{i} = (i_2 - i_1, i_3 - i_2, \ldots, i_{k-1} - i_{k-2}, i_k - i_{k-1}, m + 1 - i_k)$ and let $\mathbf{p}$ be a good composition of $\mathbf{i}$. Then the following statements hold:*

(1) *The good composition $\mathbf{p}$ of $\mathbf{i}$ is a composition associated to an outcome of a weakly increasing (m, n)-parking function with lucky set I if and only if $\mathbf{p}$ has $x + 1$ parts, where $0 \leq x \leq n - m$.*

(2) *There are*

$$\binom{|I|-1}{x}$$

good compositions of $\mathbf{i}$ *with* $x+1$ *parts, where* $0 \leq x \leq n-m$.

(3) *For each good composition* $\mathbf{p}$ *of* $\mathbf{i}$ *with* $x+1$ *parts, where* $0 \leq x \leq n-m$, *there are*

$$\binom{n-m+1}{n-m-x}$$

weak compositions that are outcomes of a weakly increasing (m,n)*-parking function with lucky set* I.

(4) *The set of weak compositions constructed from the set of good compositions of* $\mathbf{i}$ *with* $x+1$ *parts, for all* $0 \leq x \leq n-m$, *is in bijection with the set* $\mathcal{O}^{\uparrow}_{m,n}(I)$, *which is the set of outcomes of weakly increasing* (m,n)*-parking functions with lucky set* I.

Proof. Proof of Statement (1): ($\Rightarrow$) By Definition 3.24, the good compositions $\mathbf{p}$ of $\mathbf{i}$ are constructed by placing $n-m$ bars in $\mathbf{i}$. These bars can be placed at the start or end of $\mathbf{i}$, or between any two positive integers in $\mathbf{i}$. If x bars are placed in between two positive values in $\mathbf{i}$, then the good composition constructed has exactly $x+1$ parts. Thus, the most parts any good composition of $\mathbf{i}$ can have is $x+1$ where $0 \leq x \leq n-m$, as claimed.

($\Leftarrow$) This direction holds by the construction in Proposition 3.26.

Proof of Statement (2): A good composition $\mathbf{p}$ of $\mathbf{i}$ is uniquely determined by where we place the bars in $\mathbf{i}$. As there are $|I|$ entries in $\mathbf{i}$, then there are $|I|-1$ places where we can insert bars so as to affect the number of parts. By (1) we can insert x bars, where $0 \leq x \leq n-m$, among $|I|-1$ locations between two consecutive values in $\mathbf{i}$. Hence, the number of good compositions $\mathbf{p}$ of $\mathbf{i}$ with $x+1$ parts, where $0 \leq x \leq n-m$, is given by $\binom{|I|-1}{x}$, and as usual $\binom{a}{b} = 0$ whenever $b > a$.

Proof of Statement (3): Given $\mathbf{p}$, a good composition of $\mathbf{i}$ with $x+1$ parts, where $0 \leq x \leq n-m$, by Proposition 3.26, we must use x of the $n-m$ zeros to place between the entries of $\mathbf{p}$ as we construct the weak compositions with $n-m$ zeros, which correspond to outcomes of weakly increasing parking functions with lucky set I. This leaves $x+2$ positions at which we can insert the remaining $n-m-x$ zeros, i.e., between two positive values or at the start or end of $\mathbf{p}$. The number of ways we can do this is equal to the

number of weak compositions of $n-m-k$ into $x+2$ parts, which is given by

$$\binom{(n-m-x)+(x+2)-1}{n-m-x} = \binom{n-m+1}{n-m-x},$$

as claimed.

Proof of Statement (4): By (1), (2), and (3) we have constructed

$$\sum_{x=0}^{n-m} \binom{|I|-1}{x}\binom{n-m+1}{n-m-x}$$

weak compositions of m with $n-m$ zeros. By Proposition 3.26 we have confirmed that all of the constructed weak compositions are outcomes of weakly increasing (m,n)-parking functions with lucky set I. By Lemma 3.17, the number of outcomes of weakly increasing (m,n)-parking functions with lucky set I is given by $\binom{n-m+|I|}{n-m}$. As the sets are finite, it suffices to show that

$$\sum_{x=0}^{n-m} \binom{|I|-1}{x}\binom{n-m+1}{n-m-x} = \binom{n-m+|I|}{n-m}. \tag{13}$$

Equation (13) follows directly from the Chu-Vandermont identity

$$\sum_{k=0}^{\ell} \binom{r}{k}\binom{s}{\ell-k} = \binom{r+s}{\ell} \tag{14}$$

by letting $\ell = n-m$, $k = x$, $r = |I|-1$, and $s = n-m+1$. □

Example 3.28 (Continuing Example 3.25)**.** Let $m = 9$, $n = 11$, and $I = \{1,4,5\} \subseteq [9]$. Then $\mathbf{i} = (4-1, 5-1, 9+1-5) = (3,1,5)$. By Statement (2) in Proposition 3.27, we have $\binom{2}{0} = 1$ good composition of $\mathbf{i}$ with 1 part, $\binom{2}{1} = 2$ good compositions of $\mathbf{i}$ with 2 parts, and $\binom{2}{2} = 1$ good composition of $\mathbf{i}$ with 3 parts. This agrees with Equation (11). By Statement (3) in Proposition 3.27, there are $\binom{3}{2} = 3$ weak compositions associated to compositions of $\mathbf{i}$ with 1 part, $\binom{3}{1} = 3$ weak compositions associated to compositions of $\mathbf{i}$ with 2 parts, and $\binom{3}{0} = 1$ weak composition associated to compositions of $\mathbf{i}$ with 3 parts. This agrees with the data presented in (12). Lastly, by Statement (4) of Proposition 3.27, the number of outcomes of weakly increasing $(9,11)$-parking functions with lucky set $I = \{1,4,5\}$ is

$$\sum_{x=0}^{2} \tbinom{2}{x}\tbinom{3}{2-x} = \tbinom{2}{0}\tbinom{3}{2} + \tbinom{2}{1}\tbinom{3}{1} + \tbinom{2}{2}\tbinom{3}{0} = 10,$$

which agrees with the data presented in (12).

The key take-away from Proposition 3.26 is that, given a lucky set I, we can construct all of the outcomes of weakly increasing (m, n)-parking functions with lucky set I from the set of good compositions of $\mathbf{i}$ with at most $n-m+1$ parts. Next, we work towards enumerating the set of weakly increasing (m, n)-parking functions with a fixed outcome and a fixed set of lucky cars.

When $1 \leq m \leq n$, an outcome permutation $\pi \in \mathfrak{S}_{m,n}$ can be broken into subsets of the street on which the cars park, which we refer to as *occupied sub-streets*. As defined, the composition associated to π corresponds to the sequence of the lengths of the occupied sub-streets. Since the (m, n)-parking functions we are considering are weakly increasing, the cars all park on the street in order 1 to m in those occupied sub-streets. Moreover, the number of preferences of a car depends on whether the car is lucky or unlucky and also on its location within the occupied sub-street it parked in. When a car is lucky and not the first car parked in an occupied sub-street, then we need to determine how far that car is from the start of the occupied sub-street in which it parked. The following function describes this distance and plays a key role in our count for the number of weakly increasing (m, n)-parking functions with a fixed outcome and a fixed set of lucky cars.

Definition 3.29. Let $1 \leq m \leq n$ and fix the set of lucky cars $I = \{1 = i_1 < i_2 < \cdots < i_k\} \subseteq [m]$. Let $\mathbf{i} = (i_2 - i_1, i_3 - i_2, \ldots, i_{k-1} - i_{k-2}, i_k - i_{k-1}, m + 1 - i_k)$ and let $\mathbf{p}$ be a good composition of $\mathbf{i}$. Let $\mathbf{s} = (0, s_1, s_2, \ldots, s_k)$ be the partial sum list of $\mathbf{p}$. Define the function $d_{\mathbf{p}} \colon I \to \mathbb{Z}^+$ by

$$d_{\mathbf{p}}(i) = \begin{cases} 1 & \text{if } i = s_j + 1 \text{ for some } 0 \leq j \leq k, \\ i - (s_j + 1) & \text{if } s_j + 1 < i < s_{j+1} \text{ for some } 0 \leq j \leq k-1, \\ m - (s_{k-1} + 1) & \text{if } i = m. \end{cases}$$

We illustrate Definition 3.29 next.

Example 3.30 (Continuing Example 3.22)**.** Let $m = 12$, $n = 17$, and $I = \{1, 3, 5, 8, 10, 11, 12\} \subseteq [12]$. Then, $\mathbf{i} = (2, 2, 3, 2, 1, 1, 1)$. Consider $\mathbf{p} = (4, 5, 1, 2)$, which is a good composition of $\mathbf{i}$, whose partial sum list is $\mathbf{s} = (0, 4, 9, 10, 12)$. We can readily compute

$$\begin{aligned} d_{\mathbf{p}}(1) &= 1 && \text{since } 1 = 1 + s_0, \\ d_{\mathbf{p}}(3) &= 3 - 1 = 2 && \text{since } s_0 + 1 = 1 < 3 < 4 = s_1, \\ d_{\mathbf{p}}(5) &= 1 && \text{since } s_1 + 1 = 5, \end{aligned}$$

$$\begin{aligned} d_{\mathbf{p}}(8) &= 8-5=3 && \text{since } s_1+1=5<8<9=s_2,\\ d_{\mathbf{p}}(10) &= 1 && \text{since } s_2+1=10,\\ d_{\mathbf{p}}(11) &= 1 && \text{since } s_3+1=11, \text{ and}\\ d_{\mathbf{p}}(12) &= 12-10=2 && \text{since } 12=m. \end{aligned}$$

We now count the number of weakly increasing $(12,17)$-parking functions with lucky set $I=\{1,3,5,8,10,11,12\}$ and outcome

$$\pi = 1234\mathtt{X}56789\mathtt{X}10\mathtt{X}(11)(12)\mathtt{XX}$$

in two ways. First using Lemma 3.13, we have that

$$\prod_{i=1}^{n=17}\left|\mathrm{Pref}_I(\pi_i)\right| \;=\; \prod_{\pi_j\notin I\cup\{\mathtt{X}\}}\ell(\pi_j) = 1\cdot 3\cdot 1\cdot 2\cdot 4 = 24. \tag{15}$$

Now consider the product

$$\frac{\prod_{i=1}^{4}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)} = \frac{3!\ 4!\ 0!\ 2!}{1\cdot 2\cdot 1\cdot 3\cdot 1\cdot 1\cdot 2} = 24. \tag{16}$$

Hence, Equations (15) and (16) give the same count. Next, we prove that the formula given in Equation (16) holds more generally.

In what follows we let

$$\mathcal{O}^{-1,\uparrow}_{m,n}(I,\pi) = \left\{\alpha\in \mathrm{PF}^{\uparrow}_{m,n} : \mathsf{Lucky}(\alpha)=I \text{ and } \mathcal{O}(\alpha)=\pi\right\}$$

be the set of weakly increasing (m,n)-parking functions with lucky set I and outcome π.

Lemma 3.31. *Fix $\pi=\pi_1\pi_2\cdots\pi_n\in\mathfrak{S}_{m,n}$ an outcome of a weakly increasing (m,n)-parking function with lucky set I. Let*

$$\mathbf{p}=(p_1,p_2,\ldots,p_k)$$

be the composition associated to π and

$$\mathbf{s}=(0,s_1,s_2,\ldots,s_k)$$

be the partial sum list of $\mathbf{p}$. Then the number of $\alpha\in\mathrm{PF}^{\uparrow}_{m,n}$ with lucky set I and outcome π is given by

$$\left|\mathcal{O}^{-1,\uparrow}_{m,n}(I,\pi)\right| = \frac{\prod_{i=1}^{k}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)}, \tag{17}$$

where $0!=1$.

Proof. Fix $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_{m,n}$ an outcome of a weakly increasing (m,n)-parking function with lucky set I. The lengths of the occupied sub-streets on which the cars park is given by the composition associated to π, which we denote by $\mathbf{p} = (p_1, p_2, \ldots, p_k)$. As the cars park in order on an occupied sub-street of length p_i, the first has preference for a single spot and is lucky, as it is the first car parking on this occupied sub-street. Of the remaining $p_i - 1$ cars parking on the same occupied sub-street, without accounting for which cars are lucky, simply using the fact that they park in increasing order can have a total of $(p_i - 1)!$ options for their preference so as to end up parking on this sub-street. Namely, the j^{th} car parked on this sub-street can prefer any of the occupied spots to its left, of which there are $j - 1$, and taking the product of $j - 1$ over $2 \leq j \leq p_i$ yields $1 \cdot 2 \cdot 3 \cdots (p_i - 1) = (p_i - 1)!$. Repeating this process over every occupied sub-street yields the numerator

$$\prod_{i=1}^{k} (p_i - 1)!$$

appearing in Equation (17).

However, there may be cars parked on occupied sub-streets that are lucky and are not parked at the start of a sub-street. For any such car, we must reduce their number of preferences down to one possible preference, exactly the location of where they parked. So considering each lucky car $i \in I$ in order from smallest to largest index, by Definition 3.29, the value $d_{\mathbf{p}}(i)$ is the number of preferences we gave lucky car i in the product $\prod_{i=1}^{k}(p_i - 1)!$. Hence, for each lucky car $i \in I$, we must divide by $d_{\mathbf{p}}(i)$ to remove those preferences from the count. Taking the product of $d_{\mathbf{p}}(i)$ over all lucky cars reduces the preferences of those lucky cars back to one, as is necessary for them to be lucky.

Therefore, the number of weakly increasing (m,n)-parking functions with outcome π and lucky set I is given by

$$\frac{\prod_{i=1}^{k}(p_i - 1)!}{\prod_{i \in I} d_{\mathbf{p}}(i)},$$

as claimed. □

We now give our final result, a product formula for the number of weakly increasing (m,n)-parking functions with a fixed set of lucky cars.

Theorem 3.32. *Let $1 \leq m \leq n$ and fix the set of lucky cars $I = \{1 = i_1 < i_2 < \cdots < i_k\} \subseteq [m]$. Let $\mathbf{i} = (i_2 - i_1, i_3 - i_2, \ldots, i_{k-1} - i_{k-2}, i_k - i_{k-1}, m + 1 - i_k)$ and let $\mathcal{S}_x(\mathbf{i})$ denote the set of good compositions of $\mathbf{i}$ with exactly $x + 1$ parts, where $0 \leq x \leq m - n$. Then the number of weakly increasing (m, n)-parking functions with lucky set I is*

$$\left|\mathsf{LuckyPF}^{\uparrow}_{m,n}(I)\right| = \sum_{\substack{\pi=\pi_1\pi_2\cdots\pi_n \\ \pi\in\mathcal{O}^{\uparrow}_{m,n}(I)}} \left(\prod_{\pi_j \notin I \cup \{\mathtt{X}\}} \ell(\pi_j) \right) \tag{18}$$

$$= \sum_{x=0}^{n-m} \left[\sum_{\substack{\mathbf{p}=(p_1,p_2,\ldots,p_{x+1}) \\ \mathbf{p}\in\mathcal{S}_x(\mathbf{i})}} \binom{n-m+1}{n-m-x} \left(\frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)} \right) \right]. \tag{19}$$

Proof. Equation (18) follows directly from the definition of $\mathcal{O}^{\uparrow}_{m,n}(I)$ and Theorem 10. To establish Equation (19) we make the following observations. Let $\mathbf{p}$ be a good composition of $\mathbf{i}$ in $\mathcal{S}_x(\mathbf{i})$. By Statement (1) in Proposition 3.27, the number of parts of $\mathbf{p}$ must be $x+1$ with $0 \leq x \leq n-m$. By Lemma 3.31 every outcome π with the same associated composition $\mathbf{p}$ with $x+1$ parts results in the same number of weakly increasing (m, n)-parking functions with outcome π and lucky set I. By Lemma 3.31, this number is given by

$$\frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)}.$$

By Statement (2) in Proposition 3.27, the number of outcomes π with the same associated composition $\mathbf{p}$ with $x+1$ parts is given by $\binom{n-m+1}{n-m-x}$. So for each good composition in $\mathcal{S}_x(\mathbf{i})$ there are

$$\binom{n-m+1}{n-m-x} \tag{20}$$

distinct outcome permutations π. Moreover, for each outcome π, there are

$$\frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)} \tag{21}$$

(m, n)-parking functions with outcome π and lucky set I. Hence, for a particular good composition the product of Equation (20) and Equation (21) is

$$\binom{n-m+1}{n-m-x} \frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)} \tag{22}$$

and counts the number of (m,n)-parking functions with outcome π having associated composition $\mathbf{p}$ and lucky set I. The result then follows from taking Equation (22) and summing over all elements in $\mathcal{S}_x(\mathbf{i})$ and over all possible values of x, which satisfies $0 \leq x \leq n-m$. This count is given by

$$\sum_{x=0}^{n-m} \left[\sum_{\mathbf{p}=(p_1,p_2,\ldots,p_{x+1})\in\mathcal{S}_x(\mathbf{i})} \binom{n-m+1}{n-m-x} \left(\frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)} \right) \right],$$

and this completes the proof. □

We conclude this section with the following example.

Example 3.33 (Example 3.25 continued)**.** Let $m = 9$, $n = 11$, and $I = \{1,4,5\} \subseteq [9]$. Hence $\mathbf{i} = (3,1,5)$, and the good compositions of $\mathbf{i}$ are (9), $(4,5)$, $(3,6)$, and $(3,1,5)$. Next, we consider each good composition and determine what it contributes to Equation (19).

- If $\mathbf{p} = (9)$, then $x = 0$ and $\mathbf{s} = (0,9)$. Hence, $d_{\mathbf{p}}(1) = 1$, $d_{\mathbf{p}}(4) = 4-(0+1) = 3$, and $d_{\mathbf{p}}(5) = 5-(0+1) = 4$. Thus,

$$\binom{n-m+1}{n-m-x} \left(\frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)} \right) = \binom{3}{2} \left(\frac{8!}{1\cdot 3\cdot 4} \right) = 3\cdot 3360 = 10080.$$

- If $\mathbf{p} = (4,5)$, then $x = 1$ and $\mathbf{s} = (0,4,9)$. Hence, $d_{\mathbf{p}}(1) = 1$, $d_{\mathbf{p}}(4) = 4-(0+1) = 3$, and $d_{\mathbf{p}}(5) = 1$. Thus,

$$\binom{n-m+1}{n-m-x} \left(\frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)} \right) = \binom{3}{1} \left(\frac{3!\cdot 4!}{1\cdot 3\cdot 1} \right) = 3\cdot 48 = 144.$$

- If $\mathbf{p} = (3,6)$, then $x = 1$ and $\mathbf{s} = (0,3,9)$. Hence, $d_{\mathbf{p}}(1) = 1$, $d_{\mathbf{p}}(4) = 1$, and $d_{\mathbf{p}}(5) = 5-(3+1) = 1$. Thus,

$$\binom{n-m+1}{n-m-x} \left(\frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)} \right) = \binom{3}{1} \left(\frac{2!\cdot 5!}{1\cdot 1\cdot 1} \right) = 3\cdot 240 = 720.$$

- If $\mathbf{p} = (3, 1, 5)$, then $x = 2$ and $\mathbf{s} = (0, 3, 4, 9)$. Hence, $d_{\mathbf{p}}(1) = 1$, $d_{\mathbf{p}}(4) = 1$, and $d_{\mathbf{p}}(5) = 1$. Thus

$$\binom{n-m+1}{n-m-x}\left(\frac{\prod_{i=1}^{x+1}(p_i-1)!}{\prod_{i\in I} d_{\mathbf{p}}(i)}\right) = \binom{3}{0}\left(\frac{2!\cdot 0!\cdot 4!}{1\cdot 1\cdot 1}\right) = 48.$$

By Theorem 3.14, the number of weakly increasing $(9, 11)$-parking functions with lucky set $I = \{1, 4, 5\}$ is

$$\left|\mathsf{LuckyPF}^{\uparrow}_{9,11}(I)\right| = 10080 + 144 + 720 + 48 = 10992. \tag{23}$$

Alternatively, we can use the data from Example 3.25 where we established that

$$\mathcal{O}^{\uparrow}_{m,n}(I) = \left\{ \begin{array}{ll} \texttt{XX}123456789, & \texttt{X}123456789\texttt{X}, \\ 123456789\texttt{XX}, & \texttt{X}123\texttt{X}456789, \\ 123\texttt{XX}456789, & 123\texttt{X}456789\texttt{X}, \\ \texttt{X}1234\texttt{X}56789, & 1234\texttt{XX}56789, \\ 1234\texttt{X}56789\texttt{X}, & 123\texttt{X}4\texttt{X}56789 \end{array} \right\}.$$

For each $\pi \in \mathcal{O}^{\uparrow}_{m,n}(I)$, we compute the product in Equation (18):

- If $\pi = \texttt{XX}123456789$, $\texttt{X}123456789\texttt{X}$, or $123456789\texttt{XX}$, then

$$\prod_{\pi_j \notin I\cup\{\texttt{X}\}} \ell(\pi_j) = 1\cdot 2\cdot 5\cdot 6\cdot 7\cdot 8 = 3360.$$

- If $\pi = \texttt{X}123\texttt{X}456789$, $123\texttt{XX}456789$, or $123\texttt{X}456789\texttt{X}$, then

$$\prod_{\pi_j \notin I\cup\{\texttt{X}\}} \ell(\pi_j) = 1\cdot 2\cdot 2\cdot 3\cdot 4\cdot 5 = 240.$$

- If $\pi = \texttt{X}1234\texttt{X}56789, 1234\texttt{XX}56789, 1234\texttt{X}56789\texttt{X}$, or $123\texttt{X}4\texttt{X}56789$, then

$$\prod_{\pi_j \notin I\cup\{\texttt{X}\}} \ell(\pi_j) = 1\cdot 2\cdot 1\cdot 2\cdot 3\cdot 4 = 48.$$

By Equation (18), we have that

$$\left|\mathsf{LuckyPF}^{\uparrow}_{9,11}(I)\right| = 3(3360) + 3(240) + 4(48) = 10992. \tag{24}$$

The counts in Equations (23) and (24) agree.

4 Future work

Although we have a characterization of the outcomes of a parking function with a fixed lucky set (Theorem 2.7), the enumeration of these permutations is quite sensitive to the location in which the lucky cars park, and providing a general formula for the number of outcomes has proven difficult. Thus, it is an open problem to give an enumeration for the set $\left|\mathcal{O}_n(I)\right|$ for any lucky set I of PF_n. In addition, one could consider the analogous question in the case where there are more parking spots than cars.

Remark 4.1. Since the submission of this manuscript, the authors in collaboration with Ferreri and Swartz have a proof of the above open question [15].

In forthcoming work, Barreto et al. [3] consider the case of unit interval parking functions, Fubini rankings, and unit Fubini rankings, providing a variety of results both on the number of lucky cars and the number of such parking functions with a fixed set of lucky cars. One could extend these studies to other specific subsets of parking functions to include interval parking functions and ℓ-interval parking functions [1, 7] and even to generalizations such as parking sequences and assortments [6, 14].

For additional open problems related to parking functions we point the interested reader to the article by Carlson et al. [5].

Acknowledgments

We thank J. Carlos Martínez Mori, Melissa Beerbower, and Grant Shirley for initial conversations at the start of this project. P. E. Harris and L. Martinez were supported in part by the National Science Foundation award DMS-2150434. L. Martinez was also supported by the NSF Graduate Research Fellowship Program under Grant No. 2233066. P. E. Harris was also supported in part through a grant from the Simons Foundation (Travel Support for Mathematicians). The authors would like to thank the anonymous reviewer.

References

[1] T. Aguilar-Fraga, J. Elder, R. E. Garcia, K. P. Hadaway, P. E. Harris, K. J. Harry, I. B. Hogan, J. Johnson, J. Kretschmann, K. Lawson-Chavanu, J. C. M. Mori, C. D. Monroe, D. Quiñonez, D. Tolson III,

and D. A. Williams II, Interval and ℓ-interval rational parking functions, *Discrete Math. Theor. Comput. Sci.* **26**(1) (2024), Permutation Patterns 2023 #10(2024).

[2] Y. Aguillon, D. Alvarenga, P. E. Harris, S. Kotapati, J. C. M. Mori, C. D. Monroe, Z. Saylor, C. Tieu, and D. A. Williams II, On parking functions and the Tower of Hanoi, *Amer. Math. Monthly* **130**(7) (2023), 618–624.

[3] C. Barreto, M. Beerbower, J. Elder, P. E. Harris, L. Martinez, J. L. Ramírez, S. Ramírez, G. Shirley, and J. C. Vásquez, Lucky cars in Fubini rankings and unit interval parking functions, (2025), *To appear on arXiv*.

[4] S. Billey, K. Burdzy, and B. E. Sagan, Permutations with given peak set, *J. Integer Seq.* **16**(6, Article 13.6.1) (2013).

[5] J. Carlson, A. Christensen, P. E. Harris, Z. Jones, and A. Ramos Rodríguez, Parking functions: Choose your own adventure, *College Math. J.* **52**(4) (2021), 254–264, `https://doi.org/10.1080/07468342.2021.1943115`.

[6] D. M. Chen, P. E. Harris, J. C. M. Mori, E. J. Pabón-Cancel, and G. Sargent, Permutation invariant parking assortments, *Enumer. Comb. Appl.* **4**(1) (June 2023), Article S2R4, `http://dx.doi.org/10.54550/ECA2024V4S1R4`.

[7] E. Colaric, R. DeMuse, J. L. Martin, and M. Yin, Interval parking functions, *Adv. in Appl. Math.* **123** (2021), Paper No. 102129, 17, `https://doi.org/10.1016/j.aam.2020.102129`.

[8] L. Colmenarejo, A. Dawkins, J. Elder, P. E. Harris, K. J. Harry, S. Kara, D. Smith, and B. E. Tenner, On the lucky and displacement statistics of Stirling permutations, *J. Integer Seq.* **27**(6) (2024), Article 24.6.7.

[9] L. Colmenarejo, P. E. Harris, Z. Jones, C. Keller, A. R. Rodríguez, E. Sukarto, and A. R. Vindas-Meléndez, Counting k-Naples parking functions through permutations and the k-Naples area statistic, *Enumer. Comb. Appl.* **2**(2) (June 2021), Article S2R11, `https://ecajournal.kms-ks.org/Volume2021/ECA2021_S2A11.pdf`.

[10] A. Cruz, P. E. Harris, K. J. Harry, J. Kretschmann, M. McClinton, A. Moon, J. O. Museus, and E. Redmon, On some discrete statistics of parking functions, *arXiv:2312.16786* (2024).

[11] R. Davis, S. A. Nelson, T. K. Petersen, and B. E. Tenner, The pinnacle set of a permutation, *Discrete Math.* **341**(11) (2018), 3249–3270.

[12] A. Diaz-Lopez, P. E. Harris, E. Insko, and M. Omar, A proof of the peak polynomial positivity conjecture, *J. Combin. Theory Ser. A* **149** (2017), 21–29, https://doi.org/10.1016/j.jcta.2017.01.004.

[13] A. Diaz-Lopez, P. E. Harris, E. Insko, M. Omar, and B. E. Sagan, Descent polynomials, *Discrete Math.* **342**(6) (2019), 1674–1686, https://doi.org/10.1016/j.disc.2019.01.034.

[14] R. Ehrenborg and A. Happ, Parking cars of different sizes, *Amer. Math. Monthly* **123**(10) (2016), 1045–1048, https://doi.org/10.4169/amer.math.monthly.123.10.1045.

[15] M. Ferreri, P. E. Harris, L. Martinez, and E. Swartz, Enumerating vector parking functions and their outcomes based on specified lucky cars, *arXiv:2508.13917* (2025).

[16] S. J. Franks, P. E. Harris, K. Harry, J. Kretschmann, and M. Vance, Counting parking sequences and parking assortments through permutations, *Enumer. Comb. Appl.* **4**(1) (June 2024), Article S2R2, https://ecajournal.kms-ks.org/Volume2024/ECA2024_S2A2.pdf.

[17] I. Gessel and S. Seo, A refinement of Cayley's formula for trees, *Electron. J. Comb.* **11** (2004/06), R27.

[18] N. González, P. E. Harris, G. Rojas Kirby, M. Smit Vega Garcia, and B. E. Tenner, Pinnacle sets of signed permutations, *Discrete Math.* **346**(7) (2023), Paper No. 113439, 12, https://doi.org/10.1016/j.disc.2023.113439.

[19] P. E. Harris, B. M. Kamau, J. C. Martínez Mori, and R. Tian, On the outcome map of MVP parking functions: Permutations avoiding 321 and 3412, and Motzkin paths, *Enumer. Comb. Appl.* **3**(2) (2023), Paper No. S2R11, 16.

[20] P. E. Harris, J. Kretschmann, and J. C. M. Mori, Lucky cars and the Quicksort algorithm, *Amer. Math. Monthly* **131**(5) (2024), 417–423.

[21] A. G. Konheim and B. Weiss, An occupancy discipline and applications, *SIAM J. Appl. Math.* **14**(6) (1966), 1266–1274.

[22] OEIS Foundation Inc., The catalan numbers, entry A000108 in the the On-Line Encyclopedia of Integer Sequences, (2023), https://oeis.org/A000108, *Published electronically at http://oeis.org*.

[23] P. R. F. Schumacher, Descents in parking functions, *J. Integer Seq.* **21**(2) (2018), Article 18.2.3.

[24] H. Shin, A new bijection between forests and parking functions, *arXiv:0810.0427* (2008), `https://arxiv.org/pdf/0810.0427`.

[25] S. Spiro, Subset parking functions, *J. Integer Seq.* **22**(7) (2019), Article 19.7.3.

Pamela E. Harris
Department of Mathematical Sciences,
University of Wisconsin-Milwaukee,
Milwaukee, WI 53211
peharris@uwm.edu
0000-0002-3049-991X

Lucy Martinez
Department of Mathematics,
Rutgers University,
Piscataway, NJ 08854
lucy.martinez@rutgers.edu
0000-0003-3144-4053