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When is the product of claw-free perfect graphs elementary?

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Abstract. A graph G is said to be *perfect* if, for every induced subgraph H of G, we have $\chi(H) = \omega(H)$, where $\chi(H)$ and $\omega(H)$ are the chromatic number and the clique number of H, respectively. A graph is said to be *claw-free* if it does not have the complete bipartite graph $K_{1,3}$ as an induced subgraph. The class of claw-free perfect graphs is an important subclass of perfect graphs. A graph G is said to be an elementary graph if its edges can be colored with two colors such that every induced path of length two has its edges colored distinctly. In this paper, we answer the title question for the four standard graph products.

1 Introduction

All the graphs considered in this paper are finite, undirected, and without loops. The complement of a bipartite graph is called *cobipartite graph*. A graph is said to be a *multigraph* if it contains parallel edges. A graph G is said to be *perfect* if, for every induced subgraph H of G, we have $\chi(H) = \omega(H)$, where $\chi(G)$ and $\omega(H)$ are the chromatic number and the clique number of H, respectively. A graph is said to be *claw-free* if it does not have the complete bipartite graph $K_{1,3}$ as an induced subgraph. The class of claw-free perfect graphs is an important subclass of perfect graphs.

A graph G is said to be an *elementary graph* [1] if its edges can be colored with two colors such that every induced path of length two has its edges colored distinctly. Suppose the colors given to the edges are pink and green. Let the pink graph of a given elementary graph G be the spanning subgraph of G with edge set as those edges colored in pink and the green graph of G be the spanning subgraph of G with edge set as those edges colored in

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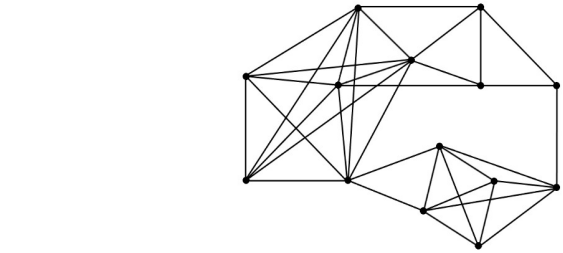
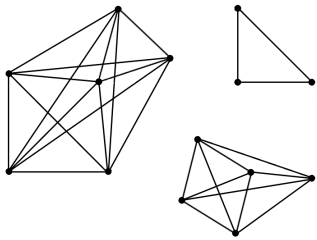
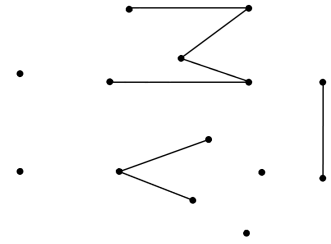


(a) An elementary graph



(b) The pink graph



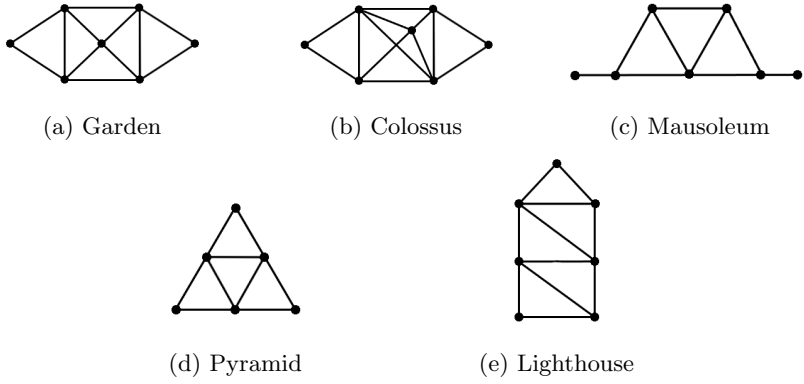
(c) The green graph

Figure 1.1: An elementary graph, its pink graph, and its green graph.

green. In Figure 1.1, we have shown an elementary graph (Figure 1.1a) and its corresponding pink graph (Figure 1.1b) and green graph (Figure 1.1c). Maffray and Reed [4] proved two necessary and sufficient conditions for an elementary graph.

Theorem 1.1 (Maffray and Reed [4])**.** *A graph G is an elementary graph if and only if it does not contain a claw, an odd hole, an odd antihole, or any of the five graphs given in Figure 1.2.*

To give the second condition on elementary graph, we define what is called augmentation of a graph G. Let $e = xy$ be an edge in G that does not lie in a triangle. Such an edge is called a *flat edge*. Let $H = (X, Y, E_{XY})$ be a cobipartite graph disjoint from G. We assume E_{XY} is non-empty and that $X \cup Y$ does not induce a clique. We construct a new graph by deleting the vertices x, y and replacing them with H such that each vertex in X is adjacent to the vertices in $N(x) - \{y\}$ and each vertex in Y is adjacent to the vertices in $N(y) - \{x\}$, where $N(x)$ and $N(y)$ denote the neighbors of x and y, respectively. Then, the resulting graph is called the *augmentation of G along xy* and H is called the *augment*



(a) Garden (b) Colossus (c) Mausoleum

(d) Pyramid (e) Lighthouse

Figure 1.2: Forbidden subgraphs for an elementary graph.

of xy. The augmentation of a graph G is the graph obtained by augmenting h pairwise non-incident flat edges $x_1y_1, x_2y_2, \ldots, x_hy_h$ by h pairwise disjoint cobipartite graphs $H_1, H_2, \ldots, H_h$.

Theorem 1.2 (Maffray and Reed [4])**.** *Let G be a graph. Then, G is elementary if and only if it is an augmentation of a line graph $L(B)$ of a bipartite multigraph B.*

A *peculiar graph* is a claw-free perfect graph. The construction of a peculiar graph is as follows. Let K_n be a complete graph whose set of vertices is split into pairwise disjoint nonempty sets $A_1, B_1, A_2, B_2, A_3, B_3$. For each $i = 1, 2, 3$, remove at least one edge with one endpoint in A_i and the other endpoint in B_{i+1} (subscript is modulo 3). Finally, add pairwise disjoint nonempty cliques Q_1, Q_2, Q_3, and, for each $i = 1, 2, 3$, make each vertex in Q_i adjacent to all the vertices in $K_n - (A_i \cup B_i)$. An example of a peculiar graph with all A_i, B_i, Q_i as K_1 is shown in Figure 1.3.

A graph G is said to have a *clique cutset* if it has a vertex cut that induces a clique. A graph may or may not have a clique cutset. In 1988, Chvatal and Sbihi [1] gave the decomposition for the class of claw-free perfect graphs as follows.

Theorem 1.3 (Chvátal and Sbihi [1])**.** *Every claw-free perfect graph that has no clique cutset is either elementary or peculiar.*

Throughout the paper, we assume all the graphs are with at least one edge. We use $d(x)$ to denote the degree of a vertex x, $\alpha(G)$ the independence

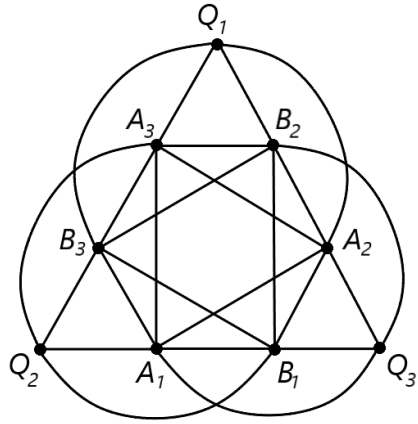


Figure 1.3: Peculiar graph with all A_i, B_i, Q_i as K_1.

number of a graph G, $N(x)$ the set of neighbors of a vertex x, and $G[S]$ the subgraph induced by the set S of vertices or edges.

2 Cartesian product and direct product

There are four standard graph products, namely Cartesian, direct, strong, and lexicographic products of graphs. The *Cartesian product* of two graphs $G = \big(V(G), E(G)\big)$ and $H = \big(V(H), E(H)\big)$, denoted by $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$ and with any two vertices (u, v) and (u', v') being adjacent if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. In 1957, Sabidussi [7] proved that $\chi(G \,\square\, H) = \max\{\chi(G), \chi(H)\}$.

It is easy to observe that if $G \,\square\, H$ is claw-free, then both G and H are claw-free. But the converse is not true. For example, P_n is claw-free; whereas, $P_n \,\square\, P_m$ is not claw-free when $m, n > 1$. We look at the condition when $G \,\square\, H$ has a claw even when G and H are claw-free and obtain the straightforward results stated in the next two lemmas.

Lemma 2.1. *If G or H is not a complete graph, then $G \,\square\, H$ has a claw.*

Lemma 2.2. *Let G and H be connected graphs. Then, $G \,\square\, H$ is claw-free if and only if both G and H are complete graphs.*

Theorem 2.3 (Ravindra and Parthasarathy [6])**.** *The product $G \square H$ is perfect if and only if it has no odd hole.*

Lemma 2.4. *Let G and H be connected claw-free perfect graphs. Then, $G \square H$ is claw-free perfect if and only if both G and H are complete graphs.*

Proof. Let $G \square H$ be claw-free perfect. By Lemma 2.2, both G and H are complete. Conversely, using the same lemma, both G and H being complete implies that $G \square H$ is claw-free. Any induced cycle in $K_m \square K_n$ is such that the edges alternately project to both layers and thus it is an even cycle. So, $K_m \square K_n$ cannot have an induced odd cycle. Hence, $G \square H$ is perfect and therefore claw-free perfect. □

Lemma 2.4 can be generalized to the disjoint union of complete graphs. That is, the graph $G \square H$ is claw-free perfect if and only if both G and H are disjoint unions of complete graphs.

Lemma 2.5. *Every cobipartite graph is elementary.*

Proof. Let G be a cobipartite graph with partite sets X and Y. Let us color the edges of the cliques X and Y with pink and the edges between X and Y with green. As any induced P_3 in G has two vertices in one partite set and the other vertex in the other partite set, the two edges involved are distinctly colored, hence being elementary. □

We now identify what type of claw-free perfect graph $K_m \square K_n$ is for any $m, n \in \mathbb{N}$. In the next theorem, we prove that $K_m \square K_n$ is elementary.

Theorem 2.6. *For any $m, n \in \mathbb{N}$, the graph $K_m \square K_n$ is elementary.*

Proof. Let us color the edges of all K_m with pink and the edges between any two K_m with green. As any induced P_3 in $K_m \square K_n$ has two vertices in one copy of K_m and the other vertex in another copy of K_m, the two edges involved are distinctly colored. Thus, $K_m \square K_n$ is elementary. □

Now, we look at direct products. The *direct product* of two graphs $G = \big(V(G), E(G)\big)$ and $H = \big(V(H), E(H)\big)$, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and with any two vertices (u, v) and (u', v') being adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$. We know that if H is claw-free, then $K_2 \times H$ is also claw-free.

Lemma 2.7. *Let G and H be connected graphs with at least three vertices. If G or H is not a complete graph, then $G \times H$ has a claw.*

Proof. Let x be a vertex in G with $d(x) \geq 2$ and $\{x_1, x_2\} \subseteq N(x)$. Suppose H is not complete and $y \in V(H)$ such that $\{y_1, y_2\}$ is an independent set of $H[N(y)]$. Then, the set of vertices $\{(x, y), (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$ induces a $K_{1,4}$ in $G \times H$. □

Lemma 2.8. *Let G and H be connected graphs. Then $G \times H$ is claw-free if and only if both G and H are complete graphs with at most 3 vertices.*

Proof. Let $G \times H$ be claw-free. Let us assume that G is not complete. Then, there exists a vertex x in G such that $\alpha\big(G(N[x])\big) \geq 2$. If $\alpha\big(G(N[x])\big) \geq 3$ then G has a claw. So for $\alpha\big(G(N[x])\big) = 2$, by Lemma 2.7, $G \times H$ has a claw, which is a contradiction. Similarly, we can prove that H is complete. It is easy to see that $K_m \times K_n$ has a claw if either m or n is greater than 3, since $\{(x_1, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4)\}$ is a claw with center (x_1, y_1), where $x_1, x_2 \in V(K_m)$ and $y_1, y_2, y_3, y_4 \in V(K_n)$. Thus, neither G nor H can have more than three vertices. Conversely, let both G and H be complete graphs with at most three vertices. We have $K_2 \times K_2 \cong 2K_2$ and $K_2 \times K_3 \cong C_6$; both are claw-free. Now for $K_3 \times K_3$, each vertex has degree 4, and for any vertex x the subgraph induced by its neighbors is $2K_2$. So, $K_3 \times K_3$ is also claw-free. □

Theorem 2.9 (Ravindra and Parthasarathy [6])**.** *The direct product $G \times H$ is perfect if and only if either of these holds:*

1. *G or H is bipartite.*
2. *Both G and H have no odd hole or paw as an induced subgraph.*

Lemma 2.10. *Let G and H be claw-free perfect graphs. Then $G \times H$ is claw-free perfect if and only if both G and H are complete graphs with at most 3 vertices.*

Proof. From Lemma 2.8, the lemma holds for claw-free graphs. From Theorem 2.9, the condition of perfectness holds. □

Lemma 2.10 can be generalized to the disjoint union of complete graphs with each component of at most three vertices. That is $G \times H$ is claw-free

perfect if and only if both G and H are disjoint unions of complete graphs with each component of at most three vertices.

We now identify what type of claw-free perfect graph $K_m \times K_n$ is for $m, n \in \{2, 3\}$. In the next theorem, we prove when $K_m \times K_n$ is elementary.

Theorem 2.11. *For any $m, n \in \{2, 3\}$, the graph $K_m \times K_n$ is elementary.*

Proof. The graphs $K_2 \times K_2$ and $K_2 \times K_3$ are elementary as they do not have any of the forbidden subgraphs in Figure 1.2. Now it remains to check $K_3 \times K_3$. We check for the existence of the five forbidden subgraphs. As a colossus has a vertex of degree 5 and $K_3 \times K_3$ is 4-regular, it does not contain a colossus. A garden has a vertex of degree 4 whose neighbors induce a C_4. A mausoleum, a lighthouse, and a pyramid each have a vertex of degree 4 whose neighbors induce a P_4, but every vertex in $K_3 \times K_3$ has degree 4 with neighbors that induce a $2K_2$. So $K_3 \times K_3$ does not contain a garden, mausoleum, lighthouse, or pyramid. Thus $K_3 \times K_3$ is an elementary graph. □

3 Strong product

The *strong product* of two graphs $G = \big(V(G), E(G)\big)$ and $H = \big(V(H), E(H)\big)$, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G \square H) \cup E(G \times H)$. Hammack et al. [3] proved that $\chi(G \boxtimes H) \leq \chi(G)\chi(H)$. Similar to the Cartesian product, the strong product $G \boxtimes H$ being claw-free would imply that G and H are claw-free. But the converse is not true, in general.

Lemma 3.1. *If connected graphs G and H are not complete, then $G \boxtimes H$ has a claw.*

Proof. Since G and H are not complete, there exists a vertex x in G such that $G[N(x)]$ is not complete, and there exists a vertex y in H such that $H[N(y)]$ is not complete. Let $\{x_1, x_2\}$ be an independent set of $G[N(x)]$ and $\{y_1, y_2\}$ an independent set of $H[N(y)]$. Then, the set of vertices $\{(x, y), (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$ induces a $K_{1,4}$ in $G \boxtimes H$. □

As the strong product is commutative, we have $G \boxtimes H \cong H \boxtimes G$. So, for the following results, it holds even when the conditions for G and H are interchanged.

Lemma 3.2. *Let G and H be connected graphs. Then $G \boxtimes H$ is claw-free if and only if G is complete and H is claw-free.*

Proof. Let $G \boxtimes H$ be claw-free. Suppose that H has a claw. Then, $G \boxtimes H$ has a claw. Let G and H be claw-free but not complete graphs. Then, by Lemma 3.1, $G \boxtimes H$ has a claw, which is a contradiction.

We know that $G \boxtimes H$ is claw-free when both G and H are complete, as $G \boxtimes H$ is complete. Suppose, G is complete and H is not complete. Let $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(H) = \{y_1, y_2, \ldots, y_n\}$. Suppose that $G \boxtimes H$ has a claw with center (x_1, y_1). Let us look at the other three vertices in the claw. The neighbors of (x_1, y_1) must be of the form (x_i, y_1) where $x_i x_1 \in E(G)$, of the form (x_1, y_j) where $y_j y_1 \in E(H)$, or of the form $(x_k, y_{k'})$ where $x_1 x_k \in E(G)$ and $y_1 y_{k'} \in E(H)$. If at least two of them are of the form (x_i, y_1) or $(x_k, y_{k'})$, it gives an independent set of size at least two in G, contradicting our assumption of G being a complete graph. If all three of them are of the form (x_1, y_j) or $(x_k, y_{k'})$, it gives a claw in H, contradicting our assumption of H being a claw-free graph. The only possibility for the choice of those three vertices is when two vertices are of the form (x_1, y_j) and one vertex is of the form (x_i, y_1). Let the vertices be (x_1, y_{j_1}), (x_1, y_{j_2}), and $(x_{i'}, y_1)$. As $x_{i'}$ is adjacent to x_1 and y_{j_1}, y_{j_2} are both adjacent to y_1, then both (x_1, y_{j_1}) and (x_1, y_{j_2}) are adjacent to $(x_{i'}, y_1)$. Hence, $G \boxtimes H$ claw-free. □

The *line clique covering number* [5] of a graph G, denoted by $\theta_1(G)$, is the minimum number of cliques containing all the edges of G. A graph G is $\theta_1(G)$-perfect if for any induced subgraph H of G, $\theta_1(H) = \alpha(H)$. Equivalently, a graph G is $\theta_1(G)$-perfect if and only if it is (P_4, C_4)-free.

Theorem 3.3 (Ravindra and Parthasarathy [5])**.** *If G and H are θ_1-perfect, then $G \boxtimes H$ is perfect.*

Lemma 3.4. *Let H be a claw-free perfect graph. Then H is θ_1-perfect if and only if H is either*

1. *a cobipartite graph such that $\overline{H}[E(\overline{H})] \cong K_{1,k}$ for some k or*
2. *the line graph of a bipartite multigraph that is (P_5, C_4)-free.*

Proof. Let H be a claw-free perfect graph. The graph H cannot be both a peculiar graph and a graph with a clique cutset since the five forbidden subgraphs in Figure 1.2 have an induced P_4. So we look at elementary

graphs. Let H be a cobipartite graph. If $\overline{H}[E(\overline{H})]$ is not a $K_{1,k}$ for any k, it has an induced P_4 or $2K_2$. Then, H must have an induced P_4 or C_4, respectively. Let H be an elementary graph other than a cobipartite graph. If H has no augmentations, then H is the line graph of some bipartite multigraph B. So H is (P_4, C_4)-free if and only if B is (P_5, C_4)-free. If H has at least one augmentation, a flat edge xy in $L(B)$ is replaced by a cobipartite graph (X, Y). At least one of X or Y is of size greater than 1. If $X \cup Y \cup \big(N(Y) - X\big) \neq V(H)$ or $X \cup Y \cup \big(N(X) - Y\big) \neq V(H)$, then H has a P_4. We check $X \cup Y \cup \big(N(Y) - X\big) = V(H)$. As H is not cobipartite, $N(Y) - X \neq \emptyset$. When $|X| = 1$, it becomes a cobipartite graph as with option 1 of the theorem statement. Let $|X| > 1$ and let $x \in X$ and $y \in Y$ be a pair of non-adjacent vertices. There exists a vertex $z \in N(Y) - X$ and a vertex other than x in X, say w. So, $xwyz$ is a P_4 in H. Therefore, H is not θ_1-perfect when H has an augmentation. □

Lemma 3.5. *Let G and H be claw-free perfect graphs. Then $G \boxtimes H$ is claw-free perfect when G is complete and H is one of the types in Lemma 3.4.*

Proof. Let G be complete and H be one of the types in Lemma 3.4. By Lemma 3.2, we have $G \boxtimes H$ is claw-free. As both G and H are θ_1-perfect, by Theorem 3.3, $G \boxtimes H$ is perfect. □

Lemma 3.5 can be generalized to the disjoint union of complete graphs and the disjoint union of graphs of form 1 or 2 in Lemma 3.4. That is, the graph $G \boxtimes H$ is claw-free perfect if and only if G is the disjoint union of complete graphs and H is the disjoint union of graphs of form 1 or 2 in Lemma 3.4.

We now identify what type of claw-free perfect graph $K_m \boxtimes H$ is, given H has one of the forms in Lemma 3.4.

Theorem 3.6. *For any claw-free perfect graph H as one of the types in Lemma 3.4, $K_m \boxtimes H$ is elementary.*

Proof.

Case 1: H is a cobipartite graph such that $\overline{H}[E(\overline{H})] \cong K_{1,k}$ for some k.

Let the partite sets of H be X and Y. Since X and Y are complete graphs, then $X \boxtimes K_m$ and $Y \boxtimes K_m$ are complete graphs. As H is cobipartite with $\overline{H}[E(\overline{H})] \cong K_{1,k}$ for some k, it has at least one pair of vertices that are not adjacent. Let $x \in X$ and $y \in Y$. If

$xy \notin E(H)$, the vertices in the copies of K_m corresponding to x and y are not adjacent. If $xy \in E(H)$, all the vertices in the copies of K_m corresponding to x and y are adjacent. Hence, the graph $K_m \boxtimes H$ is a cobipartite graph.

Case 2: H is the line graph of a bipartite multigraph that is (P_5, C_4)-free.

Let G be the line graph of a bipartite multigraph B. The graph $K_m \boxtimes G$ must be the line graph of the bipartite multigraph B' where B' is the bipartite multigraph obtained by replacing each edge in B by m multiple edges. As H is the line graph of a bipartite multigraph that is (P_5, C_4)-free, the graph $K_m \boxtimes H$ is also the line graph of some bipartite multigraph and hence elementary. □

4 Lexicographic product

The *lexicographic product* of two graphs $G = \big(V(G), E(G)\big)$ and $H = \big(V(H), E(H)\big)$, denoted by $G[H]$, is the graph with vertex set $V(G) \times V(H)$ and any two vertices $(u, v), (u', v')$ are adjacent if and only if $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. The product $G[H]$ being claw-free would imply that G and H are claw-free. But the converse is not true, in general. Let $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(H) = \{y_1, y_2, \ldots, y_n\}$. We denote y_{ij} to be the i^{th} vertex in the j^{th} copy of H in $G[H]$.

Lemma 4.1. *Let G and H be claw-free graphs. If G has at least one edge and $\alpha(H) \geq 3$, then $G[H]$ has a claw.*

Proof. Let $\{y_1, y_2, y_3\}$ be an independent set of H and x_1x_2 be an edge in G. Then, $\{y_{i1}, y_{12}, y_{22}, y_{32}\}$ induces a claw with center y_{i1} for any i. □

Lemma 4.2. *Let G and H be connected graphs. If $\alpha(H) = 2$ and G is not complete, then $G[H]$ has a claw.*

Proof. Let $\{y_1, y_2\}$ be an independent set of H. Let x_2 be the vertex in G such that $\{x_1, x_2, x_3\}$ is an induced P_3 in G. Then, $\{y_{11}, y_{12}, y_{21}, y_{31}, y_{32}\}$ induces a $K_{1,4}$ in $G[H]$. □

Lemmas 4.1 and 4.2 are true for any two graphs G and H.

Lemma 4.3. *Let G and H be connected graphs. Then $G[H]$ is claw-free if and only if either*

1. *G is a complete graph and $\alpha(H) = 2$ or*
2. *G is claw-free and H is a complete graph.*

Proof. Given G and H are connected graphs, suppose $G[H]$ is claw-free. Then by Lemma 4.1, $\alpha(H) < 3$. If $\alpha(H) = 2$, then by Lemma 4.2, $G[H]$ is claw-free only when $\alpha\big(G[N(x)]\big) = 1$ for all $x \in G$. Therefore, G is complete, and if $\alpha(H) = 1$, then H is a complete graph. Conversely, we try to construct a claw in either case. Let $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(H) = \{y_1, y_2, \ldots, y_n\}$.

Case 1: G is claw-free and $H = K_n$.

Since G is claw-free, then for any vertex x_i we have $\alpha\big(G[N(x_i)]\big) \leq 2$. If $\alpha\big(G[N(x_i)]\big) = 1$ for all x_i, then G is complete, and hence $G[H]$ is complete. Let us assume that for some vertex, say x_2, we have that $\alpha\big(G[N(x_i)]\big) = 2$ and $\{x_1, x_2, x_3\}$ induces a P_3. We try to construct a claw with center y_{12}. Let y_{11} be the first vertex. The next vertex should be y_{13}, as any other vertex in the first copy of H would induce a triangle. The third vertex should be from a copy of H other than the first and third, as choosing such a vertex would form a triangle. So, if x_2 has no neighbors other than x_1 and x_3, then $G[H]$ cannot have a claw. If x_2 has a neighbor other than x_1 and x_3, say x_4, it is adjacent to x_1 or x_3 or both. Thus, any vertex y_{i4} from the fourth copy of H would form a triangle with y_{11} or y_{13} or both, respectively. Hence, $G[H]$ cannot have a claw when $H = K_n$.

Case 2: $G = K_m$ and $\alpha(H) = 2$.

Let $\{y_1, y_2\}$ be an independent set of H. We try to construct a claw with center y_{12}. Let y_{11} be the first vertex. The next vertex should be y_{21} as it does not induce a triangle. As $\alpha(H) = 2$, any other vertex in H is adjacent to y_1 or y_2 or both. So the third vertex cannot be chosen from the first copy, as it would form a triangle. Let us choose the fourth vertex from a copy of H other than the first and second copy of H, say y_{ij}. It is adjacent to both y_{11} and y_{21}, as G is a complete graph. Hence, $G[H]$ cannot have a claw when $G = K_m$ and $\alpha(H) = 2$. □

Theorem 4.4 (Ravindra and Parthasarathy [6])**.** *Let G and H be two graphs. Then, $G[H]$ is perfect if and only if both G and H are perfect.*

Lemma 4.5. *Let H be a perfect graph. Then $\alpha(H) = 2$ if and only if H is either*

1. *a cobipartite graph other than a complete graph;*
2. *a graph where $V(H)$ can be partitioned into three cliques G_1, G_2, and G_3 such that each vertex in G_1 is adjacent to all the vertices in G_2 or G_3 and there is no adjacency between G_2 and G_3; or*
3. *a disjoint union of two complete graphs.*

Proof. It is easy to see that if the three conditions hold, then $\alpha(H) = 2$. Conversely, assume that H is connected and claw-free perfect with $\alpha(H) = 2$. The graph H cannot be peculiar as the peculiar graph has independence number 3.

Case 1: H is an elementary graph.

We show that, if H is elementary, then it is cobipartite. Let us assume that H is an elementary graph other than a cobipartite graph. If H is not cobipartite, then $\overline{H}$ contains an odd cycle. As $\alpha(H) = 2$, then $\overline{H}$ cannot contain a C_3 or C_{2n+5} for $n \in \mathbb{N}$. So the only possibility is C_5. We have $\overline{H}$ contains a C_5 if and only if H contains a C_5. But H being elementary, it cannot have an odd hole. This gives a contradiction. Hence H is cobipartite other than a complete graph.

Case 2: H has a clique cutset.

Let G_1 be a clique cutset of H. Since vertices from different components of $G - G_1$ form an independent set, $G - G_1$ should have exactly two components: G_2 and G_3. If any one of these, say G_2, has a pair of non-adjacent vertices u, v, then u, v, and a vertex in G_3 must be an independent set of size 3, which is not possible since $\alpha(H) = 2$. Therefore, both G_2 and G_3 are complete graphs. Now suppose that there is a vertex x in G_1 that is not adjacent to all vertices in G_2 or G_3. Let x be non-adjacent to y in G_2 and z in G_3. Then, $\{x, y, z\}$ forms an independent set of size 3, which is not possible. If H is claw-free perfect with $\alpha(H) = 2$ but not connected, then the only possibility is that H is the disjoint union of two complete graphs. □

Lemma 4.6. *Let G and H be perfect graphs. Then, $G[H]$ is claw-free perfect if and only if either of these holds:*

1. *G is claw-free, and H is a complete graph.*
2. *G is a complete graph, and H has one of the forms in Lemma 4.5.*

Proof. Let $G[H]$ be claw-free perfect. By Lemma 4.3, if $G[H]$ is claw-free, then either G is claw-free while H is complete or G is complete while $\alpha(H) = 2$. If H is perfect with $\alpha(H) = 2$, Lemma 4.5 gives the three different possible structures for H. Conversely, using Lemma 4.3, if G and H are perfect graphs such that $H = K_n$ is complete or G is complete and $\alpha(H) = 2$, then $G[H]$ is claw-free. By Theorem 4.4, as both G and H are perfect, $G[H]$ is perfect and hence claw-free perfect. □

We now identify what type of claw-free perfect graph $G[H]$ is. We know that any claw-free perfect graph belongs to any one of peculiar, elementary, or with clique cutset. In the next theorem, we prove $G[K_m]$ is the same type (peculiar/elementary/with clique cutset) as G.

Theorem 4.7. *If G is peculiar, elementary, or has clique cutset, then $G[K_m]$ is peculiar, elementary, or has clique cutset, respectively.*

Proof.

Case 1: G is a peculiar graph.

Gravier et al. proved a lemma that if a graph is $(K_{1,3}, C_5)$-free and contains a peculiar graph, then the graph is peculiar [2, Lemma 2.1]. As $G[K_m]$ is claw-free, perfect, and contains a peculiar graph, we can use this lemma and see that $G[K_m]$ is peculiar.

Case 2: G is an elementary graph.

Then, G is the augmentation of the line graph of some bipartite multigraph B. Let the edges augmented in $L(B)$ be $e_1, e_2, \ldots, e_k$ and let $\{v_1, v_2, \ldots, v_{k'}\} = V\big(L(B)\big) - V'$ where V' is the set of endpoints of all e_i. Let B' be the bipartite multigraph obtained from B by replacing the edges corresponding to $\{v_1, v_2, \ldots, v_{k'}\}$ in $L(B)$ by m multiple edges. Then $L(B')$ is the graph after replacing each v_i by K_m. Thus, $G[K_m]$ can be obtained by augmenting the edges $e_1, e_2, \ldots, e_k$. Thus, $G[K_m]$ is elementary.

Case 3: G has a clique cutset.

Let G have a clique cutset C. It can be seen that the subgraph induced by the copies of K_m corresponding to each vertex in C is a clique cutset in $G[K_m]$. Thus, $G[K_m]$ has a clique cutset. □

Theorem 4.8. *If H is a claw-free perfect graph of one of the types in Lemma 4.5, then $K_m[H]$ is elementary.*

Proof.

Case 1: H is a cobipartite graph with $E(\overline{H}) \neq \emptyset$.

Let H be a cobipartite graph with partite sets X and Y. Let

$$H_1 = (X_1, Y_1),\ H_2 = (X_2, Y_2),\ \ldots,\ H_m = (X_m, Y_m)$$

be the m copies of H corresponding to the m vertices of K_m. It can be seen that

$$X' = X_1 \cup X_2 \cup \cdots \cup X_m$$

and

$$Y' = Y_1 \cup Y_2 \cup \cdots \cup Y_m$$

induce complete graphs in $K_m[H]$. Thus $K_m[H]$ is the cobipartite set with partite sets X' and Y', and hence $K_m[H]$ is elementary.

Case 2: $V(H)$ can be partitioned into three cliques G_1, G_2, and G_3 such that every vertex in G_3 is complete to (i.e., adjacent to all the vertices of) G_1 and/or G_2.

We partition the vertices in G_3 into two parts: G_3' and G_3''. The vertices in G_3' are the vertices in G_3 that are complete to G_1 or both G_1 and G_2, and the vertices in G_3'' are the vertices in G_3 that are complete to G_2 but not to G_1. The m copies of $G_1 \cup G_3'$ and the m copies of $G_2 \cup G_3''$ induce cliques in $K_m[H]$. Let $X = G_1 \cup G_3'$ and $Y = G_2 \cup G_3''$. Then $K_m[H]$ is a cobipartite graph with partite sets X and Y, and thus $K_m[H]$ is elementary.

Case 3: H is the disjoint union of two complete graphs.

Let H be the disjoint union of two complete graphs say H_1 and H_2. The proof is the same as the proof in Case 1 as we can see that H is also a cobipartite graph with partite sets H_1 and H_2. Thus, $K_m[H]$ is elementary. □

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