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The geoconvex number of a graph

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Abstract. For a graph G and vertices u and v, the interval $I(u, v)$ consists of u, v, and all vertices lying on some u-v geodesic in G. A subset $S \subseteq V(G)$ is said to be *convex* if, for every $u, v \in S$, the interval $I(u, v)$ is contained in S. A u-v geodesic is called a *convex geodesic* if $I(u, v)$ is convex, i.e., $I(I(u, v)) = I(u, v)$. A set $S \subseteq V(G)$ is called a *geoconvex set* if every vertex $x \in V(G)$ lies on some u-v convex geodesic in G, where $u, v \in S$. The minimum cardinality of a geoconvex set of G is called the *geoconvex number* of G, denoted by $g_{\text{con}}(G)$. In this paper, we introduce a new parameter and establish a relationship between the geodetic number and the geoconvex number of a graph. We also characterize graphs for which $g_{\text{con}}(G) = n$. Furthermore, we prove that for every triple of positive integers a, b, and n with $2 \leq a \leq b \leq n - 1$ there exists a connected graph of order n such that $g(G) = a$ and $g_{\text{con}}(G) = b$.

1 Introduction

In this paper, we consider finite, undirected, simple graphs. For standard graph-theoretic terminology, we refer the reader to [3]. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest u-v path in G; such a path is called a *u-v geodesic*. For a vertex $v \in V(G)$, the *eccentricity* $e(v)$ is the maximum distance from v to any other vertex in G. The *radius* $\text{rad}(G)$ and *diameter* $\text{diam}(G)$ of G are respectively the minimum and maximum eccentricities among the vertices of G. A vertex v is called a *simplicial vertex* if its neighborhood induces a complete subgraph; in particular, every end-vertex (vertex of degree one) is simplicial.

The *interval* $I(u, v)$ between two vertices u and v consists of u, v, and all vertices lying on some u-v geodesic. For a subset $S \subseteq V(G)$, we define $I(S) = \bigcup_{u,v \in S} I(u, v)$. A set S is called a *geodetic set* of G if $I(S) = V(G)$,

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and the minimum cardinality of such a set is the *geodetic number* $g(G)$. This concept was introduced in [3] and has been extensively studied; see for example [1, 4, 5, 9, 13, 14]. Chartrand and Zhang [8] extended the notion to oriented graphs. A subset $S \subseteq V(G)$ is *convex* if $I(S) = S$. The *convexity number* $\mathrm{con}(G)$ is the maximum cardinality of a convex set that is not equal to $V(G)$. Convexity in graphs has been investigated in [3, 6, 7, 11]. Note that the empty set, all singletons, and $V(G)$ itself are always convex; thus, any graph G with at least three vertices has at least $|V(G)|+2$ convex sets. If this bound is tight, then G is called *g-minimal*. For instance, complete bipartite graphs are g-minimal; see [2]. Every convex set induces a *distance-preserving* subgraph, i.e., a subgraph H such that $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. In [11], the authors characterized the convexity number using the property $\mathcal{E}(k)$, which concerns the nonexistence of distance-preserving convex subgraphs of order k.

Theorem 1.1 (Buckley and Harary [3]). *Let G be a connected graph of order $n \geq 2$. Then*

(i) *$g(G) = n$ if and only if $G = K_n$;*

(ii) *$g(G) = n - 1$ if and only if $G = K_1 + \bigcup_j m_j K_j$ where $2 \leq \sum_j m_j$.*

Theorem 1.2 (Chartrand et al. [5]). *If G is a nontrivial connected graph of order $n \geq 2$ and diameter d, then $g(G) \leq n - d + 1$.*

Theorem 1.3 (Buckley and Harary [3]). *For a connected graph G, $g(G) = 2$ if and only if there exist vertices u, v such that $d(u, v) = \mathrm{diam}(G)$ and every vertex of G lies on some u-v geodesic of G.*

Theorem 1.4 (Chartrand et al. [5]). *Every geodetic set of a graph contains its simplicial vertices.*

Theorem 1.5 (Chartrand et al. [5]). *The geodetic number of a tree T is the number of end vertices in T. In particular, the set of all end vertices of T is the unique minimum geodetic set of G.*

Theorem 1.6 (Chartrand et al. [5]). *For integers $m, n \geq 2$,*

$$g(K_{m,n}) = \min\{m, n, 4\}.$$

The paper is organized as follows. In Section 2, we introduce the geoconvex number $g_{\mathrm{con}}(G)$ and investigate its basic properties. In Section 2.1, we

characterize graphs of order n for which $g_{\mathrm{con}}(G) = n$. In Section 3, we establish realization theorems by proving that for any integers a, b, and n satisfying $2 \leq a \leq b \leq n-1$ there exists a connected graph G of order n such that $g(G) = a$ and $g_{\mathrm{con}}(G) = b$. Finally, we conclude with a discussion of open problems.

2 The geoconvex number

For a graph G and any two vertices u and v, the interval $I(u, v)$—comprising u, v, and all vertices lying on some u-v geodesic—need not be a convex set, especially when u and v are non-adjacent. A graph in which $I(u, v)$ is convex for every pair $u, v \in V(G)$ is called an *interval monotone graph*. Examples of interval monotone graphs include trees and hypercubes. This class of graphs has been studied in [16], and the interval operator $I(u, v)$ remains a fundamental tool for analyzing metric properties of graphs; see [15, 17]. Various variants of geodetic sets have been investigated in the literature, such as the forcing geodetic number $f_g(G)$, the geodetic domination number $\gamma_g(G)$, the connected geodetic number $g_c(G)$, and the edge geodetic number $g_e(G)$; see [10, 12, 19, 20]. In general, for an arbitrary graph G, the only subset of $V(G)$ that is both a geodetic set and a convex set is $V(G)$ itself.

Motivated by interval monotone graphs and g-minimal graphs, we introduce a new parameter that bridges the geodetic number and convexity concepts to better understand interval-related structures in graphs. We say that a u-v geodesic is a *convex geodesic* if its interval $I(u, v)$ is convex, i.e., $I\big(I(u, v)\big) = I(u, v)$. A set $S \subseteq V(G)$ is called a *geoconvex set* if every vertex $x \in V(G)$ lies on some u-v convex geodesic with $u, v \in S$. The minimum cardinality of a geoconvex set of G is called the *geoconvex number* of G, denoted by $g_{\mathrm{con}}(G)$. For disconnected graphs, the geoconvex number is defined as the sum of the geoconvex numbers of all connected components. Hence, throughout this paper we restrict our attention to connected graphs G of order $n \geq 2$. Moreover, if G is g-minimal, then $g_{\mathrm{con}}(G) = n$. This observation highlights a close relationship between g-minimality and the geoconvex number.

Remark 2.1. *For a nontrivial graph, every geoconvex set is a geodetic set, i.e.,* $2 \leq g(G) \leq g_{\mathrm{con}}(G) \leq n$.

Lemma 2.2. *Every geoconvex set of G contains its simplicial vertices. In particular, if the set of all simplicial vertices of G is a geoconvex set of G,*

then the set of all simplicial vertices is the unique minimum geoconvex set of G.

Proof. Since every geoconvex set is a geodetic set, by Theorem 1.4, G contains all its simplicial vertices. □

The following example shows that the second inequality in Remark 2.1 is strict. Consider the graph G shown in Figure 2.1. The vertices v_1 and v_5 are simplicial vertices of G, and the set $\{v_1, v_5, v_6\}$ is a geodetic set of G, which implies that $g(G) \leq 3$. By Theorem 1.3, $g(G) \geq 3$. Hence, $g(G) = 3$.

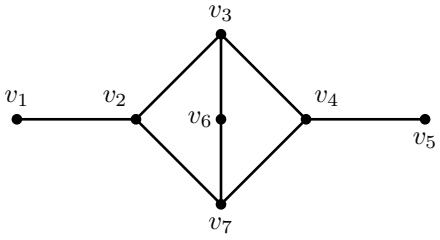

Figure 2.1: A graph G with $g(G) = 3$ and $g_{\text{con}}(G) = 4$.

Moreover, the set $\{v_1, v_5, v_6\}$ is the unique geodetic set of G. However, it is not geoconvex, since the intervals $I(v_1, v_5)$, $I(v_1, v_6)$, and $I(v_5, v_6)$ are not convex. Therefore, $g_{\text{con}}(G) \geq 4$. On the other hand, the set $\{v_1, v_5, v_3, v_7\}$ is geoconvex, so $g_{\text{con}}(G) = 4$.

Observation 2.3. For any connected graph G of order $n \leq 5$, $g_{\text{con}}(G) = g(G)$.

Observation 2.4. For any connected graph G of order $n \geq 2$, $g(G) = g_{\text{con}}(G) = n$ if and only if $G = K_n$.

Observation 2.5.

1. For cycle C_n, $g(C_n) = g_{\text{con}}(C_n)$.
2. For any nontrivial tree T, $g(T) = g_{\text{con}}(T)$.

Observation 2.6. Let G be a connected graph of order $n \geq 2$. Then $g_{\text{con}}(G) = 2$ if and only if $g(G) = 2$.

Lemma 2.7. *If G is a non-complete graph of order $n \geq 3$ and G contains a simplicial vertex, then $g_{\text{con}}(G) \leq n - 1$.*

Proof. Let v be a simplicial vertex of G. Since G is not complete, there exists a vertex x such that $xu \in E(G)$ and $xv \notin E(G)$ for some $u \in N(v)$. Then, $u \in I(x, v)$. Since v is a simplicial vertex, $I(x, v)$ is convex. Hence, $g_{\text{con}}(G) \leq n - 1$. □

Proposition 2.8. *For a complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$, where $n_1 \geq n_2 \geq \cdots \geq n_r$,*

$$g_{con}\big(K_{n_1,n_2,\ldots,n_r}\big) = \begin{cases} n_1 & \text{if } n_i = 1 \text{ for all } i \text{ except } 1; \\ 2 & \text{if } n_i = 2 \text{ for some } i; \\ n_1 + n_2 + \ldots + n_r & \text{otherwise.} \end{cases}$$

Proof. Let G be a complete multipartite graph with vertex partition $V_1, V_2, \ldots, V_r$ with $|V_i| = n_i$.

Case (i) $n_i = 2$ for some i.
Let $V_i = \{x_{i_1}, x_{i_2}\}$. Then, $I(V_i) = V(G)$, and hence it is convex in G. Therefore, $g_{\text{con}}(G) = 2$.

Case (ii) No $n_i = 2$.
Let $u_{i_1}, u_{i_2} \in V_i$ for some $n_i > 1$. Then, $I(u_{i_1}, u_{i_2}) = (V(G) \setminus V_i) \cup \{u_{i_1}, u_{i_2}\}$. But for some $x, y \in I(u_{i_1}, u_{i_2})$, $I(x, y)$ contains some vertices of V_i, so $I(u_{i_1}, u_{i_2})$ is not convex. From this, for any two non-adjacent vertices u, v of G, the interval $I(u, v)$ is not a convex set of G. Therefore, $g_{\text{con}}\big(K_{n_1,n_2,\ldots,n_r}\big) = n_1 + n_2 + n_3 + \cdots + n_r$. If each $n_i = 1$, then G is a complete graph, and $g_{\text{con}}(G) = r$. Assume $n_i = 1$ for all i except for 1. Then, the set V_1 is the unique geoconvex set of G. Thus, $g_{\text{con}}(G) = n_1$. □

2.1 Graphs with $g_{\text{con}}(G) = n$

If $g_{\text{con}}(G) = n$ and G is a non-complete graph, then for every pair of non-adjacent vertices u and v in the cycle, the interval $I(u, v)$ is not convex. Consequently, for any four vertices u, v, x, y that induce a 4-cycle $uvxyu$, there exists a vertex z adjacent either to u and v or to v and y. It was shown in [2] that if G is *g-minimal*—that is, the only convex sets in G are $\varnothing$, all singletons, all pairs of adjacent vertices, and $V(G)$ itself—then for any two non-adjacent vertices u and v there exist at least two internally disjoint u-v geodesics. Moreover, every g-minimal graph satisfies $g_{\text{con}}(G) = n$. However, the converse does not hold. For example, consider a complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$ with $r \geq 2$ and $n_i \geq 2$ for all i (i.e., not a complete graph). Such a graph is not g-minimal, yet $g_{\text{con}}(K_{n_1,n_2,\ldots,n_r}) =$

$n_1 + n_2 + \cdots + n_r = n$. These observations motivate the following lemma, which we prove following the same technique as in [2].

Lemma 2.9. *If $g_{\text{con}}(G) = n$ and G is non-complete, then for any two non-adjacent vertices u, v there exist at least two internally disjoint u-v geodesics in G.*

Proof. We prove this by induction on $d(u, v) \geq 2$. Assume $d(u, v) = 2$. Since $g_{\text{con}}(G) = n$, we have at least two internally disjoint geodesics in G; otherwise, $I(u, v)$ is convex in G. Suppose $d(u, v) = k$ and $k > 2$, and assume that for any two vertices with distance $k - 1$ there exist two internally disjoint geodesics in G. Let $u = v_0, v_1, v_2, \ldots, v_k = v$ be a u-v geodesic in G. Then, $d(u, v_{k-1}) = k - 1$, and by the induction hypothesis there are two internally disjoint u-v_{k-1} geodesics in G, say P and Q. Let x and y be the neighbors of v_{k-1} in P and Q, respectively. Since each interval is not convex, there exists an x-v geodesic $xx'v$ in G, where $x' \neq v_{k-1}$. Similarly, there exists a y-v geodesic $yy'v$ in G, where $y' \neq v_{k-1}$; otherwise, $I(x, v)$ or $I(y, v)$ is convex. Therefore, the geodesic P followed by $xx'v$ and Q followed by $yy'v$ are internally disjoint u-v geodesics in G. □

Corollary 2.10. *If G is a non-complete connected graph of order $n \geq 2$ with diameter d and $g_{\text{con}}(G) = n$, then $g(G) \leq n - 2d + 2$.*

Proof. Let $u, v \in V(G)$ with $d = \text{diam}(G) = d(u, v)$. By Lemma 2.9, there exist at least two internally disjoint u-v geodesics in G. Let it be $P\colon u = u_0, u_1, u_2, \ldots, u_d = v$ and $Q\colon u = v_0, v_1, v_2, \ldots, v_d = v$. Let $S = V(G) \setminus \{u_1, u_2, \ldots, u_{d-1}, v_1, v_2, \ldots, v_{d-1}\}$. Then $I(S) = V(G)$, and hence $g(G) \leq |S|$. It follows that $g(G) \leq n - 2d + 2$. □

Theorem 2.11. *For any graph G of order 6, $g(G)$ is either $g_{\text{con}}(G)$ or $\lceil g_{\text{con}}(G)/2 \rceil$.*

Proof. Suppose $g(G) \neq g_{\text{con}}(G)$. We must show that $g(G) = \lceil g_{\text{con}}(G)/2 \rceil$. Assume that this is not true; that is, $g(G) \neq \lceil g_{\text{con}}(G)/2 \rceil$.

Case (i) Suppose that $g_{\text{con}}(G) = 6$. Since $g(G) \neq g_{\text{con}}(G)$, by Observations 2.4 and 2.5, G is not K_6, it is not a tree, and it is not C_6. By Lemma 2.7, G does not contain simplicial vertices; then, $\text{diam}(G) \leq 3$. If $\text{diam}(G) = 3$, by Corollary 2.10, $g(G) \leq 2$. Thus $\text{diam}(G) = 2$. By Corollary 2.10 $g(G) \leq 4$. Assume that $g(G) = 4$ and $V(G) = \{v_1, v_2, v_3, v_4, x, y\}$. Let

$\{v_1, v_2, v_3, v_4\}$ be a minimum geodetic set of G. Without loss of generality, assume $x \in I(v_1, v_2)$ and $I(v_1, v_2)$ is not convex. Then, $|I(v_1, v_2)|$ is either 5 or 6. Hence, $g(G) = 3$. This leads to a contradiction to our assumption. Hence, $g(G) \leq 3$. But by our assumption, we have $g(G) < 3$. Again, this also leads to a contradiction. Hence, $g_{\text{con}}(G) \in \{3, 4, 5\}$.

Case (ii) If $g_{\text{con}}(G) = 3$, then $g(G) = 2$, which is a contradiction to our assumption.

Case (iii) Suppose $g_{\text{con}}(G) = 4$. Let S be a minimum geodetic set of cardinality 3. Let S' be a minimum geoconvex set of G. Then, $S \cap S' \neq \varnothing$. If $x, y \in S$, then $I(x, y)$ is not convex and $d(x, y) = 2$. Moreover $|I(x, y)| = 4$ and the vertices in $I(x, y)$ different from x and y, say z and w, are not adjacent. Thus G may contain $K_{2,3}$ if the vertex v, which is a vertex in $I\big(I(x, y)\big)$, is adjacent neither to x nor y.

Now, let $V(G) = \{x, y, z, w, v, t\}$. If t is simplicial, then $\{z, w, t\}$ is a minimum geoconvex set of G. Suppose t is not simplicial. Also $t \notin I(x, y)$. If t is not adjacent with x and y, then either t is simplicial (implying $g_{\text{con}}(G) = 3$) or $t \in I(z, w)$ (implying $g_{\text{con}}(G) = 2$). Thus t is adjacent with x or y (not both). Without loss of generality, assume $xt \in E(G)$. Now if $tw, tx, tv \in E(G)$, then $g_{\text{con}}(G) = 5$ (implying $I\big(I(t, y)\big) = I(t, y)$). If $tz, tx, tv \in E(G)$, then $g_{\text{con}}(G) = 5$ (implying $I\big(I(t, z)\big) = I(t, z)$). Finally, if $tx, tv \in E(G)$, then $g_{\text{con}}(G) = 2$.

Case (iv) Assume $g_{\text{con}}(G) = 5$. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $S = V \setminus \{v_6\}$ is a minimum geoconvex set of G and $v_6 \in I(x, y)$ for some $x, y \in S$. Then $I(x, y) = \{x, v_6, y\}$. From this, we can choose two vertices u and v from S (different from x and y), and we have $g(G) = 3$, a contradiction. □

Theorem 2.12. *Let G be a non-complete connected graph of order $n \geq 7$ with $g_{\text{con}}(G) = n$. If S is a minimum geodetic set of G, then $V \setminus S$ is a geodetic set of G.*

Proof. Let S be a minimum geodetic set of G. We have to show that every vertex $x \in S$ lies on some u-v geodesic in G for some u, v in $V \setminus S$. Suppose x does not lie on any u-v geodesic for all $u, v \in V \setminus S$. Then, one of the following holds: (i) $N(x)$ is contained in S. (ii) $V \setminus S$ contains exactly one neighbor of x. (iii) $V \setminus S$ contains more than one neighbor of x such that all neighbors are adjacent.

Case (i) Suppose $N(x)$ is contained in S. Since $\delta(G) \geq 2$, by Lemma 2.7 G contains no simplicial vertices, so S contains no simplicial vertices. Let $N(x) = \{u_1, u_2, u_3, \ldots, u_k\}$ where $k \geq 2$. Let $v \in V \setminus S$. Assume v lies on some x-z geodesic in G for some $x, z \in S$. By Lemma 2.9, there exist two x-z internally disjoint geodesics in G. Let them be P and Q. Suppose v lies on P. Since $N(x)$ is contained in S, we may assume u_i is on P and u_j is on Q. Then, v lies on the u_i-z geodesic and x lies on some u_i-u_j geodesic for some i and j. Thus, $S \setminus \{x\}$ is a minimum geodetic set of G, which leads to a contradiction.

Case (ii) Suppose $x' \in N(x) \cap (V \setminus S)$. **Claim:** $d(x) \geq 3$. Suppose $d(x) = 2$ and $N(x) = \{x', y\}$. Since S is not a complete set, $|S| \geq 3$, there is $z \in S$ such that $xz \notin E(G)$. By Lemma 2.9, there exist two internally disjoint geodesics between x and z, say P_1 and P_2. Without loss of generality, x' lies on P_1 and y lies on P_2. This means that $y \in I(x, z)$. Hence, by the minimality of S, there is some vertex $w \in V \setminus S$ such that $w \in I(y, z')$ for some $z' \in S$. Since y is the only neighbor of x in S, $xz' \notin E(G)$. Since every geodesic between y and z' can be extended to a geodesic between x and z', $S \setminus \{y\}$ is a minimum geodetic set of G, a contradiction. Hence, $d(x) \geq 3$. Also, by Lemma 2.7, $N(x)$ contains at least two non-adjacent vertices. Since S is a minimal geodetic set of G, there exists $w \in V \setminus S$ such that $w \in I(x, z)$ for some $z \in S$. Let $x, x_1, x_2, ..., x_k, z$ be an x-z geodesic and $w = x_i$ for some $i \neq 1$. Suppose $x_1 = x'$. Then, there exists a vertex $y \in N(x) \setminus (V \setminus S)$ such that $yx_2 \in E(G)$. Also, $yz \notin E(G)$ (If $yz \in E(G)$, then $d(x, z) = 2$, $w \notin I(x, z)$). From this, $w \in I(x, z)$ implies $w \in I(y, z)$ for some $z \in S$, and $S \setminus \{x\}$ is a geodetic set of G, a contradiction. Suppose $x_1 \neq x'$. Then, $x_1 \in N(x) \setminus (V \setminus S)$. From this, $w \in I(x_1, z)$. Thus, $S \setminus \{x\}$ is a geodetic set of G, a contradiction. Suppose $x' \in I(x, z)$. If x' is adjacent with some neighbor of x, then $S \setminus \{x\}$ is a geodetic set of G, a contradiction. Suppose not. Then, by Lemma 2.9, there exist two internally disjoint geodesics, say P_1 and P_2. Then, there exists $y \in N(x)$, and y lies on P_2. From this, $w \in I(x, z)$ implies $w \in I(y, z)$ for some $z \in S$, and $S \setminus \{x\}$ is a geodetic set of G, a contradiction.

Case (iii) Suppose $x', x'' \in N(x) \cap (V \setminus S)$ and x' and x'' are adjacent in G. Using similar techniques we used in **Case (ii)**, we arrive at a contradiction. Hence $V(G) \setminus S$ is a geodetic set of G. □

Theorem 2.13. *If G is a noncomplete connected graph G of order $n \geq 7$ with $g_{\text{con}}(G) = n$, then $g(G) \leq \frac{n}{2}$.*

Proof. Let S be a minimum geodetic set of G. By Theorem 2.12, $V(G)-S$ is a geodetic set of G. Since $|S|+|V-S|=n$ and $|S|\leq|V-S|$, it follows $g(G)\leq\frac{n}{2}$. □

Proposition 2.14. *Let G be a connected graph of order $n\geq 6$. If $\mathrm{con}(G)=2$ and $g(G)\geq 3$, then $g_{\mathrm{con}}(G)=n$.*

Proof. If $\mathrm{con}(G)=2$ and $g(G)\geq 3$, then for every non-adjacent vertices u and v in G the interval $I(u,v)(\neq V(G))$ is not convex, and hence $V(G)$ is the only geoconvex set of G. □

The *Cartesian product* $G\times H$ of two graphs G and H has vertex set $V(G\times H)=V(G)\times V(H)$ and two distinct vertices (u,v) and (x,y) of $G\times H$ are adjacent if either (1) $u=x$ and $vy\in E(H)$ or (2) $v=y$ and $ux\in E(G)$. Motivated by the construction of Theorem 3.1 in [5], i.e. $g(G)\leq g(G\times K_2)$, we have the following result.

Proposition 2.15. *If G is a connected graph of order $n\geq 4$, then $g_{\mathrm{con}}(G)\leq g_{\mathrm{con}}(G\times K_2)$. Also, the bound is sharp.*

Proof. Let S be a minimum geodetic set of $G\times K_2$. Let G_1 and G_2 be two copies of $G\times K_2$ with vertex sets V_1 and V_2, respectively. Let S' be the union of those vertices of S in G_1 and the vertices of G_1 corresponding to the vertices of G_2 that belong to S. Let $v\in V_1$. Since S is a geoconvex set, v lies in some convex geodesic x-y, for some $x,y\in S$. Then at least one of x and y belongs to V_1. If both x,y are in V_1, then x,y are in S', and hence S' is a geoconvex set in G_1. Assume x is in V_1 and y is in V_2. If y corresponds to x, then $v=x$ in S'. Therefore, y corresponds to y' in S', $y'\neq x$. Since $d(x,y)=d(x,y')+1$ and the vertex v lies on a x-y geodesic in $G\times K_2$, it follows that v lies in a x-y' geodesic in G_1. Since y' is in $I(x,y)$ and $I(x,y)$ is convex, $I(x,y')\subseteq I(x,y)$. Hence $I(x,y')$ is convex, S' is a geoconvex set of G_1 and $g_{\mathrm{con}}(G)\leq g_{\mathrm{con}}(G\times K_2)$. □

Proposition 2.16. *For integers $m,n\geq 3$, $g_{\mathrm{con}}(K_{m,n}\times K_2)=m+n$.*

Proof. Assume $m\geq n$. By Proposition 2.8, $g_{\mathrm{con}}(K_{m,n})=m+n$, and by Proposition 2.15, $g_{\mathrm{con}}(K_{m,n}\times K_2)\geq m+n$. Let G_1 and G_2 be the two copies of $K_{m,n}$ with $V(G_1)=\{x_1,x_2,\ldots,x_m\}\cup\{y_1,y_2,\ldots,y_n\}$ and $V(G_2)=\{x'_1,\ldots,x'_m\}\cup\{y'_1,\ldots,y'_n\}$. Let $S=\{x'_1,\ldots,x'_m,y_1,y_2,\ldots,y_n\}$. Then each vertex of $K_{m,n}\times K_2$ lies on some x-y geodesic, where $x,y\in S$

and $I(x, y)$ is convex in G. It follows that $g_{\text{con}}(K_{m,n} \times K_2) \leq m+n$. Hence $g_{\text{con}}(K_{m,n} \times K_2) = m + n$. □

3 Realization theorems

In this section, we construct a family of graphs that simultaneously realize prescribed values of the radius, diameter, geodetic number, and geoconvex number. Using Theorem 2.13, we obtain the following result.

Theorem 3.1. *For every positive integer a and every integer $n \geq 7$ satisfying $a \leq \frac{n}{2}$, there exists a connected graph G of order n such that $g(G) = a$ and $g_{\text{con}}(G) = n$.*

Proof. We construct a graph G as follows. Let $F = K_a$ be a complete graph on a vertices with vertex set $V(K_a) = \{v_1, v_2, \ldots, v_{\lfloor \frac{a}{2} \rfloor}\} \cup \{u_1, u_2, \ldots, u_{\lceil \frac{a}{2} \rceil}\}$. Let $H = K_{\lfloor \frac{n-a}{2} \rfloor, \lceil \frac{n-a}{2} \rceil}$ be a complete bipartite graph with vertex partition as $V(K_{\lceil \frac{n-a}{2} \rceil, \lfloor \frac{n-a}{2} \rfloor}) = V' \cup U'$ where $V' = \{v'_1, v'_2, \ldots, v'_{\lceil \frac{n-a}{2} \rceil}\}$ and $U' = \{u'_1, u'_2, \ldots, u'_{\lfloor \frac{n-a}{2} \rfloor}\}$. Then G is obtained from F and H by adding the edges in the following ways: If $|U'| \leq 2$ and $|V'| \leq 2$, then join every vertex u_i to u'_j and v_i is adjacent to v'_j. Suppose $|U'| \geq 2$ and $|V'| \geq 3$ or $|U'| \geq 3$ and $|V'| \geq 2$. If $|U'| = 2$, then join every vertex u_i to u'_j. Similarly, if $|V'| = 2$, then join every vertex v_i to v'_j. In the remaining cases, join every vertex u_i to exactly two vertices of U' such that for any distinct u_i, u_j, we have $\left|N(u_i) \cap N(u_j) \cap U'\right| = 1$. Similarly, join every vertex v_i to exactly two vertices of V' such that for any distinct v_i, v_j we have $\left|N(v_i) \cap N(v_j) \cap V'\right| = 1$. Then for any two non-adjacent vertices u, v in G, $I(u, v)$ is not convex. Thus $g_{\text{con}}(G) = n$. Let $S \subseteq V(H)$ be set of vertices such that $V(F) \subseteq I(S)$. Then S is a geodetic set of G. It follows that $g(G) \leq |S|$ and $g(G) \leq a$. Since $|U| \leq |U'|$ and $|V| \leq |V'|$, every geodetic set containing at least $\lceil \frac{a}{2} \rceil$ vertices from both $U \cup U'$ and $V \cup V'$, which implies $g(G) = a$. □

Figure 3.1 illustrates Theorem 3.1 with $g(G) = 4$ and $g_{\text{con}}(G) = 8$. Let $S = \{v'_1, v'_2, u'_1, u'_2\}$. Then, $I(S) = V(G)$. It follows that $g(G) \leq 4$ and further $g(G) \geq 3$. Every geodetic set of G contains at least two vertices from the set S, and G has a clique of order 4, which implies $g(G) \geq 4$. Therefore, $g(G) = 4$. For every pair of non-adjacent vertices u and v of G, $I(u, v)$ is not a convex subset of G, and hence $g_{\text{con}}(G) = 8$.

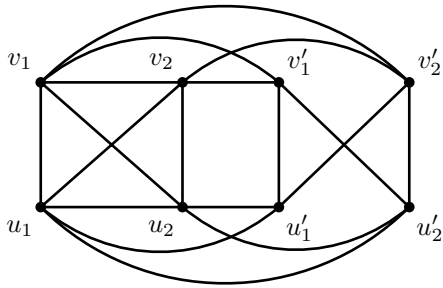

Figure 3.1: A graph G with $a = 4$ and $n = 8$.

Theorem 3.2. *For every three positive integers a, b, and n with $2 \leq a \leq b \leq n-1$, there exists a connected graph G of order n with $g(G) = a$ and $g_{\mathrm{con}}(G) = b$.*

Proof. Assume $a = b$. We take a tree on n vertices with a pendent vertices. Then the set of all simplicial vertices is a geoconvex set of G. Hence $g(G) = g_{\mathrm{con}}(G) = a$.

Assume $b = n-1$. Then the graph G is constructed as follows. Let $H = K_{2,3}$ with $V(H) = \{u_1, u_2\} \cup \{w, v_{a-1}, v_a\}$. The graph G is obtained from H by adding the new vertices $v_1, v_2, \ldots, v_{a-2}, w_1, w_2, \ldots, w_{n-a-3}$ and adding the new edges $v_i w$, $1 \leq i \leq a-2$, $v_{a-1}w_1$, $w_1 v_a$, ww_1, ww_i, $w_i v_a$, $2 \leq i \leq n-a-3$. The vertices $v_1, v_2, \ldots, v_{a-2}$ are the simplicial vertices of G; hence, we have $g(G) \geq a-2$. The set $\{v_1, v_2, \ldots, v_{a-2}, v_{a-1}, v_a\}$ is the unique minimum geodetic set of G. Thus $g(G) = a$. The only convex intervals in G are $I(v_i, w_j)$, where $1 \leq i \leq a-2$ and $1 \leq j \leq n-a-3$. Therefore, $S = V(G) \setminus \{w\}$ is the geoconvex set of G. It follows that $g_{\mathrm{con}}(G) \leq n-1$. But S is the unique minimum geoconvex set of G and hence $g_{\mathrm{con}}(G) = n-1$. By Observation 2.6, we can assume that $a \geq 3$.

Assume $b \leq n-2$. Since $a \geq 3$ and $b \neq a$, by Theorem 2.11, we have $n \geq 7$.

Case (i) Suppose $a = 3$.
Let $H = K_{n-b-1}$ with $V(H) = \{v_1, v_2, \ldots, v_{n-b-1}\}$. Let $F = K_{2,3}$ with $V(F) = \{u_1, u_2\} \cup \{w, u_3, u_4\}$. The graph G is obtained from H and F by identifying w with any vertex of H and adding the new vertices

$$x, y, w_1, w_2, \ldots, w_{b-5}$$

and adding the new edges

$$xv_1,\ u_3y,\ yu_4,\ yu_2,\ v_iw_j,\ w_ju_4,$$

where $1 \leq i \leq n-b-1$ and $1 \leq j \leq b-5$. Let $S = \{x, u_3, u_4\}$. Then S is the geodetic set of G. It follows that $g(G) \leq 3$. But S is the unique minimum geodetic set of G. Thus $g(G) = 3$. The only convex intervals are $I(x, u_1)$, $I(x, u_2)$, $I(x, y)$, and $I(x, w_j)$, where $1 \leq i \leq n-b-1$ and $1 \leq j \leq b-5$. Let $S = V(G) \setminus \{v_1, v_2, \ldots, v_{n-b-1}, u_2\}$. Then S is a geoconvex set of G. Thus $g_{\text{con}}(G) \leq b$. Further S is the unique minimum geoconvex set of G and hence $g_{\text{con}}(G) = b$.

Case (ii) Suppose $a = 4$ and $b = 5$.

Case (ii.a) Suppose $n = 7$.
The required graph G is shown in Figure 3.2. Since v_1 is a simplicial vertex of G, v_1 belongs to every geodetic set of G. Also $\big|I(v_1, v_4)\big| = 4$ and $\big|I(v_1, v_i)\big| = 3$ and $\big|I(v_i, v_j)\big| = 4$ or 5, $1 < i < j$. Hence no 3-element subset of G is a geodetic set of G. It follows that $g(G) \geq 4$. Then $S = \{v_1, v_3, v_5, v_6\}$ and $S' = \{v_1, v_2, v_4, v_7\}$ are the only geodetic sets of G. Hence $g(G) = 4$.

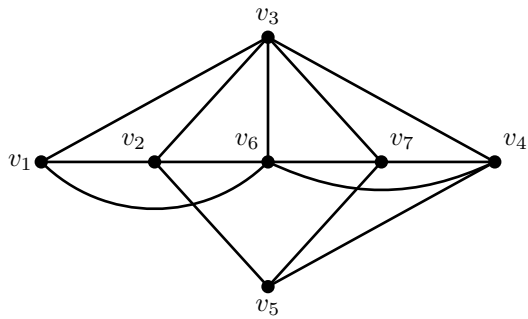

Figure 3.2: A graph G with $g(G) = 4$, $g_{\text{con}}(G) = 5$, and $n = 7$.

Since v_7 lies in v_3-v_5 geodesic in G but the intervals $I(v_3, v_5)$ and $I(v_2, v_4)$ are not convex in G, this implies $g_{\text{con}}(G) \geq 5$. The set

$$T = \{v_1, v_3, v_4, v_5, v_7\}$$

is a geoconvex set of G. Hence $g_{\text{con}}(G) = 5$.

Case (ii.b) Suppose $n \geq 8$.
The graph G is constructed as follows. Let $H = K_{2,3}$ with

$$V(H) = \{u_1, u_2\} \cup \{w, u_3, u_4\}.$$

Let $F = K_{n-7}$ with $V(F) = \{v_1, v_2, \ldots, v_{n-7}\}$. The graph G is obtained by identifying w with any vertex of F and by adding the new vertices

$$z, x, y$$

and adding the new edges

$$xv_1,\ yv_1,\ u_3z,\ zu_4,\ yz,\ zu_1,\ v_iz,\ v_iu_3,$$

where $1 \leq i \leq n-7$. Since $S = \{x, y, u_3, u_4\}$ is a geodetic set of G, we have $g(G) \leq 4$. But S is the unique minimum geodetic set of G, $g(G) = 4$. The only convex intervals in G are $I(x, z)$, $I(y, u_1)$, $I(x, u_2)$, $I(y, u_2)$, and $I(y, u_4)$. From this,

$$S = V(G) \setminus \{v_1, v_2, \ldots, v_{n-7}, u_3, z\}$$

is the unique minimum geoconvex set of G, and hence $g_{\text{con}}(G) = 5$.

Case (iii) Suppose $a = 4$ and $b \geq 6$.
The graph G is constructed as follows. Let $H = K_{2,3}$ with

$$V(H) = \{u_1, u_2\} \cup \{w, u_3, u_4\}.$$

Let $F = K_{n-b-1}$ with $V(F) = \{v_1, v_2, \ldots, v_{n-b-1}\}$. The graph G is obtained by identifying w with any vertex of F and by adding the new vertices

$$z, x, y, w_1, w_2, \ldots, w_{b-6}$$

and adding the new edges

$$xv_1,\ yv_1,\ v_iu_3,\ v_iz,\ u_3z,\ zu_4,\ w_ju_3,\ w_ju_4,$$

where $1 \leq i \leq n-b-1$ and $1 \leq j \leq b-6$. Let $S = \{x, y, u_3, u_4\}$. Then S is the unique minimum geodetic set of G. Thus $g(G) = 4$. The only convex intervals are $I(x, z)$, $I(y, u_1)$, $I(x, u_1)$, $I(x, u_2)$, $I(y, u_2)$, and $I(v_i, w_j)$, where $1 \leq i \leq n-b-1$ and $1 \leq j \leq b-6$. Then

$$S = \{x, y, z, u_1, u_2, u_4, w_1, w_2, \ldots, w_{b-6}\}$$

is the unique minimum geoconvex set of G and hence $g_{\text{con}}(G) = b$.

Case (iv) $a \geq 5$.

Case (iv.a) Suppose $b = n-2$.
Construct a graph G as follows. Let $H = K_{3,b-a+2}$ with vertex set

$V(H) = U \cup V$, where $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, \ldots, v_{b-a+2}\}$. Then the graph G is obtained from H by adding the new vertices

$$w_1, w_2, \ldots, w_{a-3}$$

and the edges

$$w_i u_1,\ w_i v_1,$$

where $1 \leq i \leq a-3$. Then $w_1, w_2, \ldots, w_{a-3}$ are simplicial vertices of G. Thus $g(G) \geq a-3$. Let $S = \{w_1, w_2, \ldots, w_{a-3}, u_1, u_2, u_3\}$. Then S is a geodetic set of G. It follows that $g(G) \leq a$. Since S is the unique minimum geodetic set, we have $g(G) = a$. Clearly $b - a + 2 \geq 3$, the intervals $I(w_i, u_j)$ and $I(w_i, v_l)$ are the only convex sets in G, $1 \leq i \leq a-3$, $1 \leq j \leq 3$, $1 \leq l \leq b-a+2$, and $S' = V(G) \setminus \{u_1, v_1\}$ is a geoconvex set of minimum cardinality of G. Hence $g_{\mathrm{con}}(G) = b$.

Case (iv.b) Assume $b < n-2$.
Construct a graph G as follows. Let $H = K_{3,b-a+2}$ with vertex set $V(H) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, \ldots, v_{b-a+2}\}$. If $n - b - 2 \neq 1$, then G is obtained from H by adding the new vertices

$$w_1, w_2, \ldots, w_{a-3}, x_1, x_2, \ldots, x_{n-b-2},$$

the edges

$$w_i u_1,\ w_i v_1,$$

where $1 \leq i \leq a-3$ and the edges

$$w_1 x_j,\ w_2 x_j,$$

where $1 \leq j \leq n-a-2$. Then $w_3, w_4, \ldots, w_{a-3}$ are simplicial vertices of G, and $S = \{w_1, w_2, \ldots, w_{a-3}, u_1, u_2, u_3\}$ is a geodetic set of G. Hence $g(G) \leq a$. Since S is the unique minimum geodetic set, we have $g(G) = a$. The only convex intervals in G are $I(w_1, w_2)$, $I(w_i, u_j)$, and $I(w_i, v_l)$, where $1 \leq i \leq a-3$, $1 \leq j \leq 3$, and $1 \leq l \leq b-a+2$. Let $S' = \{w_1, w_2, \ldots, w_{a-3}, u_2, u_3, v_2, v_3, \ldots, v_{b-a+2}\}$. Then S' is a geoconvex set of G. Thus $g_{\mathrm{con}}(G) \leq b$. Also $g(G) \leq g_{\mathrm{con}}(G)$ and $g_{\mathrm{con}}(G) = b$. If $n - b - 2 = 1$, then G is obtained from H by adding the vertices

$$w_1, w_2, \ldots, w_{a-3}, x_1,$$

the edges

$$w_i u_1,\ w_i v_1,$$

where $1 \leq i \leq a-3$, and the edges

$$w_1 x_1,\ x_1 u_2.$$

Then $g(G) = a$ and $g_{\mathrm{con}}(G) = b$. □

For every connected graph G, we have $\text{rad}(G) \leq \text{diam}(G) \leq 2\,\text{rad}(G)$. Ostrand [18] showed that for any two positive integers a and b satisfying $a \leq b \leq 2a$, there exists a connected graph with radius a and diameter b. This result has been extended to other graph parameters by the authors of [5, 19]. Following this, we present the following theorem.

Theorem 3.3. *For every three positive integers r, d, and a with $r \leq d \leq 2r$, there exists a connected graph G such that* $\text{rad}(G) = r$, $\text{diam}(G) = d$, *and* $g_{\text{con}}(G) = a$.

Proof. Assume $r = 1$. If $d = 1$, we let $G = K_a$. Then $g_{\text{con}}(G) = a$. If $d = 2$, we let $G = K_{1,a}$. Then $g_{\text{con}}(G) = a$. For $r \geq 2$ we construct the graph G as follows. Consider a cycle C_{2r} on $2r$ vertices $v_0, v_1, v_2, \ldots, v_{2r-1}$ and a path P_{d-r+1} on $d - r + 1$ vertices $u_0, u_1, u_2, u_3, \ldots, u_{d-r}$. The graph G is obtained by identifying v_0 in C_{2r} and u_0 in P_{d-r+1} and adding the new vertices $w_1, w_2, \ldots, w_{a-2}$ by joining the new edges $w_i u_{d-r-1}$, $1 \leq i \leq a-2$. Then G has $\text{rad}(G) = r$ and $\text{diam}(G) = d$. The vertices $u_{d-r}, w_1, w_2, w_3, \ldots, w_{a-2}$ are simplicial vertices of G and hence $g_{\text{con}}(G) \geq a - 1$. Let $S = \{v_r, w_1, w_2, \ldots, w_{a-2}, u_{d-r+1}\}$. Since the interval $I(v_r, w_i)$ is convex in G, for all i, $1 \leq i \leq a-2$, S is a geoconvex set of G. Hence $g_{\text{con}}(G) = a$. □

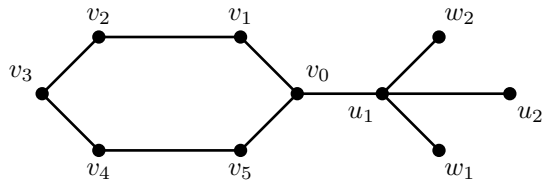

Figure 3.3: A graph G with $r = 3$, $d = 5$, and $a = 4$.

Figure 3.3 illustrates Theorem 3.3 with $\text{rad}(G) = 3$, $\text{diam}(G) = 5$, and $g_{\text{con}}(G) = 4$. The vertices w_1, w_2, u_2 are simplicial vertices of G and $\{w_1, w_2, u_2\}$ is not a geoconvex set of G, which implies $g_{\text{con}}(G) \geq 4$. Let $S = \{w_1, w_2, u_2, v_3\}$. Then for any two vertices $u, v \in S, I(u, v)$ is convex and S is a geoconvex set of G. Hence $g_{\text{con}}(G) = 4$.

Open problems

By Theorem 3.2, the geodetic number may be strictly less than the geoconvex number. This raises the following natural questions:

> **Problem 1.** Characterize all connected graphs G for which $g(G) = g_{\text{con}}(G)$.

Problem 2. Characterize all connected graphs G of order $n \geq 5$ for which $g_{\text{con}}(G) = n - 1$.

Additionally, it would be interesting to study the computational complexity of determining $g_{\text{con}}(G)$ and to explore relationships between $g_{\text{con}}(G)$ and other convexity-based graph parameters.

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