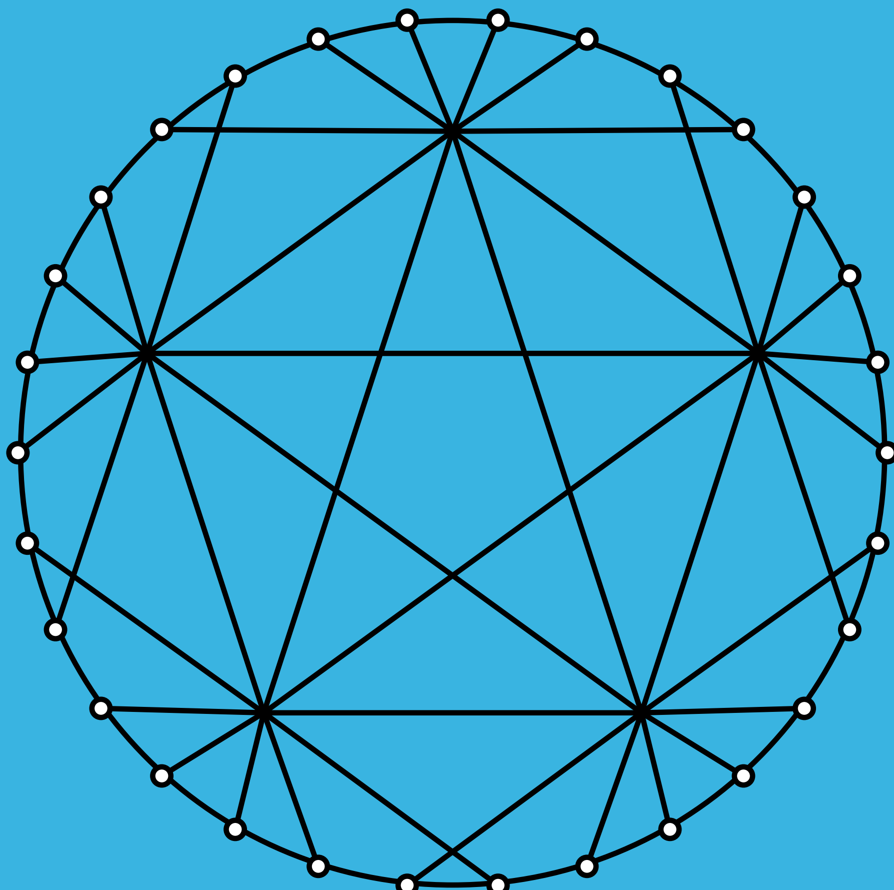
BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 83
June 2018



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **83** (2018), Pages 77–83

A *k*-partite generalization of chordal bipartite graphs

Terry A. McKee

Department of Mathematics & Statistics, Wright State University, Dayton, OH, 45435, USA
terry.mckee@wright.edu

Abstract: The traditional development of chordal bipartite graph theory is largely by analogy with chordal graph theory. But chordal bipartite graphs can be viewed, instead, within a general concept of "chordally k-partite graphs," defined to be the k-partite graphs in which every minimal vertex separator induces a complete k-partite subgraph. Chordal bipartite graphs are precisely the chordally 2-partite graphs.

A new characterization of a graph G being chordally k-partite is proved that emphasizes the graphs being properly k-colored: If each color c determines the subgraph G_c of all edges of G that have a color-c endpoint, then G is chordally k-partite if and only if each G_c is chordal bipartite and every induced nontriangular cycle of G is in exactly two G_c subgraphs.

1 Introduction

Define an ${}^{\geq}l$*-cycle* to be a cycle of length l or more. A graph can be defined to be *chordal* if every cycle long enough to have a chord—in other words, every ${}^{\geq}4$-cycle—does have a chord, and a bipartite graph can be defined to be *chordal bipartite* if every cycle long enough to have a chord—in other words, using bipartiteness, every ${}^{\geq}6$-cycle—does have a chord.

The successes of chordal graph theory help motivate chordal bipartite graphs (although they are also strongly connected with matrix analysis and in-

Received: 5 January 2018
Accepted: 24 April 2018

terconnected with strongly chordal graphs; see [10]). This approach involves looking at chordal bipartite graphs as the bipartite analogs of chordal graphs. For instance, the following types of characterizations of bipartite graphs being chordal bipartite correspond to characterizations of chordal graphs: ***(i)*** having no induced ≥4-cycle, in [3]; ***(ii)*** minimal separators always inducing complete bipartite graphs, in [3, 9]; ***(iii)*** having perfect elimination schemes, both edge elimination in [1, 3] and vertex elimination in [1, 4]; and ***(iv)*** using subtrees-of-trees representations, both intersection graphs in [5] and neighborhood trees in [6].

In spite of the successful and continuing study of chordal bipartite graphs, corresponding notions of chordal tripartite and chordal k-partite graphs have been lacking. The present paper attempts to help with this by focusing on chordal bipartite graphs as the $k = 2$ case of "chordally k-partite graphs," which are defined using a type (*ii*) definition based on [7].

A graph G is *k-partite* (equivalently, *properly k-colored*) if $V(G)$ is partitioned into *partite sets* $V_1, \ldots, V_k$ (also called *color classes*) such that vertices in the same V_i are always nonadjacent. Partite sets V_i are allowed to be empty, making k-partite graphs also k'-partite for all $k' \geq k$. Typically, the vertices of $V_1, \ldots, V_k$ will be assigned colors $c = 1, \ldots, k$, respectively. A k-partite graph is *complete k-partite* if every vertex in V_i is adjacent to every vertex in V_j whenever $i \neq j$. Say that *color c occurs in a subgraph H* of G if $V(H) \cap V_c \neq \emptyset$. If $S \subset V(G)$, let $G[S]$ denote the k-partite subgraph of G induced by *S with colors as in G*, meaning that $G[S]$ has partite sets $V_1 \cap S, \ldots, V_k \cap S$. Let $N_G(v)$ denote the *neighborhood* of v, meaning the set of vertices that are adjacent to v in G.

For nonadjacent vertices x and y in a connected graph G, an *x,y-separator* of G is a set $S \subset V(G)$ such that x and y are in different components of $G[V(G) - S]$. A *minimal x,y-separator* is an x,y-separator that is not a proper subset of another x,y-separator, and a *minimal separator* of G is a minimal x,y-separator for some $x, y \in V(G)$; see [1] for details (and for any undefined notation and terminology).

Define a *chordally k-partite graph* to be a k-partite graph G in which every minimal separator S induces a complete k-partite with colors as in G. (More general "complete-multipartite–separator graphs," not requiring G itself to be k-partite, were introduced in [8], but these may be too general and seem to lack a simple characterization.)

The graph on the left in Figure 1 is chordally 3-partite; its only minimal separators are the minimal a, h-separators $\{b, c\}$ and $\{f, g\}$, the minimal b, g-separator $\{c, d, e, f\}$, the minimal c, f-separator $\{b, d, e, g\}$, and the minimal d, e-separator $\{b, c, f, g\}$), each of which induces a complete bipartite graph (either $K_{1,1} \cong K_2$ or $K_{2,2} \cong C_4$) with colors as in G. The graph on the right is not chordally 3-partite; for instance, the minimal a, g-separator $\{c, d, e\}$ does not induce a complete 3-partite graph with colors as in G (since c and e have different colors without being adjacent).

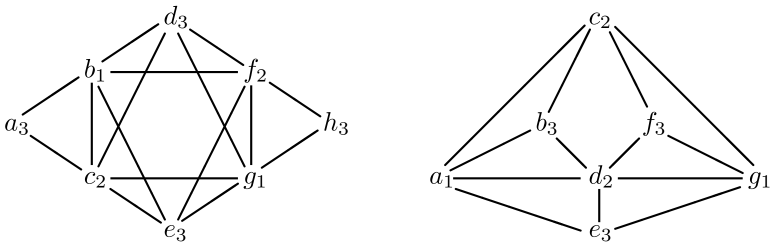


Figure 1: Attaching subscripts to indicate vertex colors, the graph on the left is chordally 3-partite, but the graph on the right is not.

Chordally k-partite graphs were introduced as "chordal multipartite graphs" in [7]. Among the characterizations in [7] (but not used below) is that a graph is chordally k-partite if and only if it is *weakly chordal*—meaning that neither G nor its complement $\overline{G}$ contains an induced ${}^{\geq}5$-cycle—and no induced subgraph is isomorphic to the order-5 graph obtained by bisecting one edge of K_4.

While the characterizations in [7] are useful, Section 2 will present a new characterization of chordally k-partite graphs that views k-partite graphs in terms of how their color-c vertices are related to their non-c colored vertices, over all colors c, by emphasizing chordal bipartite subgraphs in a simple way that does not have a previous chordal graph analog.

Since connected bipartite graphs have uniquely determined color classes, the chordally 2-partite graphs are precisely the traditional "chordal bipartite graphs" as in [2, 3, 10], even though that traditional terminology may seem to conflict with forests being the only graphs that are simultaneously chordal and bipartite. Since a graph happens to be chordal bipartite if and only if it is simultaneously weakly chordal and bipartite, see [1, 2, 9], an occasional remedy for the problematic terminology "chordal bipartite graphs" is to call them "weakly chordal bipartite graphs" instead.

This "weakly chordal" remedy would fail for chordally k-partite graphs. Although [7] shows that chordally k-partite graphs are always weakly chordal, a graph that is both weakly chordal and k-partite does not have to be chordally k-partite—any proper 3-coloring of the weakly chordal, (non-chordally) 3-partite graph obtained by bisecting one edge of K_4 is an example. All in all, the adverb "chordally" in "chordally k-partite graphs" seems to be an attractive alternative to the adjective "weakly chordal" in "weakly chordal bipartite graphs."

2 The new, color-based characterization

Lemma 1 *No induced $^{\geq}4$-cycle of a chordally k-partite graph G with colors as in G can be a $^{\geq}5$-cycle or have vertices of three or more distinct colors.*

Proof. Suppose G is a chordally k-partite graph with an induced cycle C of length $l \geq 4$ and vertices $v_1, v_2, \ldots, v_l$ in that order around C, with colors as in G. Also assume that either $l \geq 5$ or C has vertices of more than two distinct colors (arguing by contradiction). Either way, C has nonadjacent, distinctly colored vertices v_i and v_j with $1 \neq i \neq l$. But now v_i and v_j are in a common minimal v_{i-1}, v_{i+1}-separator S of G, and so $G[S]$ would not be complete k-partite with colors as in G (contradicting that G is chordally k-partite). □

For a k-partite graph G in which color c occurs, define the *color-c-based subgraph of G* to be the subgraph G_c with colors as in G that is formed by all the edges of G that have a color-c endpoint. Since the vertices of induced cycles of G_c alternate between color-c and non-c vertices of G, every induced cycle of G_c has even length, and so every color-c-based subgraph is bipartite (irrespective of the coloring of G).

If $k = 2$, then both subgraphs $G_c = G$. Figure 2 shows the G_c subgraphs of the graph on the left in Figure 1 for the colors $c = 1, 2, 3$.

Theorem 2 *A k-partite graph G is chordally k-partite if and only if every color-c-based subgraph G_c is chordal bipartite (irrespective of the coloring of G) and every induced $^{\geq}4$-cycle of G is in exactly two G_c subgraphs.*

Proof. To prove necessity, first suppose G is chordally k-partite, but also assume that some color-c-based subgraph G_c is not chordal bipartite (arguing by contradiction). Thus, the bipartite graph G_c has an even-length induced $^{\geq}6$-cycle C. By Lemma 1, C is not an induced cycle of

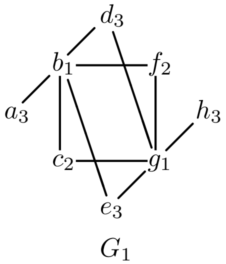


Figure 2: The subgraphs G_1, G_2, G_3 of the chordally 3-partite graph G on the left in Figure 1.

G, so there are vertices $x_1, y_1 \in V(C)$ with distinct non-c colors such that $x_1y_1 \in E(G) - E(G_c)$ is a chord of C in G. Let C_1' be the longer of the two cycles of G that have edges sets contained in $E(C) \cup \{x_1y_1\}$. Thus C_1' is a $^{\geq}4$-cycle in which at least three colors occur (c and the colors of x_1 and y_1). By Lemma 1, C_1' is not an induced cycle of G, so there are vertices $x_2, y_2 \in V(C_1')$ with distinct non-c colors such that $x_2y_2 \in E(G) - E(G_c)$ is a chord of C in G. Let C_2' be the longer of the two cycles of G that have edges sets contained in $E(C) \cup \{x_2y_2\}$.

Repeat this for successively shorter cycles C_i' of G with $V(C_i') \subset V(C)$. Lemma 1 combines with the definition of G_c to ensure that there will eventually be edges $uv, vw \in E(G) - E(G_c)$ that are chords of C in G, where u, v, w have three distinct non-c colors in G, such that uv and vw form an induced 4-cycle of G with two edges $ux, wx \in E(C)$ where x has color c. But now this induced 4-cycle of G would have vertices x, u, v with three distinct colors (contradicting Lemma 1). Therefore, every G_c is chordal bipartite.

Now suppose C is an induced $^{\geq}4$-cycle of the chordally k-partite graph G. By Lemma 1, the vertices of C must alternate between two colors c and c'. Therefore, C is a cycle of the two subgraphs G_c and $G_{c'}$, but not of a third subgraph G_d with $d \notin \{c, c'\}$.

To prove sufficiency, suppose G is k-partite with every G_c chordal bipartite and every induced $^{\geq}4$-cycle of G in exactly two G_c subgraphs. Also assume that G is not chordally k-partite (arguing by contradiction).

Thus G has a minimal x, y-separator S that contains nonadjacent, distinctly colored vertices $u, v \in S$, and so there is an induced x-to-y-path π_u that contains u but not v, and there is an induced x-to-y-path π_v that contains

v but not u. If G_x and G_y are the components of $G[V(G)-S]$ that contain, respectively, x and y, then the portion of $\pi_u \cup \pi_v$ in G_x contains an induced u-to-v-path τ_x whose interior vertices all lie in G_x; similarly, there is an induced u-to-v-path τ_y whose interior vertices all lie in G_y. The induced cycle $C = \tau_x \cup \tau_y$ is a $^{\geq}4$-cycle that contains both u and v.

Only two colors can occur in C (otherwise C would have three consecutive vertices with three distinct colors, which would contradict C being an induced cycle of exactly two G_c subgraphs). Since $u, v \in V(C)$ have different colors, C is not a 4-cycle. Therefore, C is a $^{\geq}5$-cycle whose vertices alternate between two colors c and c', and so C would be a $^{\geq}6$-cycle of G_c (which would contradict G_c being chordal bipartite). □

Figure 3 shows the G_c subgraphs of the graph G on the right in Figure 1 for the colors $c = 1, 2, 3$. This particular G is not chordally 3-partite and satisfies neither of the two conditions in Theorem 2 (since $G_3 - d$ is an induced 6-cycle of G_3, and since the cycle $G[\{a, c, e, g\}]$ is only in G_1).

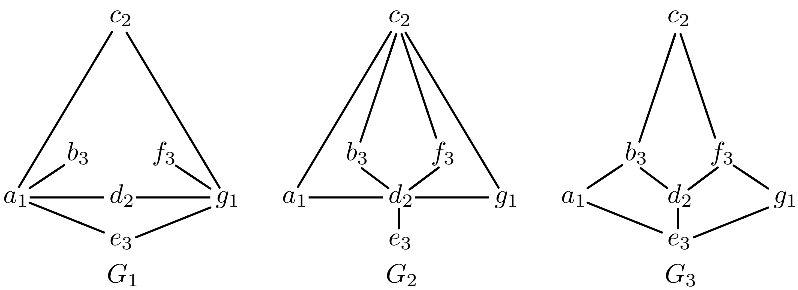


Figure 3: The subgraphs G_1, G_2, G_3 of the non-chordally 3-partite graph G on the right in Figure 1.

Moreover, a graph that is not chordally k-partite can have either one of those two conditions holding by itself, as shown by the proper 3-colorings of $\overline{C_6}$ (commonly called a "3-prism," in which the first condition holds but the second fails) and by the proper 2-coloring of C_6 (in which the second condition holds but the first fails).

References

[1] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph Classes: A Survey*, Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[2] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, second edition, Elsevier, Amsterdam, 2004.

[3] M. C. Golumbic and C. F. Goss, Perfect elimination and chordal bipartite graphs, *J. Graph Theory* **2** (1978) 155–163.

[4] P. L. Hammer, F. Maffray, and M. Preissmann, A characterization of chordal bipartite graphs, RUTCOR Research Report No. 16-89, Rutgers University New Brunswick, NJ, 1989.

[5] J. Huang, Representation characterizations of chordal bipartite graphs, *J. Combin. Theory* **B 96** (2006) 673–683.

[6] T. A. McKee, Strong clique trees, neighborhood trees, and strongly chordal graphs, *J. Graph Theory* **33** (2000) 151–160.

[7] T. A. McKee, Chordal multipartite graphs and chordal colorings, *Discrete Math.* **307** (2007) 2309–2314.

[8] T. A. McKee Requiring that minimal separators induce complete multipartite graphs, *Discuss. Math. Graph Theory* **38** (2018) 263–273.

[9] H. Müller, Recognizing interval digraphs and interval bigraphs in polynomial time, *Discrete Appl. Math.* **78** (1997) 189–205.

[10] J. P. Spinrad, *Efficient Graph Representations*, American Mathematical Society, Providence, 2003.