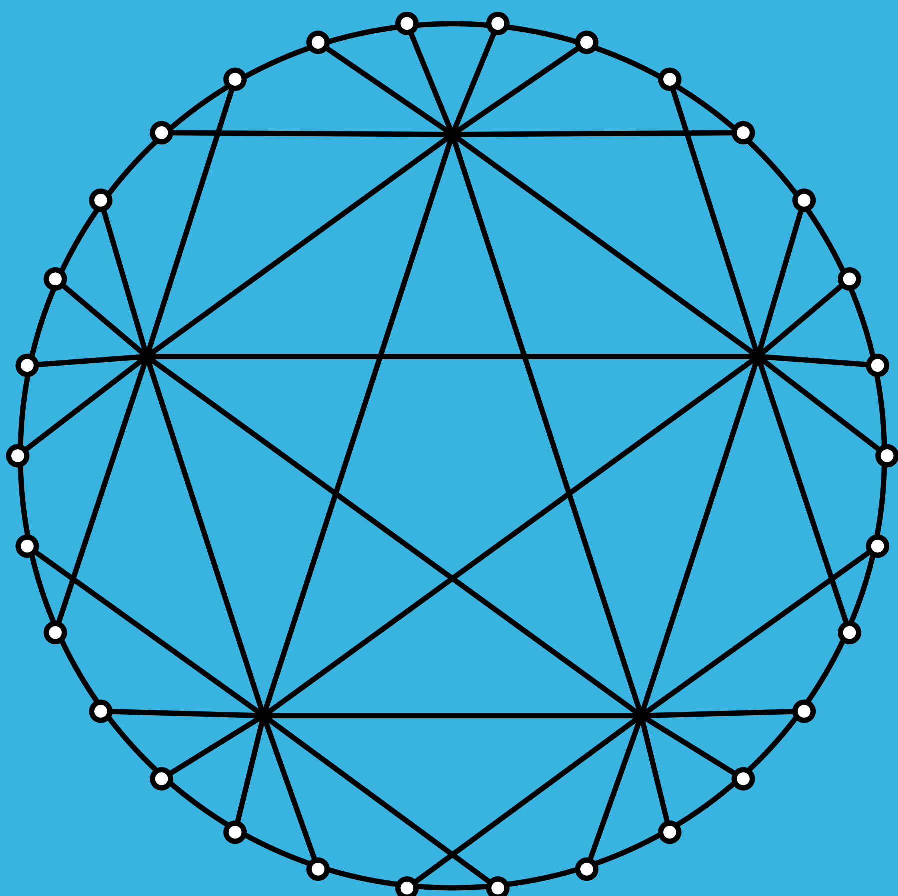
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 87
October 2019

Editors-in-Chief: Marco Buratti, Donald Kreher, Tran van Trung



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **87** (2019), Pages 47–74

Decomposition of product graphs into paths and stars with three edges

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Abstract: Let P_k and S_k respectively denote a path and a star on k vertices. Decomposition of G into p copies of H_1 and q copies of H_2 is denoted as $\{pH_1,\ qH_2\}$-decomposition. In this paper, we give necessary and sufficient conditions for the existence of a $\{pP_4,\ qS_4\}$-*decomposition* of product graphs namely cartesian product, tensor product and wreath product of graphs, where p and q are nonnegative integers.

1 Introduction

Unless stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology the readers are referred to Bondy and Murty [5]. Let $P_k,\ S_k,\ K_k$ respectively denote a path, star and complete graph on k vertices, and let $K_{m,n}$ denote the complete bipartite graph with m and n vertices in the parts. We denote a star S_k with center x_0 and end vertices $x_1, \cdots, x_{k-1}$ by $(x_0; x_1, \cdots, x_{k-1})$. A graph whose vertex set is partitioned into subsets $V_1, ..., V_m$ with edge set $\{xy : x \in V_i,\ y \in V_j,\ 1 \leq i \neq j \leq m\}$ is a *complete m-partite* graph, denoted by $K_{n_1,\ldots,n_m}$, when $|V_i| = n_i$ for all i. For $G = K_{2n}$ or $K_{n,n}$, the graph $G - I$ denotes G with a 1-factor I removed. For any integer $\lambda > 0$, λG denotes the graph consisting of λ edge-disjoint copies of G. The *complement* of the graph G is denoted by $\overline{G}$. For an arbitrary graph G, a list of edge-disjoint subgraphs $H_1, \cdots, H_k$ such that $E(G) = E(H_1) \cup \cdots \cup E(H_k)$ is called a *decomposition* of G and we write G as $G = H_1 \oplus \cdots \oplus H_k$. For $1 \leq i \leq k$, if $H_i \cong H$, we say that G has a H-*decomposition.* For two graphs G and H we define their *cartesian product* $G \square H$, *tensor product* $G \times H$ and *lexicographic or wreath product* $G \otimes H$ with vertex set $V(G) \times V(H) = \{(g, h) : g \in V(G)$ and $h \in V(H)\}$ and their

*Corresponding author.
Key words and phrases: Graph decomposition, Path, Star and Product graph
AMS (MOS) Subject Classifications: 05C51, 05C70

Received: 20 September 2018
Accepted: 2 September 2019

edge set as given below.

$$E(G \,\square\, H) = \{(g,h)(g',h') : g = g',\ hh' \in E(H),\ \text{or}\ gg' \in E(G),\ h = h'\},$$
$$E(G \times H) = \{(g,h)(g',h') : gg' \in E(G)\ \text{and}\ hh' \in E(H)\},$$
$$E(G \otimes H) = \{(g,h)(g,h') : gg' \in E(G)\ \text{or}\ g = g',\ hh' \in E(H)\}.$$

It is well known that the Cartesian product is commutative and associative and the tensor product is commutative and distributive over edge-disjoint union of graphs, i.e., if $G = G_1 \oplus \cdots \oplus G_k$, then $G \times H = (G_1 \times H) \oplus \cdots \oplus (G_k \times H)$. It is easy to observe that $K_m \otimes \overline{K_n} \cong K_{n,\cdots,n(m\ times)}$ and $K_m \otimes \overline{K_n} = (K_m \times K_n) \oplus nK_m$. If G has a decomposition into p copies of H_1 and q copies of H_2, then we say that G has a $\{pH_1,\ qH_2\}$*-decomposition.*

Study of $\{pH_1,\ qH_2\}$-decomposition of graphs is not new. Abueida et al. [1, 3] completely determined the values of n for which $K_n(\lambda)$ admits a $\{pH_1,\ qH_2\}$-decomposition such that $H_1 \cup H_2 \cong K_t$, when $\lambda \geq 1$ and $|V(H_1)| = |V(H_2)| = t$, where $t \in \{4, 5\}$. Abueida and Daven [2] proved that there exists a $\{pK_k,\ qS_{k+1}\}$-decomposition of K_n, for $k \geq 3$ and $n \equiv 0, 1 (mod\ k)$. Abueida and O'Neil [4] proved that for $k \in \{3, 4, 5\}$, there exists a $\{pC_k,\ qS_k\}$-decomposition of $K_n(\lambda)$, whenever $n \geq k + 1$ except for the ordered triples $(k, n, \lambda) \in \{(3, 4, 1), (4, 5, 1), (5, 6, 1), (5, 6, 2), (5, 6, 4), (5, 7, 1), (5, 8, 1)\}$. Shyu [8, 9] obtained a necessary and sufficient condition on (p, q) for the existence of $\{pP_4,\ qS_4\}$-decomposition of K_n and $K_{m,n}$. Priyadharsini and Muthusamy [7] established necessary and sufficient conditions for the existence of the (G_n, H_n)-multidecomposition of $K_n(\lambda)$ where $G_n, H_n \in \{C_n,\ P_{n-1},\ S_{n-1}\}$. Jeevadoss and Muthusamy [6] obtained necessary and sufficient conditions for $\{pP_5,\ qC_4\}$-decomposition of product graphs

In this paper, we show that the necessary conditions are sufficient for the existence of a $\{pP_4,\ qS_4\}$-decomposition of $K_m \square K_n$, $K_m \times K_n$ and $K_m \otimes \overline{K_n}$, where p and q are nonnegative integers. A decomposition of a graph G into p copies of a path of length k and q copies of a star with k edges for every admissible pair (p, q) will be referred to as a $(k; p, q)$-decomposition. To prove our results we state the following:

Theorem 1.1 ([9]). *Let $p, q \geq 0$, and let $0 < m \leq n$ be integers. There exists a $(3; p, q)$-decomposition of $K_{m,n}$ if and only if the following conditions hold:*

1. $3(p + q) = mn$;
2. $p \geq 1 \Rightarrow m \geq 2$;
3. $(m = 3 \vee (m = 2 \wedge n \equiv 0 \pmod 3)) \Rightarrow p \neq 1$.

Theorem 1.2 ([8]). *Let $p, q \geq 0$ and $n > 0$ be integers. There exists a $(3; p, q)$-decomposition of K_n if and only if $n \geq 6$ and $3(p+q) = \frac{n(n-1)}{2}$.*

Remark 1.1. *If G_i has a $(3; p_i, q_i)$-decomposition, for $i = 1, 2$, then $G_1 \cup G_2$ has a $(3; p_1 + p_2, q_1 + q_2)$-decomposition.*

Remark 1.2. *If two stars S_4^1 and S_4^2 with distinct centers, share at least two vertices, then $S_4^1 \oplus S_4^2$ can be decomposed into two P_4.*

Remark 1.3. *Given a star $(a; u, v, w)$, the set $\{((a,i); (u,j), (v,j), (w,j)),$ $1 \leq i \neq j \leq n\}$ provides an S_4-decomposition of $(a; u, v, w) \times K_n$.*

Remark 1.4. *Given a star $(a; u, v, w)$, the set $\{((a,i); (u,j), (v,j), (w,j)),$ $1 \leq i, j \leq n\}$ provides an S_4-decomposition of $(a; u, v, w) \otimes \overline{K_n}$.*

2 Base constructions

In this section we establish a necessary and sufficient conditions for the existence of $(3; p, q)$-decomposition in $K_{n,n} - I$.

Example 1. There exists a $(3; p, q)$-decomposition of $G_1 = K_5 \backslash E(K_2)$ and $G_2 = K_8 \backslash E(K_2)$, for every admissible pair (p, q) of nonnegative integers with $3(p+q) = |E(G_i)|$, $i = 1, 2$.

Solution: Let $V(K_r) = \{x_i : 1 \leq i \leq r\}$. We give a $(3; p, q)$-decomposition of $K_5 \backslash (E(K_2) = x_1 x_2)$ as follows:

1. $p = 0$, $q = 3$. The required stars are $(x_5; x_1, x_2, x_3)$, $(x_4; x_5, x_1, x_2)$, $(x_3; x_1, x_2, x_4)$.
2. $p = 1$, $q = 2$. The required path and stars are $x_4 x_2 x_3 x_1$ and $(x_5; x_1, x_2, x_3)$, $(x_4; x_3, x_5, x_1)$ respectively.
3. $p = 2$, $q = 1$. The required paths and star are $x_5 x_1 x_3 x_4$, $x_3 x_2 x_4 x_1$ and $(x_5; x_4, x_2, x_3)$ respectively.
4. $p = 3$, $q = 0$. The required paths and are $x_1 x_5 x_3 x_2$, $x_1 x_4 x_5 x_2$, $x_1 x_3 x_4 x_2$.

To prove the required decomposition of $K_8 \backslash E(K_2)$, first we decompose $K_8 \backslash (E(K_2) = x_1 x_4)$ into $9S_4$ as follows:

$$\begin{aligned}
&\{(x_2; x_6, \boldsymbol{x_7}, \boldsymbol{x_8}),\ (x_5; x_6, \boldsymbol{x_7}, x_1)\},\\
&\{(x_4; x_5, x_6, \boldsymbol{x_7}),\ (x_6; \boldsymbol{x_7}, \boldsymbol{x_8}, x_1)\},\\
&\{(x_3; \boldsymbol{x_4}, \boldsymbol{x_5}, x_6),\ (x_8; x_3, x_4, \boldsymbol{x_5})\},\\
&\{(x_2; \boldsymbol{x_3}, \boldsymbol{x_4}, x_5),\ (x_1; x_2, \boldsymbol{x_3}, x_8),\ (x_7; x_8, x_3, x_1)\}.
\end{aligned}$$

Now, the last three S_4 has a decomposition into either $\{1P_4,\ 2S_4\}$ or $\{3P_4\}$ as follows:

$$\{x_2 x_3 x_1 x_8,\ (x_2; x_1, x_4, x_5),\ (x_7; x_8, x_3, x_1)\}$$

or

$$\{x_7 x_8 x_1 x_3,\ x_5 x_2 x_3 x_7,\ x_7 x_1 x_2 x_4\}.$$

By Remark 1.2, required number of paths and stars for the remaining choices can be obtained from the paired stars given above. Hence $K_8 \backslash E(K_2)$ has a $(3; p, q)$-decomposition.

Example 2. There exists a $(3; p, q)$-decomposition of $G_1 = K_6 \backslash \{P_{1,1}, P_{1,2}\}$ and $G_2 = K_6 \backslash \{P_{2,1}, P_{2,2}\}$, where $P_{1,1} = x_3x_4x_6x_5$, $P_{1,2} = x_3x_5x_1x_6$, $P_{2,1} = x_3x_1x_2x_5$ and $P_{2,2} = x_1x_6x_2x_3$, for every admissible pair (p, q) of nonnegative integers with $3(p+q) = |E(G_i)|$, $i = 1, 2$.

Solution: Let $V(K_6) = \{x_i : 1 \leq i \leq 6\}$. Now, $K_6 \backslash \{P_{1,1}, P_{1,2}\}$ has a $(3; p, q)$-decomposition as follows:

1. $p = 0,\ q = 3$. The required stars are $(x_3; x_6, x_1, x_2)$, $(x_4; x_5, x_2, x_1)$, $(x_2; x_6, x_5, x_1)$.
2. $p = 1,\ q = 2$. The required path and stars are $x_1x_4x_5x_2$ and $(x_3; x_6, x_1, x_2)$, $(x_2; x_6, x_4, x_1)$ respectively.
3. $p = 2,\ q = 1$. The required paths and star are $x_1x_2x_5x_4$, $x_6x_2x_4x_1$ and $(x_3; x_6, x_1, x_2)$ respectively.
4. $p = 3,\ q = 0$. The required paths are $x_6x_3x_1x_2$, $x_3x_2x_5x_4$, $x_6x_2x_4x_1$.

The $(3; p, q)$-decomposition of $K_6 \backslash \{P_{2,1}, P_{2,2}\}$ is given below.

1. $p = 0,\ q = 3$. The required stars are $(x_3; x_6, x_5, x_4)$, $(x_4; x_6, x_2, x_1)$, $(x_5; x_6, x_4, x_1)$.
2. $p = 1,\ q = 2$. The required path and stars are $x_6x_3x_4x_5$ and $(x_4; x_6, x_2, x_1)$, $(x_5; x_6, x_4, x_1)$ respectively.
3. $p = 2,\ q = 1$. The required paths and star are $x_1x_5x_4x_2$, $x_5x_6x_4x_1$ and $(x_3; x_6, x_5, x_4)$ respectively.
4. $p = 3,\ q = 0$. The required paths are $x_1x_5x_4x_2$, $x_3x_5x_6x_4$, $x_6x_3x_4x_1$.

Lemma 2.1. *There exists a $(3; p, q)$-decomposition of $K_{4,4} - I$, for every admissible pair (p, q) of nonnegative integers with $3(p+q) = |E(K_{4,4} - I)|$ and $p \neq 1$.*

Proof. Let $V(G) = \{x_1, \cdots, x_4\} \cup \{y_1, \cdots, y_4\}$. First we decompose $K_{4,4} - I$ into $4S_4$ as follows:

$$\{(x_1; \boldsymbol{y_2}, \boldsymbol{y_3}, y_4),\ (x_2; y_1, \boldsymbol{y_3}, y_4)\},\ \{(x_3; y_1, \boldsymbol{y_2}, \boldsymbol{y_4}),\ (x_4; y_1, \boldsymbol{y_2}, y_3)\}.$$

By Remark 1.2, we have the required even number of paths and stars from the paired stars. The last $3S_4$ gives $3P_4$ as follows:

$$\{x_2y_1x_4y_3,\ y_3x_2y_4x_3,\ x_4y_2x_3y_1\}.$$

□

Lemma 2.2. *There exists a* $(3; p, q)$*-decomposition of* $K_{6,6} - I$*, for every admissible pair* (p, q) *of nonnegative integers with* $3(p+q) = |E(K_{6,6} - I)|$.

Proof. Let $V(G) = \{x_1, \cdots, x_6\} \cup \{y_1, \cdots, y_6\}$. First we decompose $K_{6,6} - I$ into $10S_4$ as follows:

$$\begin{array}{ll} \{(x_2; \boldsymbol{y_1}, \boldsymbol{y_3}, y_4),\ (x_5; \boldsymbol{y_3}, y_4, y_6)\}, & \{(x_4; y_3, \boldsymbol{y_5}, \boldsymbol{y_6}),\ (x_6; y_3, y_4, \boldsymbol{y_5})\}, \\ \{(y_5; \boldsymbol{x_1}, \boldsymbol{x_2}, x_3),\ (y_6; x_1, \boldsymbol{x_2}, x_3)\}, & \{(x_1; \boldsymbol{y_2}, \boldsymbol{y_3}, y_4),\ (x_3; y_1, \boldsymbol{y_2}, y_4)\}, \\ \{(y_1; \boldsymbol{x_4}, \boldsymbol{x_5}, x_6),\ (y_2; x_4, \boldsymbol{x_5}, x_6)\}. & \end{array}$$

Now, the last $3S_4$ can be decomposed into $3P_4$ as follows:

$$y_4x_3y_2x_6, x_6y_1x_5y_2, y_2x_4y_1x_3.$$

By Remark 1.2, the required decomposition for the remaining choices of p and q other than $p = 1$ can be obtained from the paired stars given above. For $p = 1$, the required path and stars are $x_1y_2x_3y_4$, $(x_3; y_1, y_5, y_6)$, $(x_1; y_3, y_5, y_6)$, $(x_2; y_1, y_3, y_4)$, $(y_2; x_4, x_5, x_6)$, $(y_1; x_4, x_5, x_6)$, $(y_3; x_4, x_5, x_6)$, $(y_4; x_1, x_5, x_6)$, $(y_5; x_2, x_4, x_6)$, $(y_6; x_2, x_4, x_5)$. □

Lemma 2.3. *There exists a* $(3; p, q)$*-decomposition of* $K_{7,7} - I$*, for every admissible pair* (p, q) *of nonnegative integers with* $3(p+q) = |E(K_{7,7} - I)|$.

Proof. Let $V(G) = \{x_1, \cdots, x_7\} \cup \{y_1, \cdots, y_7\}$. First we decompose $K_{7,7} - I$ into $14S_4$ as follows:

$$\begin{array}{l} \{(x_2; y_1, \boldsymbol{y_3}, \boldsymbol{y_4}),\ (x_7; y_1, \boldsymbol{y_3}, y_2)\}, \{(x_5; \boldsymbol{y_3}, \boldsymbol{y_4}, y_6),\ (x_7; \boldsymbol{y_4}, y_5, y_6)\}, \\ \{(x_1; \boldsymbol{y_5}, \boldsymbol{y_6}, y_7),\ (x_2; y_5, \boldsymbol{y_6}, y_7)\}, \{(x_3; y_5, \boldsymbol{y_6}, \boldsymbol{y_7}),\ (x_4; y_3, y_5, \boldsymbol{y_6})\}, \\ \{(x_6; \boldsymbol{y_3}, y_4, y_5),\ (x_1; \boldsymbol{y_2}, \boldsymbol{y_3}, y_4)\}, \{(x_3; \boldsymbol{y_1}, y_2, y_4),\ (x_4; \boldsymbol{y_7}, \boldsymbol{y_1}, y_2)\}, \\ \{(x_5; \boldsymbol{y_7}, \boldsymbol{y_1}, y_2),\ (x_6; \boldsymbol{y_7}, y_1, y_2)\}. \end{array}$$

Now, the last $3S_4$ can be decomposed into $3P_4$ as follows:

$$\{x_5y_7x_4y_2,\ x_6y_2x_5y_1,\ x_4y_1x_6y_7\}.$$

By Remark 1.2, the required decomposition for the remaining choices of p and q other than $p = 1$ can be obtained from the paired stars given above. For $p = 1$, the required path and stars are $x_1y_2x_3y_4$, $(x_3; y_1, y_5, y_6)$, $(x_1; y_3, y_5, y_6)$, $(x_2; y_1, y_3, y_4)$, $(y_2; x_4, x_5, x_6)$, $(y_1; x_4, x_5, x_6)$, $(y_3; x_4, x_5, x_6)$, $(y_4; x_1, x_5, x_6)$, $(y_5; x_2, x_4, x_6)$, $(y_6; x_2, x_4, x_5)$, $(x_7; y_1, y_2, y_3)$, $(x_7; y_4, y_5, y_6)$, $(y_7; x_1, x_2, x_3)$, $(y_7; x_4, x_5, x_6)$. □

Lemma 2.4. *There exists a* $(3; p, q)$*-decomposition of* $K_{9,9} - I$*, for every admissible pair* (p, q) *of nonnegative integers with* $3(p+q) = |E(K_{9,9} - I)|$.

Proof. Let $V(G) = \{x_1, \cdots, x_9\} \cup \{y_1, \cdots, y_9\}$. We can write

$$K_{9,9} - I = (K_{6,6} - I) \;\oplus\; K_{6,3} \;\oplus\; K_{3,6} \;\oplus\; (K_{3,3} - I).$$

By Lemma 2.1, $K_{6,6} - I$ has a $(3; p, q)$-decomposition. Now, decompose $G(= K_{6,3} \oplus K_{3,6} \oplus (K_{3,3} - I))$ into $14S_4$ as follows:

$$\begin{array}{ll}
\{(x_7; \boldsymbol{y_1, y_2}, y_3),\ (x_8; \boldsymbol{y_2}, y_3, y_6)\}, & \{(x_9; \boldsymbol{y_3, y_6}, y_8),\ (x_7; \boldsymbol{y_6}, y_8, y_9)\}, \\
\{(x_8; \boldsymbol{y_7, y_9}, y_1),\ (x_9; \boldsymbol{y_7}, y_1, y_2)\}, & \{(y_4; \boldsymbol{x_7, x_8}, x_9),\ (y_5; x_7, \boldsymbol{x_8}, x_9)\}, \\
\{(y_7; \boldsymbol{x_1, x_2}, x_3),\ (y_8; \boldsymbol{x_2}, x_3, x_4)\}, & \{(y_9; \boldsymbol{x_3, x_4}, x_5),\ (y_7; \boldsymbol{x_4}, x_5, x_6)\}, \\
\{(y_8; \boldsymbol{x_5, x_6}, x_1),\ (y_9; \boldsymbol{x_6}, x_1, x_2)\}. &
\end{array}$$

Now, the last $3S_4$ can be decompose into $3P_4$ as follows:

$$\{x_4y_7x_5y_8,\ x_2y_9x_6y_7,\ y_9x_1y_8x_6\}.$$

Hence by Remark 1.2, G has a $(3; p, q)$-decomposition with $p \neq 1$. Now, by Remark 1.1, we have the desired decomposition of $K_{9,9} - I$. □

Lemma 2.5. *Let p, q be nonnegative integers and G be an r-regular graph on v vertices. If G has a $(3; p, q)$-decomposition, then $rv \equiv 0 \pmod 6$.*

Proof. Since G is r-regular with v vertices, G has $rv/2$ edges. Now, assume that G has a $(3; p, q)$-decomposition. Then the number of edges in the graph must be divisible by 3, i.e., $6|rv$ and hence $rv \equiv 0 \pmod 6$. □

Theorem 2.6. *The graph $K_{n,n} - I$ has a $(3; p, q)$-decomposition for every admissible pair (p, q) of nonnegative integers with $3(p + q) = n(n - 1)$ if and only if $n \equiv 0$ or $1 \pmod 3$ with $(n, p) \neq (4, 1)$ and $q = 0$ when $n = 3$.*

Proof. **Necessity.** Since $K_{n,n} - I$ is $(n - 1)$-regular with $2n$ vertices, $n \equiv 0$ *or* $1 \pmod 3$ follows from Lemma 2.5. When $n = 3$, $K_{3,3} - I$ is 2-regular and hence it does not contains any star with 3 edges, therefore $q = 0$. Suppose there is a $\{P_4,\ 3S_4\}$-decomposition of $K_{4,4} - I$. Let $V(K_{4,4} - I) = V = V_1 \cup V_2 = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3, v_4\}$ and $I = \{u_1v_1, u_2v_2, u_3v_3, u_4v_4\}$. Without loss of generality let $P_4 = u_1v_2u_3v_1$. So $deg(u) = 3$ only for $u = u_2, u_4 \in V_1$ and $u = v_3, v_4 \in V_2$ in $(K_{4,4}-I)\backslash E(P_4)$. Then the centers of two stars are contained in exactly one partite set say V_1. So the remaining graph is not a star since $deg(u) \leq 2$ for all $u \in V$, therefore $p \neq 1$.

Sufficiency. For $n = 3$, the paths are $x_1y_2x_3y_1$, $x_1y_3x_2y_1$ and we proved such decomposition in Lemma 2.1 when $n = 4$. We construct the required decomposition for the remaining choices of n in four cases.

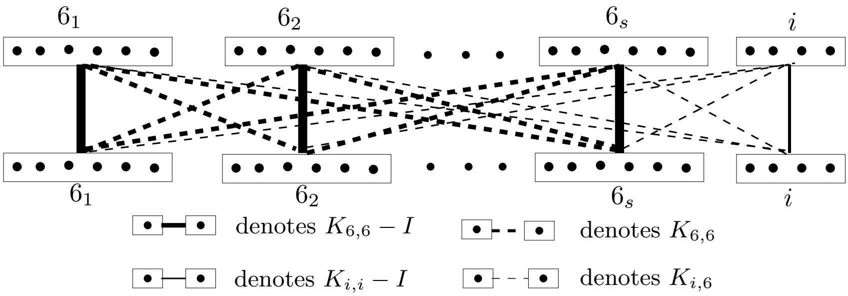

Figure 1: The graph $K_{n,n} - I$.

Case(1) $n \equiv\ 0 \pmod 6$.

Let $n = 6k,\ k > 0$ be an integer. We can write

$$K_{n,n} - I = K_{6k,6k} - I = k(K_{6,6} - I)\ \oplus\ k(k-1)K_{6,6}A$$

(See Figure 1 with $s = k,\ i = 0$). By Theorem 1.1 and Lemma 2.2, $K_{6,6} - I$ and $K_{6,6}$ have a $(3; p, q)$-decomposition. Hence by Remark 1.1, $K_{n,n} - I$ has a $(3; p, q)$-decomposition.

Case(2) $n \equiv\ 1 \pmod 6$.

Let $n = 6k + 1,\ k > 0$ be an integer. We can write

$$\begin{aligned} K_{n,n} - I &= K_{6k+1,6k+1} - I \\ &= (k-1)(K_{6,6} - I)\ \oplus\ (K_{7,7} - I) \\ &\quad \oplus\ (k-1)(k-2)K_{6,6}\ \oplus\ 2(k-1)K_{7,6} \end{aligned}$$

(See Figure 1 with $s = k - 1,\ i = 7$). By Lemmas 2.2 and 2.3, $K_{6,6} - I$ and $K_{7,7} - I$ have a $(3; p, q)$-decomposition. Also, by Theorem 1.1 $K_{6,6}$ and $K_{7,6}$ have a $(3; p, q)$-decomposition. Hence by Remark 1.1, $K_{n,n} - I$ has a $(3; p, q)$-decomposition.

Case(3) $n \equiv\ 3 \pmod 6$.

Let $n = 6k + 3,\ k > 0$ be an integer. We can write

$$\begin{aligned} K_{n,n} - I &= K_{6k+3,6k+3} - I \\ &= (k-1)(K_{6,6} - I)\ \oplus\ (K_{9,9} - I) \\ &\quad \oplus\ (k-1)(k-2)K_{6,6}\ \oplus\ 2(k-1)K_{9,6} \end{aligned}$$

(See Figure 1 with $s = k - 1,\ i = 9$). By Lemmas 2.2 and 2.4, $K_{6,6} - I$ and $K_{9,9} - I$ have a $(3; p, q)$-decomposition. Also, by Theorem 1.1 $K_{6,6}$

and $K_{9,6}$ have a $(3;p,q)$-decomposition. Hence by Remark 1.1, $K_{n,n} - I$ has a $(3;p,q)$-decomposition.

Case(4) $n \equiv 4 \pmod 6$.

Let $n = 6k+4,\ k > 0$ be an integer. We can write

$$\begin{aligned} K_{n,n} - I &= K_{6k+4,6k+4} - I \\ &= k(K_{6,6} - I) \ \oplus\ k(k-1)K_{6,6} \ \oplus\ (K_{4,4} - I) \ \oplus\ 2kK_{6,4} \end{aligned}$$

(See Figure 1 with $s = k,\ i = 4$). By Lemmas 2.1 and 2.2, $K_{4,4} - I$ and $K_{6,6} - I$ have a $(3;p,q)$-decomposition. Also, by Theorem 1.1 $K_{6,6}$ and $K_{6,4}$ have a $(3;p,q)$-decomposition. Hence by Remark 1.1, $K_{n,n} - I$ has a $(3;p,q)$-decomposition.

□

3 $(3;p,q)$-decomposition of $K_m \Box K_n$

In this section we obtain the existence of $(3;p,q)$-decomposition of Cartesian product of complete graphs.

Lemma 3.1. *There exists a* $(3;p,q)$*-decomposition of* $K_6 \Box K_5$*, for every admissible pair* (p,q) *of nonnegative integers with* $3(p+q) = |E(K_6 \Box K_5)|$.

Proof. Let $V(K_6 \Box K_5) = \{x_{i,j} : 1 \le i \le 6,\ 1 \le j \le 5\}$. We can write

$$\begin{aligned} K_6 \Box K_5 = 3K_6 \ &\oplus\ 6(K_5 \backslash E(K_2)) \oplus (K_6 \backslash \{P_{1,1}, P_{1,2}\}) \\ &\oplus\ (K_6 \backslash \{P_{2,1}, P_{2,2}\}) \ \oplus\ (P_{1,1} \oplus P_{1,2} \ \oplus\ P_{2,1} \ \oplus\ P_{2,2} \ \oplus\ 6K_2), \end{aligned}$$

where

$$\begin{aligned} P_{1,1} &= x_{3,1}x_{4,1}x_{6,1}x_{5,1}, \\ P_{1,2} &= x_{3,1}x_{5,1}x_{1,1}x_{6,1}, \\ P_{2,1} &= x_{3,2}x_{1,2}x_{2,2}x_{5,2}, \\ P_{2,2} &= x_{1,2}x_{6,2}x_{2,2}x_{3,2}. \end{aligned}$$

Now, by Examples 1 and 2:

$$6(K_5 \backslash E(K_2)),\ K_6 \backslash \{P_{1,1}, P_{1,2}\} \text{ and } K_6 \backslash \{P_{2,1}, P_{2,2}\}$$

have a $(3;p,q)$-decomposition. Also, by Theorem 1.2, K_6 has a $(3;p,q)$-decomposition. We prove $(P_{1,1} \ \oplus\ P_{1,2} \oplus P_{2,1} \oplus P_{2,2} \ \oplus\ 6K_2)$ has a $(3;p,q)$-decomposition as follows:

1. $p = 0,\ q = 6$. The required stars are
   $(x_{6,1}; x_{1,1}, x_{5,1}, x_{6,2})$, $(x_{5,1}; x_{1,1}, x_{3,1}, x_{5,2})$, $(x_{4,1}; x_{3,1}, x_{6,1}, x_{4,2})$,
   $(x_{2,2}; x_{6,2}, x_{5,2}, x_{2,1})$, $(x_{1,2}; x_{1,1}, x_{2,2}, x_{6,2})$, $(x_{3,2}; x_{3,1}, x_{2,2}, x_{1,2})$.

2. $p = 1,\ q = 5$. The required path and stars are $x_{3,1}x_{3,2}x_{2,2}x_{1,2}$ and $(x_{6,1}; x_{1,1}, x_{5,1}, x_{6,2})$, $(x_{5,1}; x_{1,1}, x_{3,1}, x_{5,2})$, $(x_{4,1}; x_{3,1}, x_{6,1}, x_{4,2})$, $(x_{2,2}; x_{6,2}, x_{5,2}, x_{2,1})$, $(x_{1,2}; x_{1,1}, x_{3,2}, x_{6,2})$ respectively.
3. $p = 2,\ q = 4$. The required paths and stars are $x_{1,1}x_{1,2}x_{3,2}x_{3,1}$, $x_{6,2}x_{1,2}x_{2,2}x_{3,2}$ and $(x_{6,1}; x_{1,1}, x_{5,1}, x_{6,2})$, $(x_{5,1}; x_{1,1}, x_{3,1}, x_{5,2})$, $(x_{4,1}; x_{3,1}, x_{6,1}, x_{4,2})$, $(x_{2,2}; x_{6,2}, x_{5,2}, x_{2,1})$ respectively.
4. $p = 3,\ q = 3$. The required paths and stars are $x_{1,1}x_{1,2}x_{2,2}x_{2,1}$, $x_{5,2}x_{2,2}x_{3,2}x_{3,1}$, $x_{3,2}x_{1,2}x_{6,2}x_{2,2}$ and $(x_{6,1}; x_{1,1}, x_{5,1}, x_{6,2})$, $(x_{5,1}; x_{1,1}, x_{3,1}, x_{5,2})$, $(x_{4,1}; x_{3,1}, x_{6,1}, x_{4,2})$ respectively.
5. $p = 4,\ q = 2$. The required paths and stars are $x_{1,1}x_{1,2}x_{2,2}x_{2,1}$, $x_{1,1}x_{5,1}x_{3,1}x_{3,2}$, $x_{5,1}x_{5,2}x_{2,2}x_{3,2}$, $x_{3,2}x_{1,2}x_{6,2}x_{2,2}$ and $(x_{6,1}; x_{1,1}, x_{5,1}, x_{6,2})$, $(x_{4,1}; x_{3,1}, x_{6,1}, x_{4,2})$ respectively.
6. $p = 5,\ q = 1$. The required paths and stars are $x_{1,1}x_{1,2}x_{2,2}x_{2,1}$, $x_{3,2}x_{1,2}x_{6,2}x_{2,2}$, $x_{6,2}x_{6,1}x_{1,1}x_{5,1}$, $x_{5,1}x_{5,2}x_{2,2}x_{3,2}$, $x_{6,1}x_{5,1}x_{3,1}x_{3,2}$ and $(x_{4,1}; x_{3,1}, x_{6,1}, x_{4,2})$ respectively.
7. $p = 6,\ q = 0$. The required paths are $x_{1,1}x_{1,2}x_{2,2}x_{2,1}$, $x_{3,2}x_{1,2}x_{6,2}x_{2,2}$, $x_{6,2}x_{6,1}x_{1,1}x_{5,1}$, $x_{5,1}x_{5,2}x_{2,2}x_{3,2}$, $x_{4,2}x_{4,1}x_{3,1}x_{3,2}$, $x_{4,1}x_{6,1}x_{5,1}x_{3,1}$.

Thus the graph $K_6 \square K_5$ has a required decomposition. □

Lemma 3.2. *There exists a* $(3; p, q)$*-decomposition of* $K_3 \square K_5$*, for every admissible pair* (p, q) *of nonnegative integers with* $3(p + q) = |E(K_3 \square K_5)|$.
Proof. Let $V(K_3 \square K_5) = \{x_{i,j} : 1 \le i \le 3,\ 1 \le j \le 5\}$. First we decompose $K_3 \square K_5$ into $15S_4$ as follows:

$$\begin{aligned}
&\{(x_{1,3}; x_{2,3}, \boldsymbol{x_{1,4}}, x_{1,5}),\ (x_{1,1}; \boldsymbol{x_{3,1}}, \boldsymbol{x_{1,4}}, x_{1,5})\},\\
&\{(x_{2,2}; \boldsymbol{x_{1,2}}, x_{2,3}, x_{2,4}),\ (x_{2,1}; \boldsymbol{x_{3,1}}, \boldsymbol{x_{2,2}}, x_{2,3})\},\\
&\{(x_{2,4}; \boldsymbol{x_{1,4}}, x_{2,5}, x_{2,1}),\ (x_{2,3}; \boldsymbol{x_{3,3}}, \boldsymbol{x_{2,4}}, x_{2,5})\},\\
&\{(x_{3,2}; \boldsymbol{x_{2,2}}, \boldsymbol{x_{3,3}}, x_{3,4}),\ (x_{3,1}; x_{3,2}, \boldsymbol{x_{3,3}}, x_{3,5})\},\\
&\{(x_{3,4}; \boldsymbol{x_{2,4}}, x_{3,5}, x_{3,1}),\ (x_{3,3}; \boldsymbol{x_{1,3}}, \boldsymbol{x_{3,4}}, x_{3,5})\},\\
&\{(x_{2,5}; \boldsymbol{x_{1,5}}, \boldsymbol{x_{2,1}}, x_{2,2}),\ (x_{3,5}; \boldsymbol{x_{1,5}}, x_{2,5}, x_{3,2})\},\\
&\{(x_{1,1}; \boldsymbol{x_{2,1}}, \boldsymbol{x_{1,2}}, x_{1,3}),\ (x_{1,2}; \boldsymbol{x_{3,2}}, x_{1,3}, x_{1,5}),\ (x_{1,4}; x_{3,4}, x_{1,5}, x_{1,2})\}.
\end{aligned}$$

Now, the last $3S_4$ can be decomposed into either $\{1P_4,\ 2S_4\}$ or $\{3P_4\}$ as follows:

$$\{x_{2,1}x_{1,1}x_{1,3}x_{1,2},\ (x_{1,2}; x_{3,2}, x_{1,1}, x_{1,5}),\ (x_{1,4}; x_{3,4}, x_{1,5}, x_{1,2})\}$$

or

$$\{x_{1,1}x_{1,2}x_{1,4}x_{3,4},\ x_{2,1}x_{1,1}x_{1,3}x_{1,2},\ x_{3,2}x_{1,2}x_{1,5}x_{1,4}\}.$$

By Remark 1.2, required number of paths and stars for remaining choices of p and q can be obtained from the paired stars given above. $\square$

Lemma 3.3. *There exists a* $(3; p, q)$*-decomposition of* $K_3 \square K_6$*, for every admissible pair* (p, q) *of nonnegative integers with* $3(p+q) = |E(K_3 \square K_6)|$.

Proof. Let $V(K_3 \square K_6) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 6\}$. First we decompose $K_3 \square K_6$ into $21S_4$ as follows:

$$\begin{aligned}
&\{(x_{3,4}; x_{1,4}, x_{3,2}, \boldsymbol{x_{3,6}}),\ (x_{2,4}; x_{1,4}, \boldsymbol{x_{3,4}}, \boldsymbol{x_{2,1}})\},\\
&\{(x_{1,6}; x_{3,6}, \boldsymbol{x_{1,1}}, \boldsymbol{x_{1,2}}),\ (x_{1,5}; x_{1,4}, \boldsymbol{x_{1,1}}, x_{1,6})\},\\
&\{(x_{1,3}; \boldsymbol{x_{1,4}}, \boldsymbol{x_{1,5}}, x_{1,6}),\ (x_{1,4}; \boldsymbol{x_{1,2}}, x_{1,1}, x_{1,6})\},\\
&\{(x_{1,2}; \boldsymbol{x_{2,2}}, x_{3,2}, x_{1,3}),\ (x_{1,1}; \boldsymbol{x_{2,1}}, x_{1,3}, \boldsymbol{x_{1,2}})\},\\
&\{(x_{3,4}; \boldsymbol{x_{3,5}}, x_{3,3}, \boldsymbol{x_{3,1}}),\ (x_{3,2}; \boldsymbol{x_{3,1}}, x_{2,2}, x_{3,3})\},\\
&\{(x_{1,5}; \boldsymbol{x_{1,2}}, \boldsymbol{x_{2,5}}, x_{3,5}),\ (x_{2,5}; x_{2,3}, \boldsymbol{x_{2,1}}, x_{3,5})\},\\
&\{(x_{3,6}; x_{3,5}, \boldsymbol{x_{3,2}}, x_{2,6}),\ (x_{3,5}; x_{3,3}, \boldsymbol{x_{3,1}}, \boldsymbol{x_{3,2}})\},\\
&\{(x_{2,6}; \boldsymbol{x_{1,6}}, x_{2,1}, x_{2,4}),\ (x_{2,3}; x_{2,1}, \boldsymbol{x_{2,6}}, \boldsymbol{x_{2,2}})\},\\
&\{(x_{2,5}; \boldsymbol{x_{2,2}}, \boldsymbol{x_{2,4}}, x_{2,6}),\ (x_{2,2}; \boldsymbol{x_{2,1}}, x_{2,4}, x_{2,6})\},\\
&\{(x_{3,1}; x_{1,1}, \boldsymbol{x_{2,1}}, \boldsymbol{x_{3,6}}),\ (x_{3,3}; x_{3,1}, \boldsymbol{x_{3,6}}, x_{1,3}),\ (x_{2,3}; x_{1,3}, x_{3,3}, x_{2,4})\}.
\end{aligned}$$

Now, the last $3S_4$ can be decomposed into either $\{1P_4,\ 2S_4\}$ or $\{3P_4\}$ as follows:

$$\begin{aligned}
&\{x_{2,3}x_{2,4}x_{1,3}x_{3,3},\ (x_{3,1}; x_{1,1}, x_{2,1}, x_{3,6}),\ (x_{3,3}; x_{3,1}, x_{3,6}, x_{2,3}))\}\\
\text{or}\quad &\{x_{2,3}x_{2,4}x_{1,3}x_{3,3}, x_{1,1}x_{3,1}x_{3,3}x_{2,3},\ x_{2,1}x_{3,1}x_{3,6}x_{3,3}\}.
\end{aligned}$$

By Remark 1.2, required number of paths and stars for remaining choices of p and q can be obtained from the paired stars given above. $\square$

Lemma 3.4. *There exists a* $(3; p, q)$*-decomposition of* $K_4 \square K_6$*, for every admissible pair* (p, q) *of nonnegative integers with* $3(p+q) = |E(K_4 \square K_6)|$.

Proof. Let $V(K_4 \square K_6) = \{x_{i,j} : 1 \leq i \leq 4,\ 1 \leq j \leq 6\}$.
We can write $K_4 \square K_6 = (6K_4 \oplus 3K_6) \oplus K_6$. First we decompose $(6K_4 \oplus 3K_6)$ into $27S_4$ as follows:

$$\begin{aligned}
&\{(x_{4,1}; x_{3,1}, \boldsymbol{x_{2,1}}, x_{1,1}),\ (x_{3,1}; x_{1,1}, \boldsymbol{x_{2,1}}, \boldsymbol{x_{3,6}})\},\\
&\{(x_{4,2}; \boldsymbol{x_{3,2}}, x_{2,2}, x_{1,2}),\ (x_{1,2}; \boldsymbol{x_{2,2}}, \boldsymbol{x_{3,2}}, x_{1,3})\},\\
&\{(x_{4,3}; \boldsymbol{x_{3,3}}, x_{2,3}, x_{1,3}),\ (x_{2,3}; \boldsymbol{x_{1,3}}, \boldsymbol{x_{3,3}}, x_{2,4})\},\\
&\{(x_{4,4}; \boldsymbol{x_{3,4}}, x_{2,4}, x_{1,4}),\ (x_{2,4}; \boldsymbol{x_{1,4}}, \boldsymbol{x_{3,4}}, x_{2,1})\},\\
&\{(x_{4,5}; \boldsymbol{x_{3,5}}, x_{2,5}, x_{1,5}),\ (x_{1,5}; x_{1,2}, \boldsymbol{x_{2,5}}, \boldsymbol{x_{3,5}})\},\\
&\{(x_{4,6}; x_{3,6}, x_{2,6}, \boldsymbol{x_{1,6}}),\ (x_{2,6}; \boldsymbol{x_{1,6}}, \boldsymbol{x_{2,1}}, x_{2,4})\},\\
&\{(x_{3,4}; x_{1,4}, \boldsymbol{x_{3,2}}, x_{3,6}),\ (x_{3,6}; \boldsymbol{x_{3,5}}, \boldsymbol{x_{3,2}}, x_{2,6})\},\\
&\{(x_{1,6}; x_{3,6}, \boldsymbol{x_{1,1}}, \boldsymbol{x_{1,2}}),\ (x_{1,5}; x_{1,4}, \boldsymbol{x_{1,1}}, x_{1,6})\},\\
&\{(x_{3,3}; \boldsymbol{x_{3,1}}, \boldsymbol{x_{3,6}}, x_{1,3}),\ (x_{3,5}; x_{3,3}, \boldsymbol{x_{3,1}}, x_{3,2})\},\\
&\{(x_{3,4}; \boldsymbol{x_{3,5}}, x_{3,3}, \boldsymbol{x_{3,1}}),\ (x_{3,2}; \boldsymbol{x_{3,1}}, x_{2,2}, x_{3,3})\},\\
&\{(x_{2,5}; x_{2,3}, \boldsymbol{x_{2,1}}, x_{3,5}),\ (x_{2,3}; \boldsymbol{x_{2,1}}, \boldsymbol{x_{2,6}}, x_{2,2})\},\\
&\{(x_{2,5}; \boldsymbol{x_{2,2}}, \boldsymbol{x_{2,4}}, x_{2,6}),\ (x_{2,2}; \boldsymbol{x_{2,1}}, x_{2,4}, x_{2,6})\},\\
&\{(x_{1,3}; \boldsymbol{x_{1,4}}, \boldsymbol{x_{1,5}}, x_{1,6}),\ (x_{1,4}; \boldsymbol{x_{1,2}}, x_{1,1}, x_{1,6}), (x_{1,1}; x_{2,1}, x_{1,3}, x_{1,2})\}.
\end{aligned}$$

Now, the last $3S_4$ can be decomposed into either $\{1P_4,\ 2S_4\}$ or $\{3P_4\}$ as follows:

$$\{x_{1,5}x_{1,3}x_{1,6}x_{1,4},\ (x_{1,4};x_{1,2},x_{1,1},x_{1,3}),\ (x_{1,1};x_{2,1},x_{1,3},x_{1,2})\}$$

or

$$\{x_{1,5}x_{1,3}x_{1,6}x_{1,4},\ x_{1,3}x_{1,1}x_{1,2}x_{1,4}, x_{2,1}x_{1,1}x_{1,4}x_{1,3}\}.$$

By Remark 1.2, required number of paths and stars for remaining choices of p and q can be obtained from the paired stars given above. Hence $(6K_4 \oplus 3K_6)$ has a $(3;p,q)$-decomposition. Also, by Theorem 1.2, K_6 has a $(3;p,q)$-decomposition. Hence by Remark 1.1, the graph $K_4 \Box K_6$ has the desired decomposition. $\Box$

Lemma 3.5. *There exists a* $(3;p,q)$*-decomposition of* $K_3 \Box K_8$*, for every admissible pair* (p,q) *of nonnegative integers with* $3(p+q) = |E(K_3 \Box K_8)|$.

Proof. Let $V(K_3 \Box K_8) = \{x_{i,j} : 1 \le i \le 3,\ 1 \le j \le 8\}$. First we decompose $K_3 \Box K_8$ into $36S_4$ as follows:

$$\begin{array}{l}
\{(x_{3,4};\boldsymbol{x_{1,4}},\boldsymbol{x_{3,2}},x_{3,6}),\ (x_{2,4};\boldsymbol{x_{1,4}},x_{3,4},x_{2,1})\},\\
\{(x_{1,6};\boldsymbol{x_{3,6}},\boldsymbol{x_{1,1}},x_{1,2}),\ (x_{1,1};x_{2,1},\boldsymbol{x_{1,3}},x_{1,2})\},\\
\{(x_{3,1};\boldsymbol{x_{1,1}},\boldsymbol{x_{3,6}},x_{2,1}),\ (x_{3,3};x_{3,1},\boldsymbol{x_{3,6}},x_{1,3})\},\\
\{(x_{2,3};x_{1,3},\boldsymbol{x_{3,3}},\boldsymbol{x_{2,4}}),\ (x_{2,8};x_{2,6},\boldsymbol{x_{2,4}},x_{2,3})\},\\
\{(x_{3,4};\boldsymbol{x_{3,5}},\boldsymbol{x_{3,3}},x_{3,1}),\ (x_{3,2};x_{3,1},x_{2,2},\boldsymbol{x_{3,3}})\},\\
\{(x_{1,5};x_{1,2},x_{2,5},\boldsymbol{x_{3,5}}),\ (x_{2,5};\boldsymbol{x_{2,3}},\boldsymbol{x_{3,5}},x_{2,1})\},\\
\{(x_{2,6};\boldsymbol{x_{1,6}},\boldsymbol{x_{2,3}},x_{2,5}),\ (x_{2,2};x_{2,1},\boldsymbol{x_{2,3}},x_{2,6})\},\\
\{(x_{2,1};x_{2,8},\boldsymbol{x_{2,3}},\boldsymbol{x_{2,7}}),\ (x_{2,6};\boldsymbol{x_{2,7}},x_{2,4},x_{2,1})\},\\
\{(x_{2,4};\boldsymbol{x_{2,2}},x_{2,5},x_{2,7}),\ (x_{2,5};x_{2,8},\boldsymbol{x_{2,2}},\boldsymbol{x_{2,7}})\},\\
\{(x_{1,7};\boldsymbol{x_{1,8}},x_{2,7},x_{3,7}),\ (x_{3,8};x_{3,7},\boldsymbol{x_{2,8}},\boldsymbol{x_{1,8}})\},\\
\{(x_{2,7};x_{3,7},\boldsymbol{x_{2,3}},\boldsymbol{x_{2,2}}),\ (x_{2,8};x_{2,7},x_{1,8},\boldsymbol{x_{2,2}})\},\\
\{(x_{3,7};x_{3,1},\boldsymbol{x_{3,2}},\boldsymbol{x_{3,3}}),\ (x_{3,8};x_{3,1},\boldsymbol{x_{3,2}},x_{3,3})\},\\
\{(x_{3,7};x_{3,4},\boldsymbol{x_{3,5}},\boldsymbol{x_{3,6}}),\ (x_{3,8};x_{3,4},x_{3,5},\boldsymbol{x_{3,6}})\},\\
\{(x_{1,2};\boldsymbol{x_{2,2}},x_{3,2},x_{1,3}),\ (x_{1,7};\boldsymbol{x_{1,1}},\boldsymbol{x_{1,2}},x_{1,3})\},\\
\{(x_{1,8};x_{1,1},\boldsymbol{x_{1,2}},\boldsymbol{x_{1,3}}),\ (x_{1,4};\boldsymbol{x_{1,2}},x_{1,1},x_{1,6})\},\\
\{(x_{1,3};x_{1,4},x_{1,5},\boldsymbol{x_{1,6}}),\ (x_{1,5};x_{1,4},\boldsymbol{x_{1,1}},\boldsymbol{x_{1,6}})\},\\
\{(x_{1,7};\boldsymbol{x_{1,4}},\boldsymbol{x_{1,5}},x_{1,6}),\ (x_{1,8};x_{1,4},\boldsymbol{x_{1,5}},x_{1,6})\},\\
\{(x_{3,6};x_{3,5},\boldsymbol{x_{3,2}},x_{2,6}),\ (x_{3,5};x_{3,3},\boldsymbol{x_{3,1}},\boldsymbol{x_{3,2}})\}.
\end{array}$$

Now, the last $2S_4$ decompose into $\{1P_4,\ 1S_4\}$ as follows:

$$\{x_{2,6}x_{3,6}x_{3,2}x_{3,5},\ (x_{3,5};x_{3,3},x_{3,1},x_{3,6})\}.$$

By Remark 1.2, required number of paths and stars for remaining choices of p and q can be obtained from the paired stars given above. $\Box$

Lemma 3.6. *There exists a* $(3;p,q)$*-decomposition of* $K_6\square K_8$*, for every admissible pair* (p,q) *of nonnegative integers with* $3(p+q)=|E(K_6\square K_8)|$*.*

Proof. Let $V(K_6\square K_8)=\{x_{i,j}:1\leq i\leq 6,\ 1\leq j\leq 8\}$. We can write

$$\begin{aligned}K_6\square\ K_8 &= 6K_6\ \oplus\ 6(K_8\backslash E(K_2))\ \oplus\ (K_6\backslash\{P_{1,1},P_{1,2}\})\\ &\oplus\ (K_6\backslash\{P_{2,1},P_{2,2}\})\ \oplus\ (P_{1,1}\oplus\ P_{1,2}\ \oplus\ P_{2,1}\ \oplus\ P_{2,2}\ \oplus\ 6K_2),\end{aligned}$$

where $P_{1,1}=x_{3,1}x_{4,1}x_{6,1}x_{5,1}$, $P_{1,2}=x_{3,1}x_{5,1}x_{1,1}x_{6,1}$, $P_{2,1}=x_{3,2}x_{1,2}x_{2,2}x_{5,2}$, $P_{2,2}=x_{1,2}x_{6,2}x_{2,2}x_{3,2}$. Now, by Examples 1 and 2,

$$6(K_8\backslash E(K_2)),K_6\backslash\{P_{1,1},P_{1,2}\}\ \text{and}\ K_6\backslash\{P_{2,1},P_{2,2}\}$$

have a $(3;p,q)$-decomposition. Also by Theorem 1.2, K_6 has a $(3;p,q)$-decomposition. We proved that $(P_{1,1}\ \oplus\ P_{1,2}\ \oplus\ P_{2,1}\ \oplus\ P_{2,2}\ \oplus\ 6K_2)$ has a $(3;p,q)$-decomposition in Lemma 3.1. Hence $K_6\square K_8$ has a $(3;p,q)$-decomposition. □

Lemma 3.7. *There exists a* $(3;p,q)$*-decomposition of* $K_3\square K_4$*, for every admissible pair* (p,q) *of nonnegative integers with* $3(p+q)=|E(K_3\square K_4)|$*.*

Proof. Let $V(K_3\square K_4)=\{x_{i,j}:1\leq i\leq 3,\ 1\leq j\leq 4\}$. First we decompose $K_3\square K_4$ into $10S_4$ as follows:

$$\begin{array}{l}\{(x_{1,1};\boldsymbol{x_{1,4}},\boldsymbol{x_{1,2}},x_{1,3}),\ (x_{1,2};\boldsymbol{x_{3,2}},x_{1,3},x_{1,4})\},\\ \{(x_{1,4};x_{1,3},\boldsymbol{x_{2,4}},x_{3,4}),\ (x_{2,3};\boldsymbol{x_{2,2}},\boldsymbol{x_{2,4}},x_{1,3})\},\\ \{(x_{3,2};x_{2,2},\boldsymbol{x_{3,3}},x_{3,4}),\ (x_{3,4};\boldsymbol{x_{3,1}},\boldsymbol{x_{3,3}},x_{2,4})\},\\ \{(x_{2,2};x_{2,1},x_{1,2},\boldsymbol{x_{2,4}}),\ (x_{2,1};\boldsymbol{x_{1,1}},\boldsymbol{x_{2,4}},x_{2,3})\},\\ \{(x_{3,1};x_{1,1},x_{2,1},x_{3,2}),(x_{3,3};x_{2,3},x_{1,3},x_{3,1})\}.\end{array}$$

From the last $4S_4$ we have either $\{3S_4,\ 1P_4\}$ or $\{1S_4,\ 3P_4\}$ or $\{4P_4\}$ as follows:

$$\begin{Bmatrix}x_{1,2}x_{2,2}x_{2,4}x_{2,1}, & (x_{2,1};x_{1,1},x_{2,2},x_{2,3}),\\ (x_{3,1};x_{1,1},x_{2,1},x_{3,2}), & (x_{3,3};x_{2,3},x_{1,3},x_{3,1})\end{Bmatrix}$$

or

$$\begin{Bmatrix}(x_{2,2};x_{2,1},x_{1,2},x_{2,4}), & x_{1,3}x_{3,3}x_{2,3}x_{2,1},\\ x_{3,2}x_{3,1}x_{1,1}x_{2,1}, & x_{3,3}x_{3,1}x_{2,1}x_{2,4}\end{Bmatrix}$$

or

$$\begin{Bmatrix}x_{1,3}x_{3,3}x_{2,3}x_{2,1}, & x_{3,2}x_{3,1}x_{1,1}x_{2,1},\\ x_{3,3}x_{3,1}x_{2,1}x_{2,2}, & x_{1,2}x_{2,2}x_{2,4}x_{2,1}\end{Bmatrix}$$

By Remark 1.2, required number of paths and stars for remaining choices of p and q can be obtained from the paired stars given above. □

Lemma 3.8. *There exists a* $(3;p,q)$*-decomposition of* $K_4\square K_4$*, for every admissible pair* (p,q) *of nonnegative integers with* $3(p+q)=|E(K_4\square K_4)|$*.*

Proof. Let $V(K_4 \Box K_4) = \{x_{i,j} : 1 \leq i \leq 4,\ 1 \leq j \leq 4\}$. First we decompose $K_4 \Box K_4$ into $16S_4$ as follows:

$$\begin{array}{l}
\left\{(x_{1,3}; \boldsymbol{x_{2,3}}, \boldsymbol{x_{3,3}}, x_{1,4}),\quad (x_{4,3}; x_{1,3}, x_{2,3}, \boldsymbol{x_{3,3}})\right\}, \\
\left\{(x_{2,2}; \boldsymbol{x_{1,2}}, \boldsymbol{x_{2,3}}, x_{2,4}),\quad (x_{4,2}; \boldsymbol{x_{1,2}}, x_{2,2}, x_{3,2})\right\}, \\
\left\{(x_{4,1}; \boldsymbol{x_{2,1}}, \boldsymbol{x_{4,3}}, x_{4,4}),\quad (x_{1,1}; \boldsymbol{x_{2,1}}, x_{3,1}, x_{4,1})\right\}, \\
\left\{(x_{4,4}; x_{4,3}, \boldsymbol{x_{2,4}}, x_{1,4}),\quad (x_{4,2}; \boldsymbol{x_{4,1}}, \boldsymbol{x_{4,4}}, x_{4,3})\right\}, \\
\left\{(x_{1,2}; \boldsymbol{x_{1,3}}, \boldsymbol{x_{1,4}}, x_{3,2}),\quad (x_{1,1}; x_{1,2}, \boldsymbol{x_{1,3}}, x_{1,4})\right\}, \\
\left\{(x_{2,4}; x_{1,4}, \boldsymbol{x_{3,4}}, x_{2,3}),\quad (x_{2,1}; \boldsymbol{x_{2,2}}, \boldsymbol{x_{2,4}}, x_{2,3})\right\}, \\
\left\{\begin{array}{ll} (x_{3,4}; \boldsymbol{x_{1,4}}, \boldsymbol{x_{3,1}}, x_{4,4}), & (x_{3,3}; x_{2,3}, x_{3,4}, \boldsymbol{x_{3,1}}), \\ (x_{3,2}; x_{2,2}, x_{3,3}, x_{3,4}), & (x_{3,1}; x_{2,1}, x_{4,1}, x_{3,2}) \end{array}\right\}.
\end{array}$$

From the last $4S_4$ we have either $\{3S_4,\ 1P_4\}$ or $\{1S_4,\ 3P_4\}$ or $\{4P_4\}$ as follows:

$$\left\{\begin{array}{ll} (x_{3,2}; x_{2,2}, x_{3,3}, x_{3,4}), & (x_{3,1}; x_{2,1}, x_{4,1}, x_{3,2}), \\ (x_{3,4}; x_{1,4}, x_{3,3}, x_{4,4}), & x_{2,3}x_{3,3}x_{3,1}x_{3,4} \end{array}\right\}$$

or

$$\left\{\begin{array}{ll} (x_{3,1}; x_{2,1}, x_{3,4}, x_{3,3}), & x_{2,2}x_{3,2}x_{3,1}x_{4,1}, \\ x_{1,4}x_{3,4}x_{3,2}x_{3,3}, & x_{2,3}x_{3,3}x_{3,4}x_{4,4} \end{array}\right\}$$

or

$$\left\{\begin{array}{ll} x_{2,2}x_{3,2}x_{3,1}x_{4,1}, & x_{2,3}x_{3,3}x_{3,4}x_{4,4}, \\ x_{3,4}x_{3,2}x_{3,3}x_{3,1}, & x_{1,4}x_{3,4}x_{3,1}x_{2,1} \end{array}\right\}.$$

By Remark 1.2, required number of paths and stars for remaining choices of p and q can be obtained from the paired stars given above. □

Lemma 3.9. *There exists a* $(3; p, q)$*-decomposition of* $K_3 \Box K_3$*, for every admissible pair* (p, q) *of nonnegative integers with* $3(p + q) = |E(K_3 \Box K_3)|$ *and* $p \neq 0$.

Proof. Let $V(K_3 \Box K_3) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 3\}$. First we decompose $K_3 \Box K_3$ into $5S_4$ and $1P_4$ as follows:

$$\begin{array}{l}
\left\{(x_{3,2}; \boldsymbol{x_{3,1}}, \boldsymbol{x_{2,2}}, x_{3,3}),\ (x_{1,2}; \boldsymbol{x_{2,2}}, x_{3,2}, x_{1,3})\right\}, \\
\left\{(x_{2,1}; \boldsymbol{x_{1,1}}, \boldsymbol{x_{2,3}}, x_{2,2}),\ (x_{2,3}; x_{1,3}, \boldsymbol{x_{3,3}}, x_{2,2})\right\}, \\
\left\{(x_{1,1}; x_{1,2}, x_{1,3}, x_{3,1}),\ x_{1,3}x_{3,3}x_{3,1}x_{2,1}\right\}.
\end{array}$$

The graphs in the last bracket has a P_4 decomposition as $\{x_{1,1}x_{1,3}x_{3,3}x_{3,1},\ x_{2,1}x_{3,1}x_{1,1}x_{1,2}\}$. By Remark 1.2, required number of paths and stars for remaining choices of p and q can be obtained from the paired stars given above.. □

Lemma 3.10. *There exists a $(3;p,q)$-decomposition of $K_3 \Box K_2$, for every admissible pair (p,q) of nonnegative integers with $3(p+q) = |E(K_3 \Box K_2)|$ and $p \neq 0$.*

Proof. Let $V(K_3 \Box K_2) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 2\}$. We prove $K_3 \Box K_2$ has a $(3;p,q)$-decomposition as follows:

1. $p = 1,\ q = 2$. The required paths and stars are $x_{3,1}x_{2,1}x_{2,2}x_{1,2}$ and $(x_{1,1}; x_{1,2}, x_{2,1}, x_{3,1})$, $(x_{3,2}; x_{3,1}, x_{2,2}, x_{1,2})$ respectively.
2. $p = 2,\ q = 1$. The required paths and stars are $x_{2,1}x_{2,2}x_{1,2}x_{3,2}$, $x_{2,2}x_{3,2}x_{3,1}x_{2,1}$ and $(x_{1,1}; x_{1,2}, x_{2,1}, x_{3,1})$ respectively.
3. $p = 3,\ q = 0$. The required paths are $x_{3,2}x_{3,1}x_{1,1}x_{2,1}$, $x_{1,1}x_{1,2}x_{3,2}x_{2,2}$, $x_{3,1}x_{2,1}x_{2,2}x_{1,2}$.

□

Lemma 3.11. *There exists a $(3;p,q)$-decomposition of $K_6 \Box K_2$, for every admissible pair (p,q) of nonnegative integers with $3(p+q) = |E(K_6 \Box K_2)|$.*

Proof. Let $V(K_6 \Box K_2) = \{x_{i,j} : 1 \leq i \leq 6,\ 1 \leq j \leq 2\}$. We can write

$$K_6 \Box K_2 = (K_6 \backslash \{P_{1,1}, P_{1,2}\}) \oplus (K_6 \backslash \{P_{2,1}, P_{2,2}\}) \oplus (P_{1,1} \oplus P_{1,2} \oplus P_{2,1} \oplus P_{2,2} \oplus 6K_2),$$

where $P_{1,1} = x_{3,1}x_{4,1}x_{6,1}x_{5,1}$, $P_{1,2} = x_{3,1}x_{5,1}x_{1,1}x_{6,1}$, $P_{2,1} = x_{3,2}x_{1,2}x_{2,2}x_{5,2}$, $P_{2,2} = x_{1,2}x_{6,2}x_{2,2}x_{3,2}$. Now, by Examples 1 and 2, $K_6 \backslash \{P_{1,1}, P_{1,2}\}$ and $K_6 \backslash \{P_{2,1}, P_{2,2}\}$ have a $(3;p,q)$-decomposition. We can prove $(P_{1,1} \oplus P_{1,2} \oplus P_{2,1} \oplus P_{2,2} \oplus 6K_2)$ has a $(3;p,q)$-decomposition as in Lemma 3.1. Hence $K_6 \Box K_2$ has a $(3;p,q)$-decomposition. □

Theorem 3.12. *The graph $K_m \Box K_n$ has a $(3;p,q)$-decomposition for every admissible pair (p,q) of nonnegative integers with $3(p+q) = E(K_m \Box K_n)$ if and only if $mn(m+n-2) \equiv 0 \pmod 6$.*

Proof. **Necessity.** Since $K_m \Box K_n$ is $(m+n-2)$-regular with mn vertices, the necessity follows from Lemma 2.5.
Sufficiency. To construct the required decomposition, we consider the following two cases.

Case(1) $m, n \equiv 0$ *or* $1 \pmod 3$.

We can write $K_m \Box K_n = nK_m \oplus mK_n$. By Theorem 1.2, K_m and K_n have a $(3;p,q)$-decomposition for $m, n \geq 6$. For $m, n < 6$, $K_m \Box K_n$ has a $(3;p,q)$-decomposition, by Lemmas 3.7 to 3.9.

Without loss of generality, assume that $m < 6$ and $n > 6$. To construct the required decomposition, we consider the following four subcases.

Subcase 1(i) $m = 3$ and $n = 3k$.

If $n = 6l$ and $l \in \mathbb{Z}^+$, then we can write $K_m \square K_n = l(K_3 \square K_6) \oplus \frac{3l(l-1)}{2} K_{6,6}$. By Theorem 1.1 and Lemma 3.3, $K_{6,6}$ and $K_3 \square K_6$ have a $(3; p, q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(3; p, q)$-decomposition.

If $n = 6l + 3$ and $l \in \mathbb{Z}^+$, then we can write $K_m \square K_n = l(K_3 \square K_6) \oplus (K_3 \square K_3) \oplus \frac{3l(l-1)}{2} K_{6,6} \oplus 3lK_{3,6}$. By Lemma 3.3 and Theorem 1.1, $K_3 \square K_6$, $K_{6,6}$ and $K_{3,6}$ have a $(3; p, q)$-decomposition. Also by Lemma 3.9, $K_3 \square K_3$ has a $(3; p, q)$-decomposition with $p \neq 0$. Hence by Remark 1.1, $K_m \square K_n$ has a $(3; p, q)$-decomposition with $p \neq 0$. For $p = 0$, consider $K_m \square K_n$ as $(l-1)(K_3 \square K_6) \oplus (K_3 \square K_9) \oplus \frac{3(l-1)(l-2)}{2} K_{6,6} \oplus 3(l-1)K_{6,9}$. By Lemma 3.3 and Theorem 1.1, $K_3 \square K_6$, $K_{6,6}$ and $K_{6,9}$ have a $(3; p, q)$-decomposition. So it is enough to prove that $K_3 \square K_9$ possess a S_4-decomposition. Let $V(K_3 \square K_9) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 9\}$. Now,

$$(x_{i,j}; x_{i+1,j}, x_{i,j+1} x_{i,j+2}),$$

where $i = 1, 2, 3$ and $j = 1, 2, \cdots, 9$ and

$$\begin{aligned} &(x_{i,1}; x_{i,4}, x_{i,5}, x_{i,7}), && (x_{i,2}; x_{i,6}, x_{i,7}, x_{i,8}), \\ &(x_{i,3}; x_{i,7}, x_{i,8}, x_{i,9}), && (x_{i,4}; x_{i,7}, x_{i,8}, x_{i,9}), \\ &(x_{i,5}; x_{i,2}, x_{i,8}, x_{i,9}), && (x_{i,6}; x_{i,1}, x_{i,3}, x_{i,9}), \end{aligned}$$

where $i = 1, 2, 3$ and the subscripts in the first coordinate are taken modulo 3 with residues $\{1, 2, 3\}$ and the subscripts in the second coordinate are taken modulo 9 with residues $\{1, 2, \cdots, 9\}$, gives a required S_4-decomposition of $K_3 \square K_9$. Hence by Remark 1.1, $K_m \square K_n$ has a $(3; p, q)$-decomposition.

Subcase 1(ii) $m = 3$ and $n = 3k + 1$.

If $n = 7$, then we can write $K_m \square K_n = (K_3 \square K_4) \oplus (K_3 \square K_3) \oplus 3K_{3,4}$. By Lemma 3.7 and Theorem 1.1, $K_3 \square K_4$ and $K_{3,4}$ have a $(3; p, q)$-decomposition. Also by Lemma 3.9, $K_3 \square K_3$ has a $(3; p, q)$-decomposition with $p \neq 0$. Hence by Remark 1.1, $K_m \square K_n$ has a $(3; p, q)$-decomposition with $p \neq 0$. For $p = 0$ the S_4-decomposition of $K_3 \square K_7$ with

$$V(K_3 \square K_7) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 7\}$$

is given below.

$(x_{1,1}; x_{1,2}, x_{2,1}, x_{3,1})$, $(x_{3,1}; x_{2,1}, x_{3,1}, x_{3,2})$, $(x_{1,2}; x_{2,2}, x_{1,3}, x_{1,4})$,
$(x_{3,2}; x_{2,2}, x_{1,2}, x_{3,3})$, $(x_{1,3}; x_{1,4}, x_{2,3}, x_{3,3})$, $(x_{3,3}; x_{2,3}, x_{3,4}, x_{3,5})$,
$(x_{1,4}; x_{2,4}, x_{1,1}, x_{1,5})$, $(x_{3,4}; x_{3,5}, x_{1,4}, x_{2,4})$, $(x_{1,5}; x_{1,2}, x_{1,6}, x_{2,5})$,
$(x_{1,6}; x_{1,2}, x_{1,4}, x_{2,6})$, $(x_{1,7}; x_{1,2}, x_{1,6}, x_{2,7})$, $(x_{2,5}; x_{2,6}, x_{2,7}, x_{3,5})$,
$(x_{2,6}; x_{2,4}, x_{2,7}, x_{3,6})$, $(x_{2,7}; x_{2,3}, x_{2,4}, x_{3,7})$, $(x_{3,5}; x_{3,2}, x_{3,6}, x_{1,5})$,
$(x_{3,6}; x_{3,3}, x_{3,4}, x_{1,6})$, $(x_{3,7}; x_{3,5}, x_{3,6}, x_{1,7})$, $(x_{1,1}; x_{1,5}, x_{1,6}, x_{1,7})$,
$(x_{1,3}; x_{1,1}, x_{1,5}, x_{1,6})$, $(x_{1,7}; x_{1,3}, x_{1,4}, x_{1,5})$, $(x_{2,1}; x_{2,3}, x_{2,6}, x_{2,7})$,
$(x_{2,2}; x_{2,1}, x_{2,6}, x_{2,7})$, $(x_{2,3}; x_{2,2}, x_{2,4}, x_{2,6})$, $(x_{2,4}; x_{2,1}, x_{2,2}, x_{2,5})$,
$(x_{2,5}; x_{2,1}, x_{2,2}, x_{2,3})$, $(x_{3,1}; x_{3,4}, x_{3,5}, x_{3,6})$, $(x_{3,2}; x_{3,4}, x_{3,6}, x_{3,7})$,
$(x_{3,7}; x_{3,1}, x_{3,3}, x_{3,4})$.

If $n = 6l + 1$ and $l \geq 2$ is an integer, then we can write

$$K_m \Box K_n = (K_3 \Box K_{6(l-1)+3}) \oplus (K_3 \Box K_4) \oplus 3K_{6(l-1)+3,4}.$$

By Lemma 3.7 and Theorem 1.1, $K_3 \Box K_4$ and $K_{6(l-1)+3,4}$ have a $(3; p, q)$-decomposition. Also by Subcase 1(i), $K_3 \Box K_{6(l-1)+3}$ has a $(3; p, q)$-decomposition. Hence by Remark 1.1, $K_m \Box K_n$ has a $(3; p, q)$-decomposition.

If $n = 6l + 4$ and $l \geq 1$ is an integer, then we can write $K_m \Box K_n = (K_3 \Box K_{6l}) \oplus (K_3 \Box K_4) \oplus 3K_{6l,4}$. By Lemma 3.7 and Theorem 1.1, $K_3 \Box K_4$ and $K_{6l,4}$ have a $(3; p, q)$-decomposition. Also by Subcase 1(i), $K_3 \Box K_{6l}$ has a $(3; p, q)$-decomposition. Hence by Remark 1.1, $K_m \Box K_n$ has a $(3; p, q)$-decomposition.

Subcase 1(iii) $m = 4$ and $n = 3k$.

We can write

$$K_m \Box K_n = k(K_4 \Box K_3) \oplus 2k(k-1)K_{3,3}.$$

By Theorem 1.1 and Lemma 3.7, $K_{3,3}$ and $K_4 \Box K_3$ have a $(3; p, q)$-decomposition. Hence by Remark 1.1, $K_m \Box K_n$ has a $(3; p, q)$-decomposition.

Subcase 1(iv) $m = 4$ and $n = 3k + 1$.

We can write

$$K_m \Box K_n = (k-1)(K_4 \Box K_3) \oplus (K_4 \Box K_4) \oplus 2(k-1)(k-2)K_{3,3} \oplus 4(k-1)K_{3,4}.$$

By Theorem 1.1, $K_{3,3}$ and $K_{3,4}$ have a $(3; p, q)$-decomposition. Also by Lemmas 3.7 and 3.8, $K_4 \Box K_3$ and $K_4 \Box K_4$ have a $(3; p, q)$-decomposition. Hence by Remark 1.1, $K_m \Box K_n$ has a $(3; p, q)$-decomposition.

Case(2) $m \equiv 0 \pmod 3$, $n \equiv 2 \pmod 3$.

We can write

$$K_m \square K_n = nK_m \ \oplus \ mK_n.$$

To construct the required decomposition, we consider the following four subcases.

Subcase 2(i) $m \equiv 0 \pmod 6$, $n \equiv 5 \pmod 6$.

Let $m = 6k$, $k \in \mathbb{Z}^+$ and $n = 6l + 5$, $l \geq 0$ be an integer. We can write

$$K_m \square K_n = (K_{6k} \square K_{6l}) \ \oplus \ (K_{6k} \square K_5) \ \oplus \ 6kK_{6l,5} =$$
$$(K_{6k} \square K_{6l}) \ \oplus \ k(K_6 \square K_5) \ \oplus \ \frac{5k(k-1)}{2} K_{6,6} \ \oplus \ 6kK_{6l,5}.$$

By Lemma 3.1 and Theorem 1.1, $K_6 \square K_5$, $K_{6,6}$ and $K_{6l,5}$ have a $(3;p,q)$-decomposition. Also by Case 1, $K_{6k} \square K_{6l}$ has a $(3;p,q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(3;p,q)$-decomposition.

Subcase 2(ii) $m \equiv 0 \pmod 6$, $n \equiv 2 \pmod 6$.

When $m = 6k$, $k \in \mathbb{Z}^+$ and $n = 2$, $K_m \square K_n = k(K_6 \square K_2) \ \oplus \ k(k-1)K_{6,6}$. By Theorem 1.1 and Lemma 3.11, $K_m \square K_n$ has a $(3;p,q)$-decomposition. When $n > 2$, let $m = 6k$, $n = 6l + 2$, $k, l \in \mathbb{Z}^+$. We can write

$$K_m \square K_n = (K_{6k} \square K_{6(l-1)}) \ \oplus \ (K_{6k} \square K_8) \ \oplus \ 6kK_{6(l-1),8}$$
$$= (K_{6k} \square K_{6(l-1)}) \ \oplus \ k(K_6 \square K_8) \ \oplus \ 4k(k-1)K_{6,6} \ \oplus \ 6kK_{6(l-1),8}$$

By Theorem 1.1 and Lemma 3.6, $K_{6,6}$, $K_{6(l-1),8}$ and $K_6 \square K_8$ have a $(3;p,q)$-decomposition. Also by Case 1, $K_{6k} \square K_{6(l-1)}$ has a $(3;p,q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(3;p,q)$-decomposition.

Subcase 2(iii) $m \equiv 3 \pmod 6$, $n \equiv 5 \pmod 6$.

Let $m = 6k + 3$ and $n = 6l + 5$, $k, l \geq 0$ be integers. We can write

$$K_m \square K_n = (K_{6k+3} \square K_{6l}) \ \oplus \ (K_{6k+3} \square K_5) \ \oplus \ (6k+3)K_{6l,5}$$
$$= (K_{6k+3} \square K_{6l}) \ \oplus \ k(K_6 \square K_5) \ \oplus \ (K_3 \square K_5)$$
$$\oplus \ \frac{5k(k-1)}{2} K_{6,6} \ \oplus \ 5kK_{3,6} \ \oplus \ (6k+3)K_{6l,5}.$$

By Lemmas 3.1, 3.2, 3.3 and Theorem 1.1, $K_6 \square K_5$, $K_3 \square K_6$, $K_3 \square K_5$, $K_{6,6}$, $K_{3,6}$ and $K_{6l,5}$ have a $(3;p,q)$-decomposition. Also by Case 1, $K_{6k+3} \square K_{6l}$ has a $(3;p,q)$-decomposition. Hence by Remark 1.1, $K_m \square K_n$ has a $(3;p,q)$-decomposition.

Subcase 2(iv) $m \equiv 3 \pmod 6$, $n \equiv 2 \pmod 6$.

When $m = 3$ and $n = 2$, $K_m \Box K_n$ has a $(3; p, q)$-decomposition, by Lemma 3.10.

When $m = 6k+3$ with $k \in \mathbb{Z}^+$ and $n = 2$, $K_m \Box K_n = (K_{6k} \Box K_2) \oplus (K_3 \Box K_2) \oplus 2K_{6k,3}$. By Theorem 1.1 and Subcase 2(ii), $K_{6k,3}$ and $K_{6k} \Box K_2$ have a $(3; p, q)$-decomposition. Also by Lemma 3.11, $K_3 \Box K_2$ has a $(3; p, q)$-decomposition with $p \neq 0$. Hence by Remark 1.1, $K_m \Box K_n$ has a $(3; p, q)$-decomposition with $p \neq 0$. For $p = 0$, consider $K_m \Box K_n$ as $(K_{6(k-1)} \Box K_2) \oplus (K_9 \Box K_2) \oplus 2K_{(6k-1),3}$. By Theorem 1.1 and Subcase 2(ii), $K_{6(k-1),3}$ and $K_{6(k-1)} \Box K_2$ have a $(3; p, q)$-decomposition. So it is enough to prove that $K_9 \Box K_2 (\cong K_2 \Box K_9)$ has a S_4-decomposition. Consider $K_2 \Box K_9$ as $9K_2 \oplus 2K_9 = (9K_2 \oplus K_9) \oplus K_9$. Now, K_9 has a S_4-decomposition, by Theorem 1.2 with $p = 0$. Let $V(K_2 \Box K_9) = \{x_{i,j} : 1 \le i \le 2,\ 1 \le j \le 9\}$. Now,

$$\begin{array}{ll} (x_{1,1}; x_{1,4}, x_{1,5}, x_{1,7}), & (x_{1,2}; x_{1,6}, x_{1,7}, x_{1,8}), \\ (x_{1,3}; x_{1,7}, x_{1,8}, x_{1,9}), & (x_{1,4}; x_{1,7}, x_{1,8}, x_{1,9}), \\ (x_{1,5}; x_{1,2}, x_{1,8}, x_{1,9}), & (x_{1,6}; x_{1,1}, x_{1,3}, x_{1,9}) \end{array}$$

and $(x_{1,j}; x_{2,j}, x_{1,j+1} x_{1,j+2})$, for $j = 1, 2, \cdots, 9$, where the subscripts in the second coordinate are taken modulo 9 with residues $\{1, 2, \cdots, 9\}$, gives the S_4-decomposition of $9K_2 \oplus K_9$. Hence $K_m \Box K_n$ has a $(3; p, q)$-decomposition.

When $n > 2$, let $m = 6k+3$ and $n = 6l+2$, where $k \ge 0$, $l > 0$ are integers. We can write

$$\begin{aligned} K_m \Box K_n &= (K_{6k} \Box K_{6l+2}) \oplus (K_3 \Box K_{(6l+2)}) \oplus (6l+2) K_{3,6k} \\ &= (K_{6k} \Box K_{6l+2}) \oplus (K_3 \Box K_{6(l-1)}) \oplus (K_3 \Box K_8) \\ &\qquad \oplus 3K_{6(l-1),8} \oplus (6l+2) K_{3,6k}. \end{aligned}$$

By Lemma 3.5 and Theorem 1.1, $K_3 \Box K_8$, $K_{6(l-1),8}$ and $K_{3,6k}$ have a $(3; p, q)$-decomposition. Also by Case 1 and Subcase 2(ii), $K_3 \Box K_{6(l-1)}$ and $K_{6k} \Box K_{6l+2}$ have a $(3; p, q)$-decomposition. Hence by Remark 1.1, $K_m \Box K_n$ has a $(3; p, q)$-decomposition.

□

4 $(3; p, q)$-decomposition of $K_m \times K_n$

In this section we investigate the existence of $(3; p, q)$-decomposition of tensor product of complete graphs.

Lemma 4.1. *Let G be an S_4-decomposible graph and $p, q \ge 0$ be integers with $3(p+q) = |E(G \times K_n)|$ and $p \neq 1$. Then $G \times K_n$ has a $(3; p, q)$-decomposition for all odd n and every admissible pair (p, q).*

Proof. Let $V(G \times K_n) = \{x_{g,i} : g \in V(G) \text{ and } 1 \leq i \leq n\}$. Since G is S_4-decomposible graph, for each star $(a; u, v, w)$ in G, we have the following pair of stars in $G \times K_n$:

- for each $j \in \{1, 3, \cdots, n-2\}$

$$\{(x_{a,j}; x_{u,i}, \boldsymbol{x_{v,i}}, \boldsymbol{x_{w,i}}),\ (x_{a,j+1}; x_{u,i}, x_{v,i}, \boldsymbol{x_{w,i}})\},$$

  where $1 \leq i \leq n$ and $i \neq j, j+1$;

- for $1 \leq i \leq n-1$,

$$\{(x_{a,n}; x_{u,i-1}, \boldsymbol{x_{v,i-1}}, \boldsymbol{x_{w,i-1}}),\ (x_{a,i}; x_{u,i-1}, \boldsymbol{x_{v,i-1}}, \boldsymbol{x_{w,i-1}})\},$$

  if i is even and

$$\{(x_{a,n}; x_{u,i+1}, \boldsymbol{x_{v,i+1}}, \boldsymbol{x_{w,i+1}}),\ (x_{a,i}; x_{u,i+1}, \boldsymbol{x_{v,i+1}}, \boldsymbol{x_{w,i+1}})\},$$

  if i is odd.

Then by applying remark 1.2 to the pairs of stars mentioned above we obtained all possible even number of paths and stars of $G \times K_n$. Now, consider$\{(x_{a,1}; x_{u,2}, x_{v,2}, x_{w,2}),\ (x_{a,1}; x_{u,3}, x_{v,3}, x_{w,3}),\ (x_{a,2}; x_{u,3}, x_{v,3}, x_{w,3})\}$ and decompose it into $3P_4$ as given below. $\{x_{u,2}x_{a,1}x_{u,3}x_{a,2},\ x_{v,2}x_{a,1}x_{v,3}x_{a,2},\ x_{w,2}x_{a,1}x_{w,3}x_{a,2}\}$. The remaining number of paths and stars can be obtained from the remaining pairs of stars given above except when $p = 1$. □

Lemma 4.2. *There exists a $(3; p, q)$-decomposition of $K_3 \times K_3$, for every admissible pair (p, q) of nonnegative integers with $3(p+q) = |E(K_3 \times K_3)|$.*

Proof. Let $V(K_3 \times K_3) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 3\}$. Now, $K_3 \times K_3$ has a $(3; p, q)$-decomposition as follows:

1. $p = 0,\ q = 6$. The required stars are
   $(x_{1,1}; x_{2,2}, x_{2,3}, x_{3,3})$, $(x_{1,2}; x_{2,1}, x_{2,3}, x_{3,1})$, $(x_{1,3}; x_{2,1}, x_{2,2}, x_{3,2})$, $(x_{3,1}; x_{1,3}, x_{2,2}, x_{2,3})$, $(x_{3,2}; x_{1,1}, x_{2,1}, x_{2,3})$, $(x_{3,3}; x_{1,2}, x_{2,1}, x_{2,2})$.
2. $p = 1,\ q = 5$. The required path and stars are
   $x_{2,2}x_{1,1}x_{2,3}x_{1,2}$ and $(x_{2,1}; x_{1,2}, x_{3,2}, x_{3,3})$, $(x_{1,3}; x_{2,1}, x_{2,2}, x_{3,1})$, $(x_{3,1}; x_{1,2}, x_{2,2}, x_{2,3})$, $(x_{3,2}; x_{1,1}, x_{1,3}, x_{2,3})$, $(x_{3,3}; x_{1,2}, x_{1,1}, x_{2,2})$ respectively.
3. $p = 2,\ q = 4$. The required paths and stars are
   $x_{3,3}x_{1,1}x_{2,3}x_{1,2}$, $x_{1,1}x_{2,2}x_{3,3}x_{1,2}$ and $(x_{2,1}; x_{1,2}, x_{3,2}, x_{3,3})$, $(x_{1,3}; x_{2,1}, x_{2,2}, x_{3,1})$, $(x_{3,1}; x_{1,2}, x_{2,2}, x_{2,3})$, $(x_{3,2}; x_{1,1}, x_{1,3}, x_{2,3})$ respectively.

4. $p = 3,\ q = 3$. The required paths and stars are
$x_{3,3}x_{1,1}x_{2,3}x_{1,2}$, $x_{2,2}x_{3,3}x_{1,2}x_{3,1}$, $x_{2,3}x_{3,1}x_{2,2}x_{1,1}$ and
$(x_{2,1}; x_{1,2}, x_{3,2}, x_{3,3})$, $(x_{1,3}; x_{2,1}, x_{2,2}, x_{3,1})$, $(x_{3,2}; x_{1,1}, x_{1,3}, x_{2,3})$
respectively.

5. $p = 4,\ q = 2$. The required paths and stars are
$x_{3,3}x_{1,1}x_{2,3}x_{1,2}$, $x_{2,2}x_{3,3}x_{1,2}x_{3,1}$, $x_{3,1}x_{2,2}x_{1,1}x_{3,2}$,
$x_{3,1}x_{2,3}x_{3,2}x_{1,3}$ and $(x_{2,1}; x_{1,2}, x_{3,2}, x_{3,3})$, $(x_{1,3}; x_{2,1}, x_{2,2}, x_{3,1})$
respectively.

6. $p = 5,\ q = 1$. The required paths and star are
$x_{3,3}x_{1,1}x_{2,3}x_{1,2}$, $x_{2,2}x_{3,3}x_{1,2}x_{3,1}$, $x_{3,1}x_{2,2}x_{1,1}x_{3,2}$,
$x_{2,3}x_{3,2}x_{1,3}x_{2,2}$ $x_{2,1}x_{1,3}x_{3,1}x_{2,3}$ and $(x_{2,1}; x_{1,2}, x_{3,2}, x_{3,3})$
respectively.

7. $p = 6,\ q = 0$. The required paths are
$x_{1,1}x_{2,3}x_{1,2}x_{2,1}$, $x_{3,2}x_{2,1}x_{3,3}x_{1,1}$, $x_{2,2}x_{3,3}x_{1,2}x_{3,1}$,
$x_{3,1}x_{2,2}x_{1,1}x_{3,2}$, $x_{2,3}x_{3,2}x_{1,3}x_{2,2}$ $x_{2,1}x_{1,3}x_{3,1}x_{2,3}$.

□

Lemma 4.3. *There exists a* $(3; p, q)$*-decomposition of* $K_3 \times K_4$*, for every admissible pair* (p, q) *of nonnegative integers with* $3(p+q) = |E(K_3 \times K_4)|$.

Proof. Let $V(K_3 \times K_4) = \{x_{i,j} : 1 \le i \le 3,\ 1 \le j \le 4\}$. First we decompose $K_3 \times K_4$ into $12S_4$ as follows:

$$\begin{array}{l}\{(x_{1,1}; \boldsymbol{x_{2,2}}, \boldsymbol{x_{2,3}}, x_{2,4}),\ (x_{1,2}; x_{2,1}, \boldsymbol{x_{2,3}}, x_{2,4})\},\\ \{(x_{2,1}; \boldsymbol{x_{3,2}}, \boldsymbol{x_{3,3}}, x_{3,4}),\ (x_{2,2}; x_{3,1}, \boldsymbol{x_{3,3}}, x_{3,4})\},\\ \{(x_{2,3}; x_{3,1}, \boldsymbol{x_{3,2}}, \boldsymbol{x_{3,4}}),\ (x_{2,4}; x_{3,1}, \boldsymbol{x_{3,2}}, x_{3,3})\},\\ \{(x_{3,3}; x_{1,1}, \boldsymbol{x_{1,2}}, \boldsymbol{x_{1,4}}),\ (x_{3,4}; x_{1,1}, \boldsymbol{x_{1,2}}, x_{1,3})\},\\ \{(x_{3,1}; \boldsymbol{x_{1,2}}, \boldsymbol{x_{1,3}}, x_{1,4}),\ (x_{3,2}; x_{1,1}, \boldsymbol{x_{1,3}}, x_{1,4})\},\\ \{(x_{1,3}; x_{2,1}, \boldsymbol{x_{2,2}}, \boldsymbol{x_{2,4}}),\ (x_{1,4}; x_{2,1}, \boldsymbol{x_{2,2}}, x_{2,3})\}.\end{array}$$

Now, the last $3S_4$ can be decomposed into $3P_4$ as follows:

$$\{x_{1,1}x_{3,2}x_{1,3}x_{2,4},\ x_{3,2}x_{1,4}x_{2,1}x_{1,3},\ x_{1,3}x_{2,2}x_{1,4}x_{2,3}\}.$$

Decomposition for the remaining choices of $p \ne 1$ can be obtained from the paired stars given above, by Remark 1.2. When $p = 1$, the required path and stars are

$$\begin{array}{lll}(x_{1,1}; x_{3,3}, x_{2,3}, x_{3,2}), & (x_{2,4}; x_{1,1}, x_{1,2}, x_{3,3}), & (x_{2,1}; x_{1,2}, x_{1,3}, x_{1,4}),\\ (x_{2,3}; x_{1,2}, x_{1,4}, x_{3,2}), & (x_{2,1}; x_{3,2}, x_{3,3}, x_{3,4}), & (x_{3,1}; x_{2,2}, x_{2,3}, x_{2,4}),\\ (x_{3,1}; x_{1,2}, x_{1,3}, x_{1,4}), & (x_{3,2}; x_{1,3}, x_{1,4}, x_{2,4}), & (x_{3,3}; x_{2,2}, x_{1,2}, x_{1,4}),\\ (x_{1,3}; x_{2,2}, x_{3,4}, x_{2,4}), & (x_{3,4}; x_{2,2}, x_{1,2}, x_{2,3}), & x_{3,4}x_{1,1}x_{2,2}x_{1,4}.\end{array}$$

□

Lemma 4.4. *There exists a $(3;p,q)$-decomposition of $K_3 \times K_5$, for every admissible pair (p,q) of nonnegative integers with $3(p+q) = |E(K_3 \times K_5)|$.*

Proof. Let $V(K_3 \times K_5) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 5\}$. First we decompose $K_3 \times K_5$ into $20S_4$ as follows:

$$\begin{array}{c}
\{(x_{1,1}; x_{2,2}, \boldsymbol{x_{2,3}}, \boldsymbol{x_{2,4}}),\ (x_{1,3}; x_{2,1}, x_{2,2}, \boldsymbol{x_{2,4}})\}, \\
\{(x_{1,1}; x_{3,2}, \boldsymbol{x_{3,3}}, \boldsymbol{x_{3,4}}),\ (x_{1,3}; x_{3,1}, x_{3,2}, \boldsymbol{x_{3,4}})\}, \\
\{(x_{1,4}; x_{2,1}, \boldsymbol{x_{2,5}}, \boldsymbol{x_{2,2}}),\ (x_{1,5}; x_{2,1}, \boldsymbol{x_{2,2}}, x_{2,4})\}, \\
\{(x_{1,4}; x_{3,1}, \boldsymbol{x_{3,2}}, \boldsymbol{x_{3,5}}),\ (x_{1,5}; x_{3,1}, \boldsymbol{x_{3,2}}, x_{3,4})\}, \\
\{(x_{2,3}; x_{1,4}, \boldsymbol{x_{1,5}}, \boldsymbol{x_{3,1}}),\ (x_{3,3}; x_{1,4}, \boldsymbol{x_{1,5}}, x_{2,1})\}, \\
\{(x_{2,5}; \boldsymbol{x_{1,1}}, \boldsymbol{x_{1,2}}, x_{1,3}),\ (x_{3,5}; x_{1,1}, \boldsymbol{x_{1,2}}, x_{1,3})\}, \\
\{(x_{2,1}; \boldsymbol{x_{3,2}}, \boldsymbol{x_{3,4}}, x_{3,5}),\ (x_{2,2}; x_{3,1}, \boldsymbol{x_{3,4}}, x_{3,5})\}, \\
\{(x_{2,4}; x_{3,1}, \boldsymbol{x_{3,2}}, \boldsymbol{x_{3,5}}),\ (x_{2,5}; x_{3,1}, \boldsymbol{x_{3,2}}, x_{3,4})\}, \\
\{(x_{2,3}; x_{3,2}, x_{3,4}, x_{3,5}),\ (x_{3,3}; x_{2,2}, x_{2,4}, x_{2,5})\}, \\
\{(x_{1,2}; x_{2,1}, x_{2,3}, x_{2,4}),\ (x_{1,2}; x_{3,1}, x_{3,3}, x_{3,4})\}.
\end{array}$$

Now, the last $4S_4$ can be decomposed into either $\{1P_4,\ 3S_4\}$ or $\{2P_4,\ 2S_4\}$ or $\{3P_4,\ 1S_4\}$ or $\{4P_4\}$ as follows:

$$\begin{Bmatrix} x_{3,3}x_{1,2}x_{3,4}x_{2,3}, & (x_{2,3}; x_{3,2}, x_{1,2}, x_{3,5}), \\ (x_{3,3}; x_{2,2}, x_{2,4}, x_{2,5}), & (x_{1,2}; x_{2,1}, x_{3,1}, x_{2,4}) \end{Bmatrix}$$

or

$$\begin{Bmatrix} x_{2,2}x_{3,3}x_{1,2}x_{3,1}, & x_{2,5}x_{3,3}x_{2,4}x_{1,2}, \\ (x_{2,3}; x_{3,2}, x_{3,4}, x_{3,5}), & (x_{1,2}; x_{2,1}, x_{2,3}, x_{3,4}) \end{Bmatrix}$$

or

$$\begin{Bmatrix} x_{2,2}x_{3,3}x_{1,2}x_{3,1}, & x_{2,5}x_{3,3}x_{2,4}x_{1,2}, \\ x_{2,3}x_{3,4}x_{1,2}x_{2,1}, & (x_{2,3}; x_{3,2}, x_{1,2}, x_{3,5}) \end{Bmatrix}$$

or

$$\begin{Bmatrix} x_{2,2}x_{3,3}x_{1,2}x_{3,1}, & x_{2,5}x_{3,3}x_{2,4}x_{1,2}, \\ x_{3,2}x_{2,3}x_{3,4}x_{1,2}, & x_{2,1}x_{1,2}x_{2,3}x_{3,5} \end{Bmatrix}.$$

By Remark 1.2, required number of paths and stars for the remaining choices of p and q can be obtained from the paired stars given above. □

Lemma 4.5. *There exists a $(3;p,q)$-decomposition of $K_3 \times K_6$, for every admissible pair (p,q) of nonnegative integers with $3(p+q) = |E(K_3 \times K_6)|$.*

Proof. We can write $K_3 \times K_6 = (K_3 \times K_3) \oplus (K_3 \times K_3) \oplus (K_3 \times K_{3,3})$. By Theorem 1.1 and Lemma 4.1, $K_3 \times K_{3,3} (\cong K_{3,3} \times K_3)$ has a $(3;p,q)$-decomposition with $p \neq 1$. Also, by Lemma 4.2, we have a $(3;p,q)$-decomposition of $K_3 \times K_3$. Hence by Remark 1.1, the graph $K_3 \times K_6$ has the desired decomposition. □

Lemma 4.6. *There exists a $(3;p,q)$-decomposition of $K_3 \times K_8$, for every admissible pair (p,q) of nonnegative integers with $3(p+q) = |E(K_3 \times K_8)|$.*

Proof. We know that $K_3 \times K_8 = K_{8,8,8} \backslash E(8K_3)$. Let $V(K_{8,8,8}) = X(= \{x_{1,j} : 1 \leq j \leq 8\}) \cup Y(= \{x_{2,j} : 1 \leq j \leq 8\}) \cup Z(= \{x_{3,j} : 1 \leq j \leq 8\})$ and $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, $Z = Z_1 \cup Z_2$, where $X_1 = \{x_{1,j} : 1 \leq j \leq 4\}$, $X_2 = \{x_{1,j} : 5 \leq j \leq 8\}$, $Y_1 = \{x_{2,j} : 1 \leq j \leq 4\}$, $Y_2 = \{x_{2,j} : 5 \leq j \leq 8\}$, $Z_1 = \{x_{3,j} : 1 \leq j \leq 4\}$, $Z_2 = \{x_{3,j} : 5 \leq j \leq 8\}$. We can view $K_3 \times K_8$ as $(K_{X_1,Y_1,Z_1} \backslash E(4K_3)) \oplus (K_{X_2,Y_2,Z_2} \backslash E(4K_3)) \oplus K_{X_1,Y_2} \oplus K_{Y_2,Z_1} \oplus K_{Z_1,X_2} \oplus K_{X_2,Y_1} \oplus K_{Y_1,Z_2} \oplus K_{Z_2,X_1}$. Hence $K_3 \times K_8 = G_1 \oplus G_2$, where $G_1 \cong G_2 \cong (K_{4,4,4} \backslash E(4K_3) \oplus K_{X_1,Y_2} \oplus K_{Y_2,Z_1} \oplus K_{Z_1,X_2})$. Now, $K_{4,4,4} \backslash E(4K_3) = K_3 \times K_4$ has a $(3;p,q)$-decomposition, by Lemma 4.3. Further $K_{X_1,Y_2} \oplus K_{Y_2,Z_1} \oplus K_{Z_1,X_2}$ can be decomposed into $16S_4$ as follows:

$$\begin{gathered}
\{(x_{1,3}; \boldsymbol{x_{2,5}}, \boldsymbol{x_{2,6}}, x_{2,8}),\ (x_{3,1}; \boldsymbol{x_{2,6}}, x_{2,7}, x_{2,8})\},\\
\{(x_{2,8}; x_{1,2}, \boldsymbol{x_{1,4}}, \boldsymbol{x_{3,2}}),\ (x_{2,5}; x_{3,1}, x_{1,2}, \boldsymbol{x_{1,4}})\},\\
\{(x_{2,5}; \boldsymbol{x_{1,1}}, \boldsymbol{x_{3,2}}, x_{3,3}),\ (x_{1,5}; x_{3,1}, \boldsymbol{x_{3,2}}, x_{3,3})\},\\
\{(x_{2,7}; \boldsymbol{x_{1,3}}, \boldsymbol{x_{3,3}}, x_{1,1}),\ (x_{2,8}; x_{3,4}, \boldsymbol{x_{3,3}}, x_{1,1})\},\\
\{(x_{3,1}; \boldsymbol{x_{1,6}}, \boldsymbol{x_{1,7}}, x_{1,8}),\ (x_{3,2}; x_{1,6}, \boldsymbol{x_{1,7}}, x_{1,8})\},\\
\{(x_{3,3}; \boldsymbol{x_{1,6}}, \boldsymbol{x_{1,7}}, x_{1,8}),\ (x_{3,4}; x_{1,6}, \boldsymbol{x_{1,7}}, x_{1,8})\},\\
\{(x_{2,6}; x_{1,2}, \boldsymbol{x_{3,2}}, \boldsymbol{x_{3,3}}),\ (x_{2,7}; x_{1,2}, x_{1,4}, \boldsymbol{x_{3,2}})\},\\
\{(x_{3,4}; x_{2,7}, x_{2,5}, x_{1,5}),\ (x_{2,6}; x_{1,1}, x_{1,4}, x_{3,4})\}.
\end{gathered}$$

From the last $4S_4$ we have either $\{1P_4,\ 3S_4\}$ or $\{3P_4,\ 1S_4\}$ or $\{4P_4\}$ as follows:

$$\left\{\begin{matrix} x_{2,7}x_{1,4}x_{2,6}x_{1,1}, & (x_{2,6}; x_{1,2}, x_{3,2}, x_{3,3}), \\ (x_{3,4}; x_{2,6}, x_{2,5}, x_{1,5}), & (x_{2,7}; x_{1,2}, x_{3,4}, x_{3,2}) \end{matrix}\right\}$$

or

$$\left\{\begin{matrix} x_{2,7}x_{1,4}x_{2,6}x_{1,1}, & x_{2,6}x_{1,2}x_{2,7}x_{3,4}, \\ x_{3,3}x_{2,6}x_{3,2}x_{2,7}, & (x_{3,4}; x_{2,6}, x_{2,5}, x_{1,5}) \end{matrix}\right\}$$

or

$$\left\{\begin{matrix} x_{2,7}x_{1,4}x_{2,6}x_{1,1}, & x_{3,3}x_{2,6}x_{3,2}x_{2,7}, \\ x_{1,2}x_{2,7}x_{3,4}x_{2,5}, & x_{1,2}x_{2,6}x_{3,4}x_{1,5} \end{matrix}\right\}.$$

By Remark 1.2, required number of paths and stars for the remaining choices of p and q can be obtained from the paired stars given above. □

Theorem 4.7. *The graph $K_m \times K_n$ has a $(3;p,q)$-decomposition for every admissible pair (p,q) of nonnegative integers with $3(p+q) = E(K_m \times K_n)$ if and only if $mn(m-1)(n-1) \equiv 0 \pmod 6$, $(p,q) = (2,0)$ when $(m,n) = (2,3)$ or $(m,n) = (3,2)$ and $p \neq 1$ when $(m,n) = (2,4)$ or $(m,n) = (4,2)$.*

Proof. When $m = 2$ and $n = 3, 4$ or $m = 3, 4$ and $n = 2$, the result follows from Theorem 2.6.

Necessity. Since $K_m \times K_n$ is $(n-1)(m-1)$-regular with mn vertices, the necessity follows from Lemma 2.5.

Sufficiency. To construct the required decomposition, we consider the following two cases.

Case(1) $n \equiv 0$ *or* $1 \pmod 3$.

The graph $K_m \times K_n$ can be viewed as edge-disjoint union of $m(m-1)/2$ copies of $K_{n,n} - I$. Since $n \equiv 0$ *or* $1 \pmod 3$, by Theorem 2.6, the graph $K_{n,n} - I$ has a $(3; p, q)$-decomposition except when $(n, p) = (4, 1)$ or when $n = 3$ and $q > 0$. Hence by Remark 1.1, the graph $K_m \times K_n$ has the desired decomposition except $(n, p) = (4, 1)$ and $q > 0$ when $n = 3$. We prove the required decomposition for $(n, p) = (4, 1)$ and $q > 0$ when $n = 3$ in two subcases.

Subcase 1(i) $m \equiv 0$ *or*$1 \pmod 3$.

Since $K_m \times K_n \cong K_n \times K_m$, the graph $K_n \times K_m$ can be viewed as edge-disjoint union of $n(n-1)/2$ copies of $K_{m,m} - I$. Since $m \equiv 0$ *or* 1 (mod 3), by Theorem 2.6, the graph $K_{m,m} - I$ has a $(3; p, q)$-decomposition except when $(m, p) = (4, 1)$ and $m = 3, q > 0$. Hence by Remark 1.1, the graph $K_m \times K_n$ has the desired decomposition except when $(m, p) = (4, 1)$ and $q > 0$ when $m = 3$. Here $K_3 \times K_3$ and $K_3 \times K_4$ have a $(3; p, q)$-decomposition, by Lemmas 4.2 and 4.3. So it is enough to prove the required decomposition for $(m, n, p) = (4, 4, 1)$. We can write $K_4 \times K_4 = (K_3 \times K_4) \oplus (S_4 \times K_4)$. By Remark 1.3, $S_4 \times K_4$ has an S_4-decomposition. Also, by Lemma 4.3, $K_3 \times K_4$ has a $(3; p, q)$-decomposition and hence by Remark 1.1, the graph $K_4 \times K_4$ has the desired decomposition.

Subcase 1(ii) $m \equiv 2 \pmod 3$.

When $n = 4$, if $m = 6k + 2, k \in \mathbb{Z}^+$, then $K_m \times K_4 = (K_8 \times K_4) \oplus (K_{6(k-1)} \times K_4) \oplus (K_{8,6(k-1)} \times K_4) = (K_8 \times S_4) \oplus (K_8 \times K_3) \oplus (K_{6(k-1)} \times K_4) \oplus (K_{8,6(k-1)} \times K_4)$. By Theorem 1.1 and Remark 1.3, $K_8 \times S_4$ and $K_{8,6(k-1)} \times K_4$ have an S_4-decomposition. Also by Lemma 4.6, $K_8 \times K_3$ has a $(3; p, q)$-decomposition. Since $K_{6(k-1)} \times K_4$ has a $(3; p, q)$-decomposition (by Subcase 1(i)), by Remark 1.1, the graph $K_m \times K_4$ has the desired decomposition.

If $m = 6k + 5, k \geq 0$ is an integer, then $K_m \times K_4 = (K_5 \times K_4) \oplus (K_{6k} \times K_4) \oplus (K_{5,6k} \times K_4) = (K_5 \times S_4) \oplus (K_5 \times K_3) \oplus (K_{6k} \times K_4) \oplus (K_{5,6k} \times K_4)$. By Theorem 1.1 and Remark 1.3, $K_5 \times S_4$ and $K_{5,6k} \times K_4$ have a S_4-decomposition. Also by Lemma 4.4, $K_5 \times K_3$ has a $(3; p, q)$-decomposition. Since $K_{6k} \times K_4$ has a $(3; p, q)$-decomposition (by Subcase 1(i)), by Remark 1.1, the graph $K_m \times K_4$ has the desired decomposition.

When $n=3$, if $m=6k+2, k \in \mathbb{Z}^+$, $K_m \times K_3 = (K_8 \times K_3) \oplus (K_{6(k-1)} \times K_3) \oplus (K_{6(k-1),8} \times K_3)$. By Lemma 4.6, $K_8 \times K_3$ has a $(3;p,q)$-decomposition and by Theorem 1.1 and Lemma 4.1, $K_{6(k-1),8} \times K_3$ has a $(3;p,q)$-decomposition with $p \neq 1$. Since $K_{6(k-1)} \times K_3$ has a $(3;p,q)$-decomposition (by Subcase 1(i)), by Remark 1.1, the graph $K_m \times K_3$ has the desired decomposition with $p \neq 1$. For $p=1$, the required decomposition can be obtained from a $(3;1,q)$-decomposition of $K_8 \times K_3$ and $(3;0,q)$-decomposition of the remaining graphs.

If $m=6k+5$, $k \geq 0$ is an integer, $K_m \times K_3 = (K_5 \times K_3) \oplus (K_{6k} \times K_3) \oplus (K_{6k,5} \times K_3)$. By Lemma 4.4, $K_5 \times K_3$ has a $(3;p,q)$-decomposition and by Theorem 1.1 and Lemma 4.1, $K_{6k,5} \times K_3$ has a $(3;p,q)$-decomposition with $p \neq 1$. Since $K_{6k} \times K_3$ has a $(3;p,q)$-decomposition, by Remark 1.1, the graph $K_m \times K_3$ has the desired decomposition with $p \neq 1$. For $p=1$, the required decomposition can be obtained from a $(3;1,q)$-decomposition of $K_5 \times K_3$ and $(3;0,q)$-decomposition of the remaining graphs.

Case(2) $m \equiv 0$ *or* $1 \pmod 3$ and $n \equiv 2 \pmod 3$.

Since tensor product is commutative, $K_m \times K_n \cong K_n \times K_m$. By Case 1, $K_n \times K_m$ has a $(3;p,q)$-decomposition.

□

5 $(3;p,q)$-decomposition of $K_m \otimes \overline{K_n}$

In this section we obtain the existence of $(3;p,q)$-decomposition of complete multipartite graph as follows:

Lemma 5.1. *The graph $K_3 \otimes \overline{K_2}$ has a $(3;p,q)$-decomposition, for every admissible pair (p,q) of nonnegative integers with $3(p+q) = \left|E(K_3 \otimes \overline{K_2})\right|$.*

Proof. Let $V(K_3 \otimes \overline{K_2}) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 2\}$. Now, $K_3 \otimes \overline{K_2}$ has a $(3;p,q)$-decomposition as follows:

1. $p=0,\ q=4$. The required stars are
   $(x_{1,1}; x_{2,1}, x_{2,2}, x_{3,2})$, $(x_{1,2}; x_{2,1}, x_{2,2}, x_{3,1})$, $(x_{3,1}; x_{1,1}, x_{2,1}, x_{2,2})$, $(x_{3,2}; x_{1,2}, x_{2,1}, x_{2,2})$.
2. $p=1,\ q=3$. The required path and stars are
   $x_{3,1}x_{2,1}x_{3,2}x_{2,2}$ and $(x_{1,1}; x_{3,2}, x_{2,1}, x_{3,1})$, $(x_{1,2}; x_{3,1}, x_{2,1}, x_{3,2})$, $(x_{2,2}; x_{1,1}, x_{1,2}, x_{3,1})$ respectively.
3. $p=2,\ q=2$. The required paths and stars are
   $x_{3,1}x_{2,1}x_{3,2}x_{1,2}$, $x_{3,2}x_{2,2}x_{3,1}x_{1,1}$ and $(x_{1,1}; x_{2,1}, x_{2,2}, x_{3,2})$, $(x_{1,2}; x_{2,1}, x_{2,2}, x_{3,1})$ respectively.

4. $p = 3,\ q = 1$. The required paths and star are $x_{1,1}x_{3,1}x_{1,2}x_{2,1}$, $x_{1,2}x_{3,2}x_{1,1}x_{2,1}$, $x_{3,1}x_{2,1}x_{3,2}x_{2,2}$ and $(x_{2,2}; x_{1,1}, x_{1,2}, x_{3,1})$ respectively.
5. $p = 4,\ q = 0$. The required paths are $x_{1,1}x_{3,1}x_{1,2}x_{2,1}$, $x_{1,2}x_{3,2}x_{1,1}x_{2,1}$, $x_{2,1}x_{3,2}x_{2,2}x_{1,2}$, $x_{1,1}x_{2,2}x_{3,1}x_{2,1}$. □

Lemma 5.2. *The graph $K_3 \otimes \overline{K_3}$ has a $(3; p, q)$-decomposition, for every admissible pair (p, q) of nonnegative integers with $3(p+q) = \left|E(K_3 \otimes \overline{K_3})\right|$.*

Proof. Let $V(K_3 \otimes \overline{K_3}) = \{x_{i,j} : 1 \leq i, j \leq 3\}$. Since $K_3 \otimes \overline{K_3} = 3K_{3,3}$, $K_3 \otimes \overline{K_3}$ has a $(3; p, q)$-decomposition with $p \neq 1$, by Theorem 1.1. For $p = 1$, the required path and stars are
$x_{2,1}x_{1,2}x_{2,3}x_{3,2}$, $(x_{1,1}; x_{2,1}, x_{2,2}, x_{2,3})$, $(x_{1,1}; x_{3,1}, x_{3,2}, x_{3,3})$, $(x_{1,2}; x_{3,1}, x_{3,2}, x_{2,2})$, $(x_{1,3}; x_{3,1}, x_{3,2}, x_{2,2})$, $(x_{2,1}; x_{3,1}, x_{3,2}, x_{1,3})$, $(x_{2,2}; x_{3,1}, x_{3,2}, x_{3,3})$, $(x_{2,3}; x_{3,1}, x_{3,3}, x_{1,3})$, $(x_{3,3}; x_{1,2}, x_{1,3}, x_{2,1})$. □

Lemma 5.3. *The graph $K_3 \otimes \overline{K_4}$ has a $(3; p, q)$-decomposition, for every admissible pair (p, q) of nonnegative integers with $3(p+q) = \left|E(K_3 \otimes \overline{K_4})\right|$.*

Proof. Since $K_3 \otimes \overline{K_4} = K_{4,4,4}$, let $V(K_{4,4,4}) = V_1 \cup V_2 \cup V_3$, where $V_i = V_i^1(= \{x_{i,1},\ x_{i,2}\}) \cup V_i^2(= \{x_{i,3},\ x_{i,4}\})$. We can view $K_{4,4,4}$ as $(K_3 \otimes \overline{K_2}) \oplus (K_3 \otimes \overline{K_2}) \oplus_{i \neq j \in \{1,2,3\}} K_{V_i^1, V_j^2}$. Now, $\oplus_{i \neq j \in \{1,2,3\}} K_{V_i^1, V_j^2}$ has a S_4-decomposition as follows: $\{(x_{i,1}; x_{2,3}, \boldsymbol{x_{2,4}, x_{j,3}}),\ (x_{i,2}; x_{2,3}, \boldsymbol{x_{2,4}}, x_{j,4})\}$, $\{(x_{i,3}; x_{2,1}, \boldsymbol{x_{2,2}, x_{j,2}}),\ (x_{i,4}; x_{2,1}, \boldsymbol{x_{2,2}}, x_{j,1})\}$, $i = 1,\ j = 3$ and $i = 3,\ j = 1$. By Remark 1.2, we can use these pairs of stars to construct the required decomposition into an even number of paths and stars. For odd p and q, we decompose $K_3 \otimes \overline{K_2}$ into odd number of paths and stars using Lemma 5.1. Hence by Remark 1.1, the graph $K_3 \otimes \overline{K_4}$ has the desired decomposition. □

Lemma 5.4. *Let G be an S_4-decomposible graph and $p, q \geq 0$ be integers with $3(p+q) = \left|E(G \otimes \overline{K_n})\right|$ and $p \neq 1$. Then $G \otimes \overline{K_n}$ has a $(3; p, q)$-decomposition for all even n and every admissible pair (p, q).*

Proof. Since G is S_4-decomposible graph, for each star $(a; u, v, w)$ in G, we have the following pairs of stars in $G \otimes K_n$; for each $j \in \{1, 3, \cdots, n-1\}$, $\{(x_{a,j}; x_{u,i}, \boldsymbol{x_{v,i}, x_{w,i}}),\ (x_{a,j+1}; x_{u,i}, x_{v,i}, \boldsymbol{x_{w,i}})\}$, where $1 \leq i \leq n$. Then by applying remark 1.2 to the pairs of stars mentioned above we obtained all possible even number of paths and stars of $G \otimes \overline{K_n}$. Now, consider

$$\{(x_{a,1}; x_{u,1}, x_{v,1}, x_{w,1}),\ (x_{a,1}; x_{u,2}, x_{v,2}, x_{w,2}),\ (x_{a,2}; x_{u,1}, x_{v,1}, x_{w,1})\}$$

and decompose it into $3P_4$ as given below. $\{x_{u,2}x_{a,1}x_{u,1}x_{a,2},\ x_{v,2}x_{a,1}x_{v,1}x_{a,2},\ x_{w,2}x_{a,1}x_{w,1}x_{a,2}\}$. The remaining number of paths and stars can be obtained from the remaining pairs of stars given above except when $p = 1$. □

Theorem 5.5. *Let p and q be nonnegative integers, and let $n > 1$. Then $K_m \otimes \overline{K_n}$ has a $(3; p, q)$-decomposition for every admissible pair (p, q) with $3(p+q) = E(K_m \otimes \overline{K_n})$ if and only if $mn^2(m-1) \equiv 0 \pmod 6$ and $p \neq 1$ when $(m, n) = (2, 3)$.*

Proof. When $(m, n) = (2, 3)$, the result follows from Theorem 1.1.

Necessity. Since $K_m \otimes \overline{K_n}$ is $n(m-1)$-regular with mn vertices, the necessity follows from Lemma 2.5.

Sufficiency. To construct the required decomposition, we consider the following two cases.

Case(1) $n \equiv 0 \pmod 3$.

The graph $K_m \otimes \overline{K_n}$ can be viewed as edge-disjoint union of $m(m-1)/2$ copies of $K_{n,n}$. Since $n \equiv 0 \pmod 3$, by Theorem 1.1, the graph $K_{n,n}$ has a $(3; p, q)$-decomposition except $p = 1$ when $n = 3$. Hence by Remark 1.1, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition except when$(n, p) = (3, 1)$.

Subcase 1(i) $m \equiv 0$ *or* $1 \pmod 3$.

We can write $K_m \otimes \overline{K_3} = 3K_m \oplus (K_m \times K_3)$. Since $m \equiv 0$ *or* $1 \pmod 3$, by Theorem 1.2, the graph K_m has a $(3; p, q)$-decomposition, whenever $m \geq 6$. Also by Theorem 4.7, $K_m \times K_3$ has a $(3; p, q)$-decomposition. Hence by Remark 1.1, the graph $K_m \otimes \overline{K_3}$ has the desired decomposition whenever $m \geq 6$. Since $K_4 \otimes \overline{K_3} = (K_3 \otimes \overline{K_3}) \oplus (S_4 \otimes \overline{K_3})$, by Remark 1.4, $S_4 \otimes \overline{K_3}$ has an S_4-decomposition and by Lemma 5.2, $K_3 \otimes \overline{K_3}$ has a $(3; p, q)$-decomposition and hence we have the required decomposition for $m = 3, 4$.

Subcase 1(ii) $m \equiv 2 \pmod 3$.

Let $m = 3k + 2$, $k \geq 0$ be an integer, $K_m \otimes \overline{K_3} = (K_{3k} \otimes \overline{K_3}) \oplus (K_2 \otimes \overline{K_3}) \oplus (K_{3k,2} \otimes \overline{K_3})$. By Theorem 1.1 and Remark 1.4, $K_{3k,2} \otimes \overline{K_3}$ and $K_2 \otimes \overline{K_3} \cong (K_{3,3})$ have a S_4-decomposition. By Subcase 1(i), we have that $K_{3k} \otimes \overline{K_3}$ has a required decomposition and hence by Remark 1.1, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition.

Case(2) $m \equiv 0$ *or* $1 \pmod 3$ and $n \equiv 1$ *or* $2 \pmod 3$.

We can write $K_m \otimes \overline{K_n} = nK_m \oplus (K_m \times K_n)$. Since $m \equiv 0$ *or* $1 \pmod 3$, by Theorem 1.2, the graph K_m has a $(3; p, q)$-decomposition, where $m \geq 6$. Also by Theorem 4.7, $K_m \times K_n$ has a $(3; p, q)$-decomposition. Hence by Remark 1.1, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition whenever $m \geq 6$. For $m < 6$ i.e. when $m = 3, 4$, to construct the required decomposition, we consider the following two subcases.

Subcase 2(i) $m = 3$.

When $n = 3k+1 \geq 4$, we write $K_m \otimes \overline{K_n} = K_3 \otimes \overline{K_{3k+1}} = (K_3 \otimes \overline{K_4}) \oplus (K_3 \otimes \overline{K_{3(k-1)}}) \oplus 6K_{4,3(k-1)}$. By Lemma 5.3 and Case 1, $K_3 \otimes \overline{K_4}$ and $K_3 \otimes \overline{K_{3(k-1)}}$ have a $(3; p, q)$-decomposition. Also, by Theorem 1.1, $K_{4,3(k-1)}$ has a $(3; p, q)$-decomposition with $p \neq 1$ when $k = 2$. Hence by Remark 1.1, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition with $p \neq 1$ when $k = 2$. For $p = 1$, the required decomposition can be obtained from a $(3; 1, q)$-decomposition of $K_3 \otimes \overline{K_4}$ and $(3; 0, q)$-decomposition of the remaining graphs.

When $n = 3k + 2$, $K_m \otimes \overline{K_n} = K_3 \otimes \overline{K_{3k+2}} = (K_3 \otimes \overline{K_2}) \oplus (K_3 \otimes \overline{K_{3k}}) \oplus 6K_{2,3k}$. By Lemma 5.1 and Case 1, $K_3 \otimes \overline{K_2}$ and $K_3 \otimes \overline{K_{3k}}$ have a $(3; p, q)$-decomposition. Also, by Theorem 1.1, $K_{2,3k}$ has a $(3; p, q)$-decomposition with $p \neq 1$. Hence by Remark 1.1, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition with $p \neq 1$. For $p = 1$, the required decomposition can be obtained from a $(3; 1, q)$-decomposition of $K_3 \otimes \overline{K_2}$ and $(3; 0, q)$-decomposition of the remaining graphs.

Subcase 2(ii) $m = 4$.

When $n = 3k+1 \geq 4$, we write $K_m \otimes \overline{K_n} = K_4 \otimes \overline{K_{3k+1}} = (K_4 \otimes \overline{K_4}) \oplus (K_4 \otimes \overline{K_{3(k-1)}}) \oplus 12K_{4,3(k-1)} = (K_3 \otimes \overline{K_4}) \oplus (S_4 \otimes \overline{K_4}) \oplus (K_4 \otimes \overline{K_{3(k-1)}}) \oplus 12K_{4,3(k-1)}$. By Lemmas 5.3 and 5.4 and Case 1, $K_3 \otimes \overline{K_4}$, $S_4 \otimes \overline{K_4}$ and $K_4 \otimes \overline{K_{3(k-1)}}$ have a $(3; p, q)$-decomposition. Also, by Theorem 1.1, $K_{4,3(k-1)}$ has a $(3; p, q)$-decomposition with $p \neq 1$ when $k = 2$. Hence by Remark 1.1, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition (as in Subcase 2(i)).

When $n = 3k + 2$, we write $K_m \otimes \overline{K_n} = K_4 \otimes \overline{K_{3k+2}} = (K_3 \otimes \overline{K_2}) \oplus (S_4 \otimes \overline{K_2}) \oplus (K_4 \otimes \overline{K_{3k}}) \oplus 12K_{2,3k}$. By Lemmas 5.1 and 5.4 and Case 1, $K_3 \otimes \overline{K_2}$, $S_4 \otimes \overline{K_2}$ and $K_4 \otimes \overline{K_{3k}}$ have a $(3; p, q)$-decomposition. Also by Theorem 1.1, $K_{2,3k}$ has a $(3; p, q)$-decomposition with $p \neq 1$. Hence by Remark 1.1, the graph $K_m \otimes \overline{K_n}$ has the desired decomposition (as in Subcase 2(i)). □

Acknowledgement

The authors thank the Department of Science and Technology, Government of India, New Delhi for its financial support through the Grant No. DST/ SR/ S4/ MS:828/13. The second author thank the University Grant Commission for its support through the Grant No. F.510/7/DRS-I/2016(SAP-I). The authors would like to thank the referees for their valuable suggestions and comments, which improved the presentation of the paper.

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